Green function of black holes

ADRIEN KUNTZ

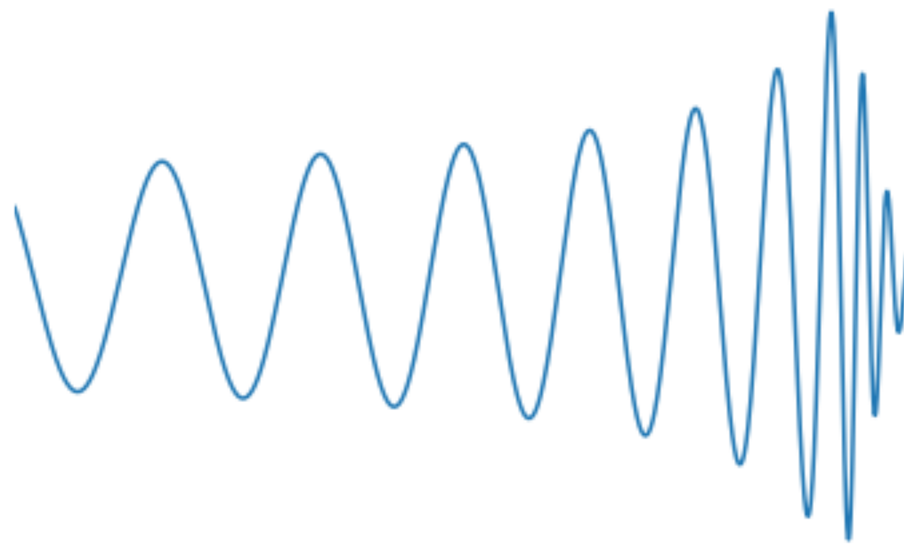
CENTRA, Lisboa

Image: DALL.E

TÉCNICO LISBOA

fct Fundação para a Ciência e a Tecnologia

MOTIVATIONS



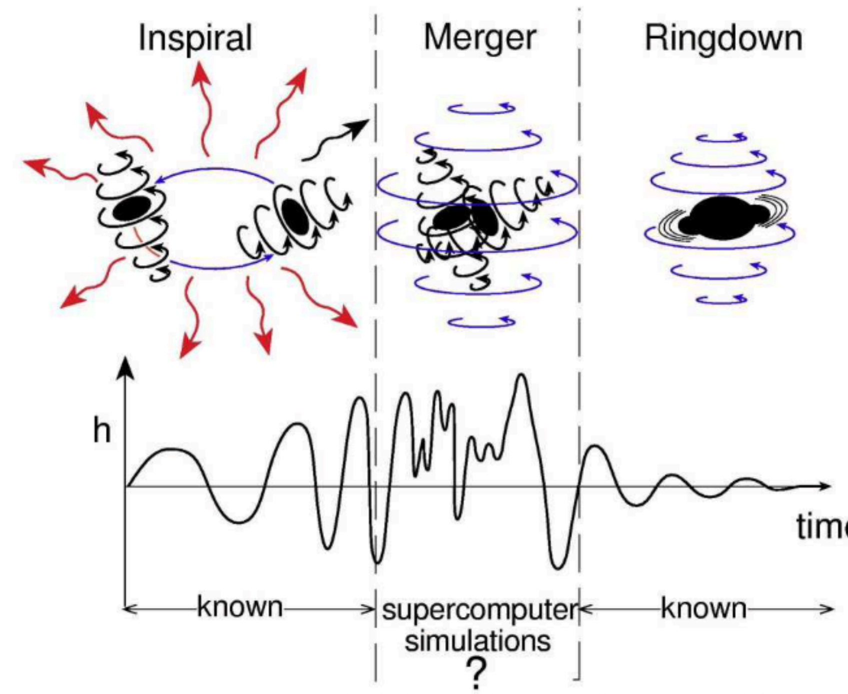
Ringdown waveform: $$\mathfrak{h}_+ - i\mathfrak{h}_\times = \frac{M}{r}\sum_{\ell m \mathcal{N}} \mathscr{A}_{\ell m \mathcal{N}} e^{i\omega_{\ell m \mathcal{N}}(r_* - t)} {}_{-2}Y^{\ell m}(\theta, \phi)$$

- When is the “starting time” ?
- Amplitudes should be time-dependent $\mathscr{A}_{\ell m \mathcal{N}}(t)$?
- Other kind of signals around merger?

Motivations

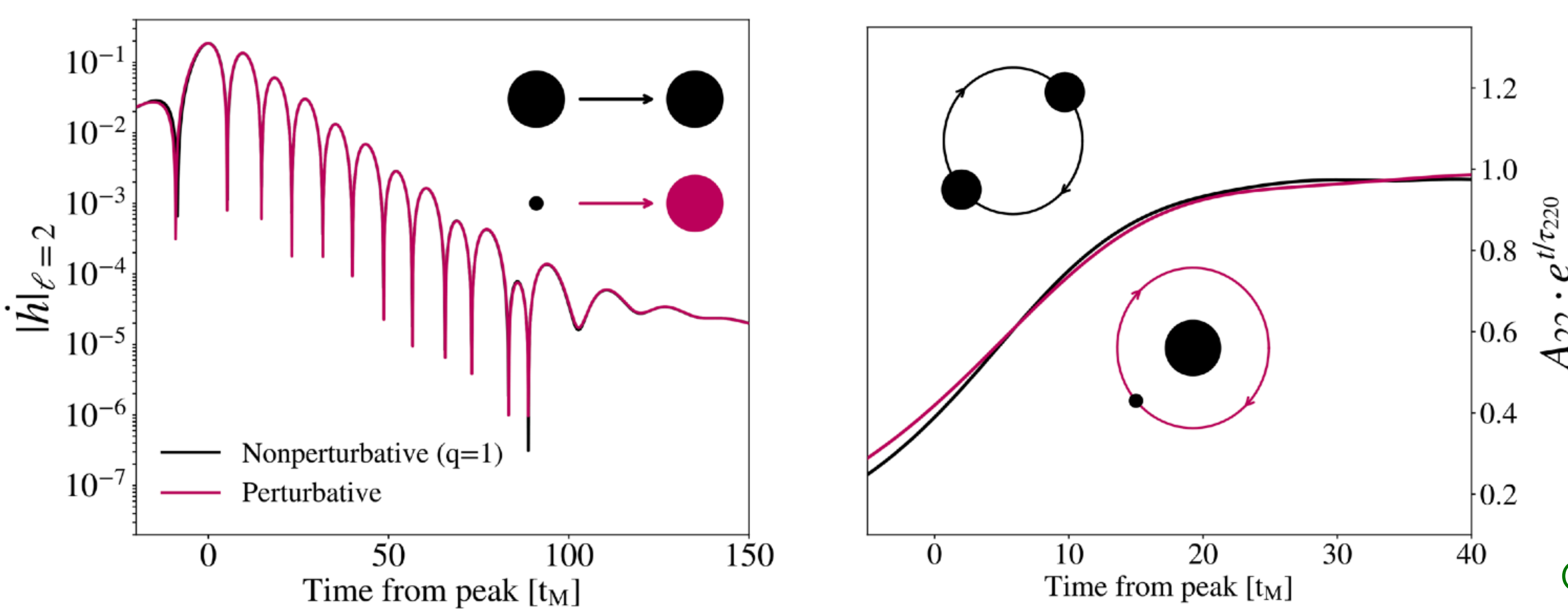
What we often hear:

"Signal is very nonlinear at merger"



Credits: K. Thorne 2004

But the 2-body waveform is extremely similar to the point-particle case!



Credits: G. Carullo

Motivations

A fundamental paper: Leaver 1986

Spectral decomposition of the perturbation response of the Schwarzschild geometry

Edward W. Leaver*
Department of Physics and The College of Science Computer, University of Utah, Salt Lake City, Utah 84112
(Received 10 February 1986)

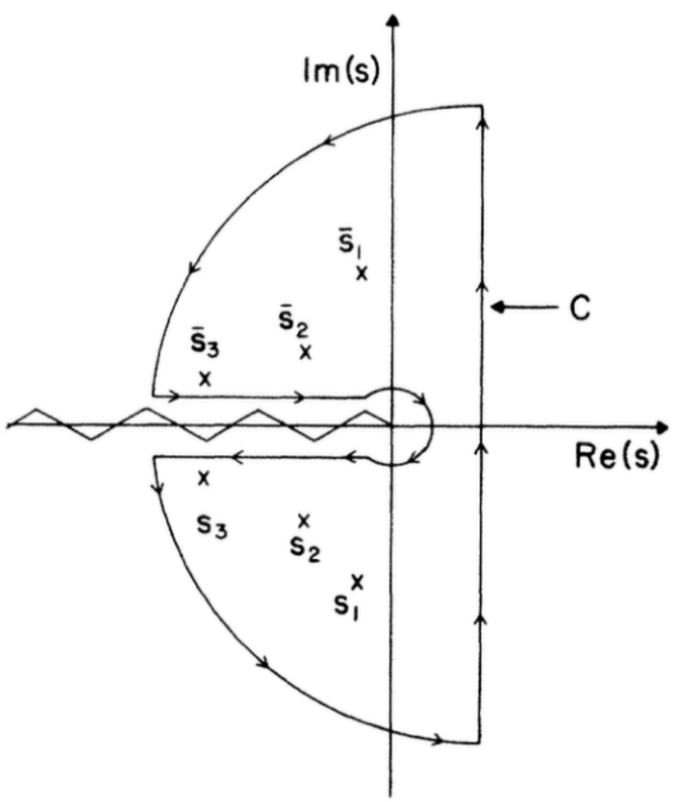


FIG. 1. Inversion contour for

$$G(r_*,t\,|\,r'_*,t')=(2\pi i)^{-1}\int_{\epsilon-i\infty}^{\epsilon+i\infty} g(r_*,r'_*,s)e^{s(t-t')}ds\ ,$$

The contour for the inversion integral (7) for $G(r_*,t\,|\,r'_*,t')$ may be deformed as illustrated in Fig. 1, and the time-domain Green's function expressed as three distinct terms:

$$G(r_*,t\,|\,r'_*,t')=G_F(r_*,t\,|\,r'_*,t')+G_Q(r_*,t\,|\,r'_*,t') + G_B(r_*,t\,|\,r'_*,t')\ , \tag{22}$$

where G_Q is the sum of the residues at the poles of $g(r_*,r'_*,s)$, G_B is the integral of $g(r_*,r'_*,s)$ around the branch cut in s, and G_F is the integral along the large $|s|$ quarter circles. It is G_F that propagates the high-frequency response, and which reduces to the free-space Green's function in the limit as the mass of the black hole goes to zero. The low-frequency ringing and late-time decay tails, that together give radiation phenomena involving black holes its distinct character, are, respectively, described by G_Q and G_B, and it is these two functions that are the subjects of the present investigation. Figure 1

Open problem: What is G_F?

MOTIVATIONS

A fundamental paper: Leaver 1986

Spectral decomposition of the perturbation response of the Schwarzschild geometry

Edward W. Leaver*
Department of Physics and The College of Science Computer, University of Utah, Salt Lake City, Utah 84112
(Received 10 February 1986)

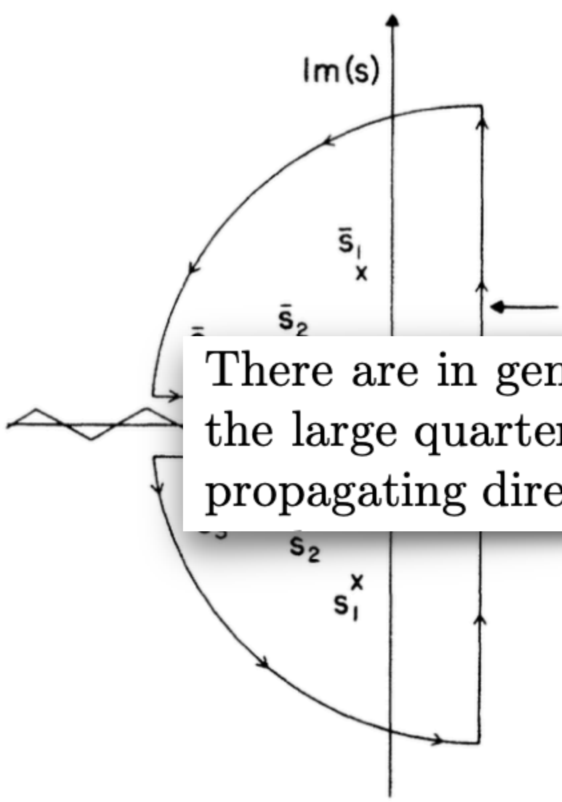


FIG. 1. Inversion contour for

$$G(r_*,t\mid r'_*,t')=(2\pi i)^{-1}\int_{\epsilon-i\infty}^{\epsilon+i\infty} g(r_*,r'_*,s)e^{s(t-t')}ds \ ,$$

The contour for the inversion integral (7) for $G(r_*,t\mid r'_*,t')$ may be deformed as illustrated in Fig. 1, and the time-domain Green's function expressed as three distinct terms:

$$G(r_*,t\mid r'_*,t')=G_F(r_*,t\mid r'_*,t')+G_Q(r_*,t\mid r'_*,t') +G_B(r_*,t\mid r'_*,t') \ , \qquad (22)$$

Berti, Cardoso, Starinets '09

There are in general three different contributions to the integral. The integral along the large quarter-circles is the flat-space analogue of the prompt response, i.e. waves propagating directly from the source to the observer at the speed of light. Depending

frequency response, and which reduces to the free-space Green's function in the limit as the mass of the black hole goes to zero. The low-frequency ringing and late-time decay tails, that together give radiation phenomena involving black holes its distinct character, are, respectively, described by G_Q and G_B, and it is these two functions that are the subjects of the present investigation. Figure 1

Open problem: What is G_F?

MOTIVATIONS

A fundamental paper: Leaver 1986

Spectral decomposition of the perturbation response of the Schwarzschild geometry

Edward W. Leaver*
Department of Physics and The College of Science Computer, University of Utah, Salt Lake City, Utah 84112
(Received 10 February 1986)

The contour for the inversion integral (7) for $G(r_*,t\,|\,r'_*,t')$ may be deformed as illustrated in Fig. 1, and the time-domain Green's function expressed as three distinct terms:

$$G(r_*,t\,|\,r'_*,t') = G_F(r_*,t\,|\,r'_*,t') + G_Q(r_*,t\,|\,r'_*,t')$$

(2) G_F comes from the arcs of the semi-infinite circle of the integration contour (green long-dashed lines in Fig. 4.1), and its contribution includes information about high-magnitude frequency contributions and asymptotically far signals, that propagate effectively in free space since they are insensitive to the potential. For this reason, G_F is expected to be responsible for the prompt response in ringdown solutions, which will mostly depend on the details of the initial conditions.

QNM review 2025

Green's function in the limit as the mass of the black hole goes to zero. The low-frequency ringing and late-time decay tails, that together give radiation phenomena involving black holes its distinct character, are, respectively, described by G_Q and G_B, and it is these two functions that are the subjects of the present investigation. Figure 1

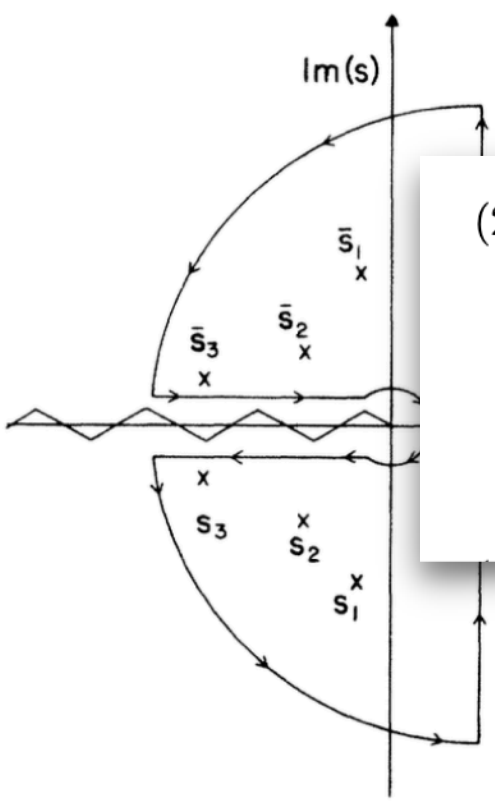


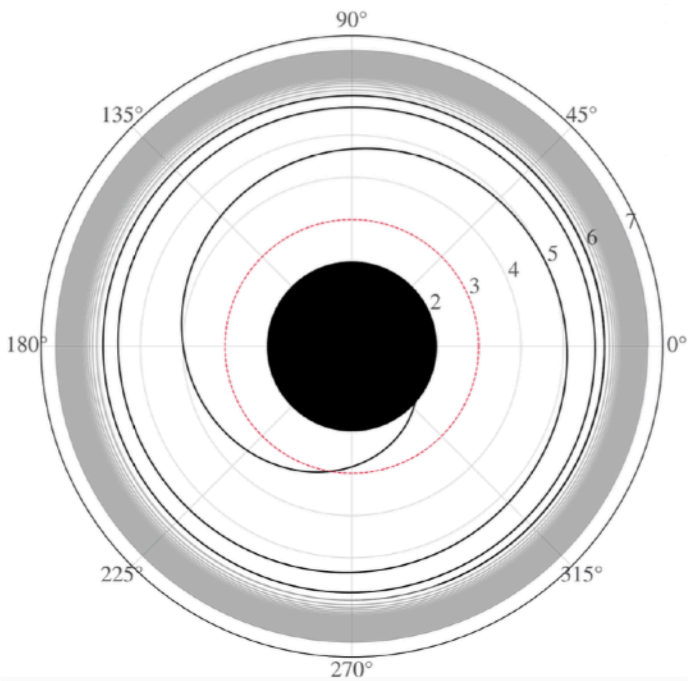
FIG. 1. Inversion contour for
$$G(r_*,t\,|\,r'_*,t') = (2\pi i)^{-1}\int_{\epsilon-i\infty}^{\epsilon+i\infty} g(r_*,r'_*,s)e^{s(t-t')}ds\ ,$$

Open problem: What is G_F?

PLAN

- SOME BASICS ON THE GREEN FUNCTION (GF)
- THE POSCHL-TELLER GF
- SCHWARZSCHILD GF
- REDSHIFT MODES

Some basics



$$\bar{g}_{\mu\nu}\mathrm{d}x^{\mu}\mathrm{d}x^{\nu} = -f(r)\mathrm{d}t^2 + f^{-1}(r)\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2\right)$$

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \varepsilon h^{(1)}_{\mu\nu} \qquad \Rightarrow G^{(1)}_{\mu\nu}\left[\mathbf{h}^{(1)}\right] = 0$$

Using background symmetries and fixing gauge:

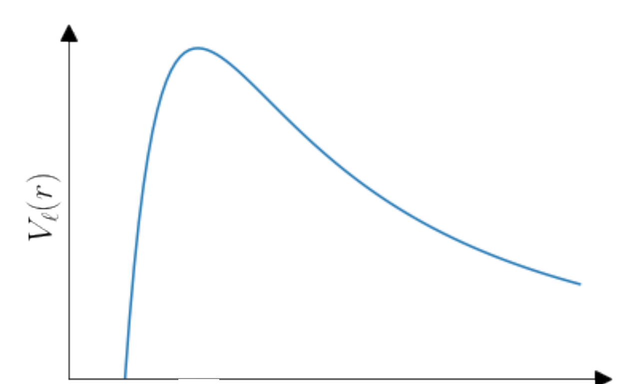
$$h \sim \sum_{\ell,m} \psi_{\ell m}(t,r) Y_{\ell m}(\theta,\phi)$$

$$\longrightarrow \qquad -\frac{\partial^2\psi}{\partial t^2} + \frac{\partial^2\psi}{\partial x^2} - V\psi = -S$$

$$x = r + \log(r-1) \quad (M = 1/2)$$

$$V_\ell(r) = \left(1 - \frac{2GM}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} - \frac{6GM}{r^3}\right)$$

$$S = f(x_{\mathrm{P}}(t))\delta(x - x_{\mathrm{P}}(t)) + g(x_{\mathrm{P}}(t))\delta'(x - x_{\mathrm{P}}(t))$$



Green function: definition

Laplace transform: $\tilde{\psi}(x,\omega) = \int_{t_0}^{\infty} \psi(x,t) e^{i\omega t} \mathrm{d}t \quad \longleftrightarrow \quad \psi(x,t) = \int_{i\gamma-\infty}^{i\gamma+\infty} \tilde{\psi}(x,\omega) e^{-i\omega t} \frac{\mathrm{d}\omega}{2\pi}$

GF in frequency domain, definition:

$$\tilde{G}(x,\bar{x},\omega) = \frac{\Theta(\bar{x}-x)}{W} \psi_{\infty}^{+}(\bar{x},\omega)\psi_{H}^{-}(x,\omega) + \frac{\Theta(x-\bar{x})}{W} \psi_{H}^{-}(\bar{x},\omega)\psi_{\infty}^{+}(x,\omega)$$

Where the homogeneous solutions are

$$\psi_{H}^{-}(x,\omega) \simeq e^{-i\omega x} \; (x \to -\infty) \,, \quad \psi_{\infty}^{+}(x,\omega) \simeq e^{i\omega x} \; (x \to \infty) \qquad W(\omega) = \psi_{H}^{-}(\psi_{\infty}^{+})' - (\psi_{H}^{-})'\psi_{\infty}^{+}$$

In time domain, the GF satisfies

$$-\frac{\partial^2 G}{\partial t^2} + \frac{\partial^2 G}{\partial x^2} - VG = -\delta(x-\bar{x})\delta(t-\bar{t})$$

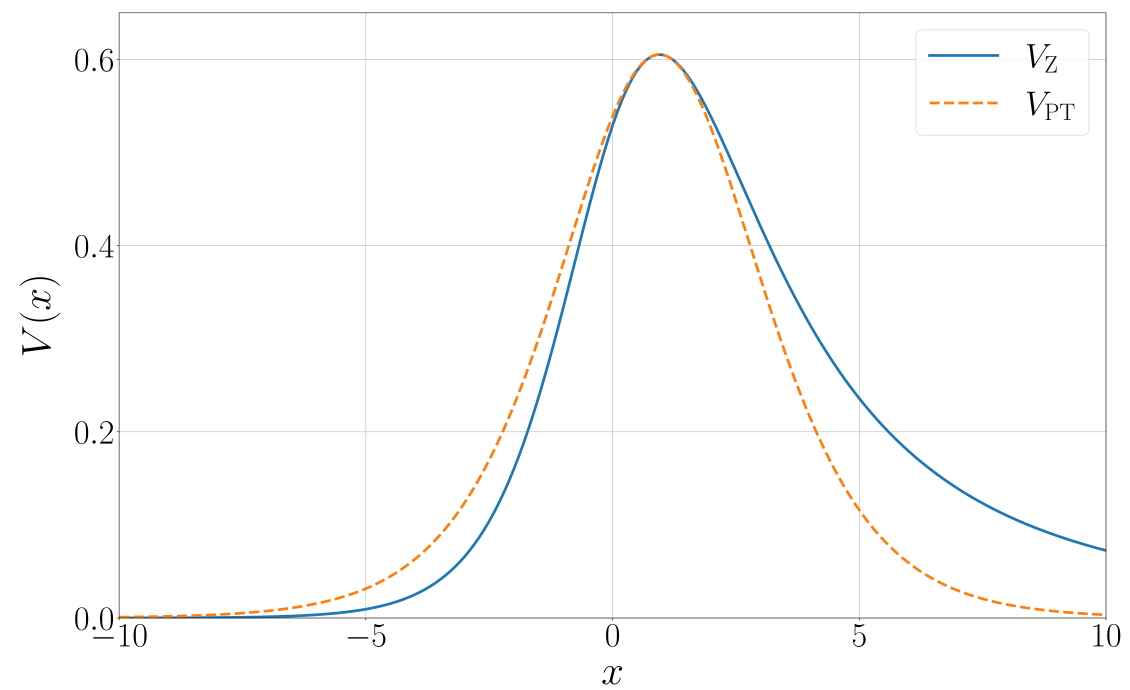
GREEN FUNCTION : DEFINITION

From the GF one can get back the field as

$$\psi(x,t) = \int_{-\infty}^{\infty} \mathrm{d}\bar{x} \int_{t_0}^{\infty} \mathrm{d}\bar{t}\; G(x,\bar{x},t-\bar{t})S(\bar{x},\bar{t}) - \int_{-\infty}^{\infty} \mathrm{d}\bar{x} \left[G(x,\bar{x},t-t_0)\dot{\psi}_0(\bar{x}) + \partial_t G(x,\bar{x},t-t_0)\psi_0(\bar{x})\right]$$

The Poschl-Teller GF

$$V_{PT}(x) = \frac{V_0}{\cosh^2 \alpha(x - x_m)}$$



Homogeneous solutions are hypergeometric:

$$\psi_H^-(x,\omega) = e^{-i\omega x} F\left(\lambda, \bar{\lambda}, 1 - i\omega/\alpha, \left(1 + e^{-2\alpha(x-x_m)}\right)^{-1}\right)$$

$$\psi_\infty^+(x,\omega) = e^{i\omega x} F\left(\lambda, \bar{\lambda}, 1 - i\omega/\alpha, e^{-2\alpha(x-x_m)}\left(1 + e^{-2\alpha(x-x_m)}\right)^{-1}\right) \qquad \lambda = \frac{1}{2}\left(1 + i\sqrt{\frac{4V_0}{\alpha^2} - 1}\right) \simeq 0.5 + 2.09i$$

$$\psi_\infty^-(x,\omega) = e^{-i\omega x} F\left(\lambda, \bar{\lambda}, 1 + i\omega/\alpha, e^{-2\alpha(x-x_m)}\left(1 + e^{-2\alpha(x-x_m)}\right)^{-1}\right)$$

Analytic structure of the GF

Connection coefficients: $\psi_H^-(x,\omega) = A^-(\omega)\psi_\infty^-(x,\omega) + A^+(\omega)\psi_\infty^+(x,\omega)$

$$A^- = \frac{\Gamma\big(1 - i\omega/\alpha\big)\Gamma\big(-i\omega/\alpha\big)}{\Gamma\big(\lambda - i\omega/\alpha\big)\Gamma\big(\bar{\lambda} - i\omega/\alpha\big)} \qquad A^+ = \frac{2i\pi}{|\Gamma\big(\lambda\big)|^2\left(e^{-\pi\omega/\alpha} - e^{\pi\omega/\alpha}\right)}$$

For $x > \bar{x}$ the GF is: $$\tilde{G}(x,\bar{x},\omega) = \frac{\psi_H^-(\bar{x},\omega)\psi_\infty^+(x,\omega)}{2i\omega A^-}$$

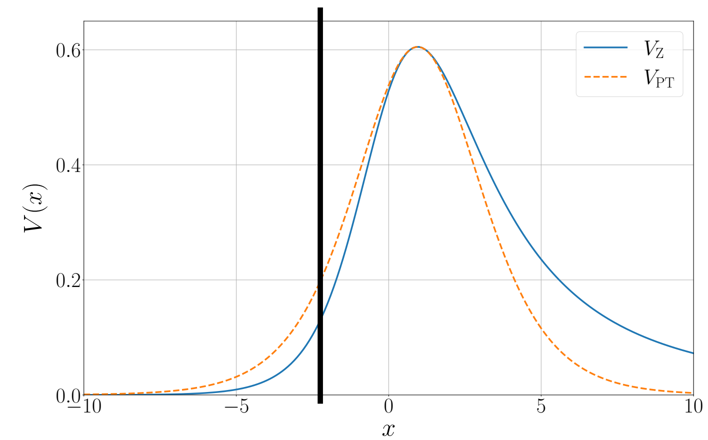
- A^- has double poles at $\omega = -i\alpha(k+1) \quad (k \geq 0)$ (Matsubara)
- Both ψ_H^- and ψ_∞^+ have simple poles at the Matsubaras
- A^- has zeros at the QNMs $\omega_n = -i\alpha(n+\lambda) \quad (n \geq 0)$

The GF itself only has poles at QNMs and nothing (no 0, no pole) at Matsubaras

Kuntz (2025)

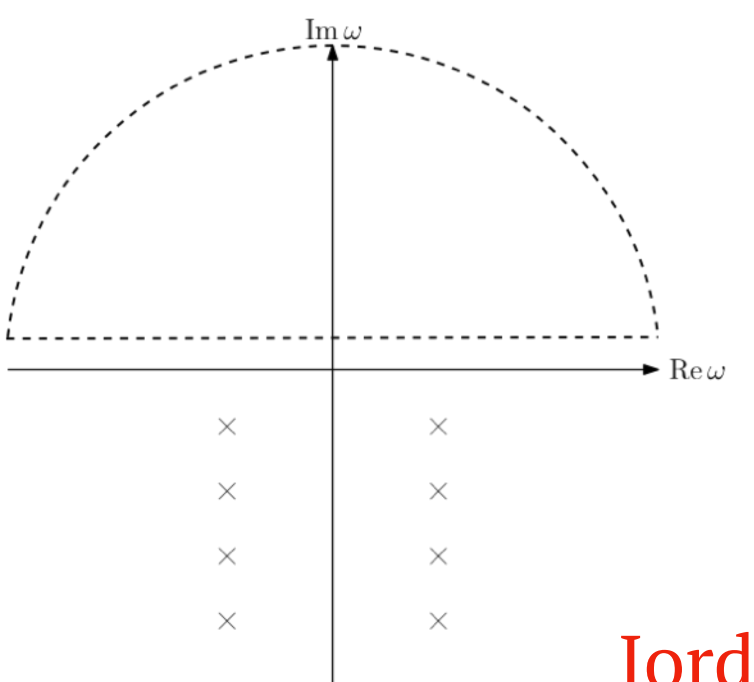
INVERSE LAPLACE TRANSFORM

$$G(x,\bar{x},t) = \int_{i\gamma-\infty}^{i\gamma+\infty} \tilde{G}(x,\bar{x},\omega)e^{-i\omega t}\frac{\mathrm{d}\omega}{2\pi}$$

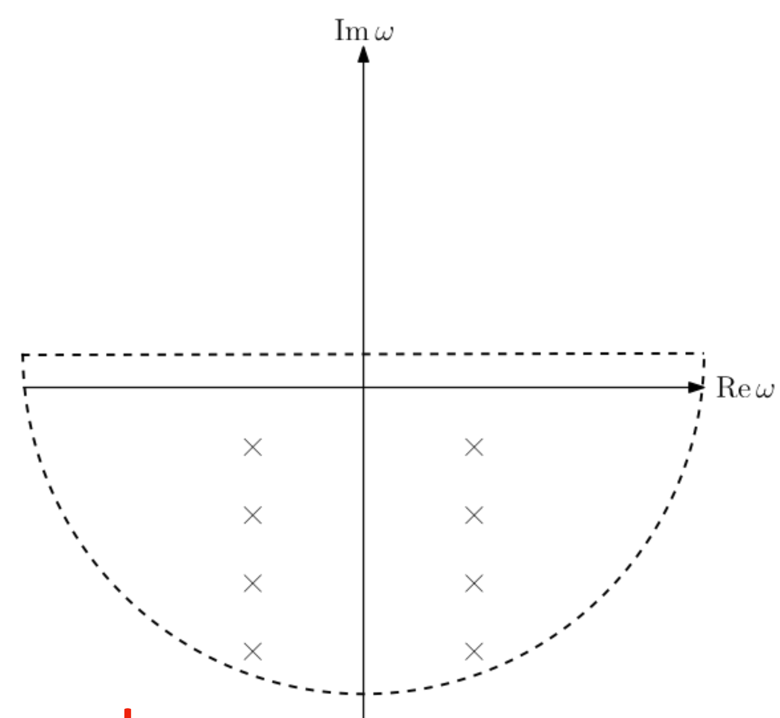


First case: $x < x_m$

$$\tilde{G}e^{-i\omega t} \underset{|\omega|\to\infty}{\sim} e^{i\omega(x-\bar{x}-t)}/\omega$$



$t < x - \bar{x}$



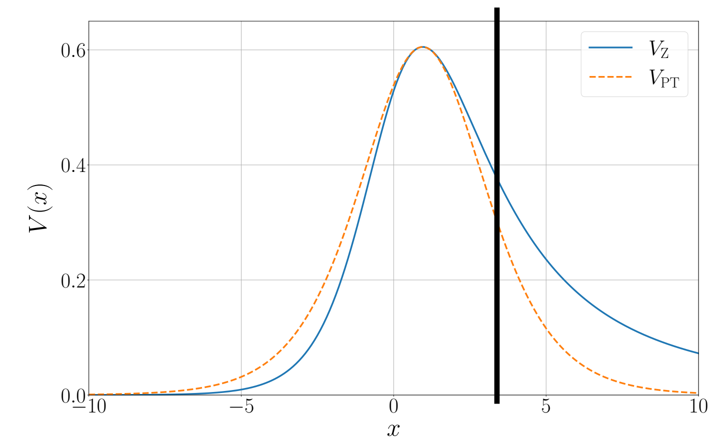
$t > x - \bar{x}$

Jordan's lemma: arc is zero!

Kuntz (2025)

Inverse Laplace transform

Second case: $x > x_m$



Less trivial limit for $|\omega| \to \infty$!

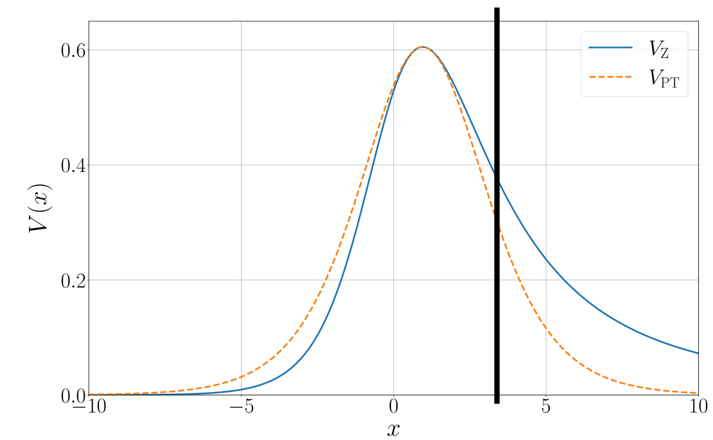
$$\psi_H^-(x,\omega) = e^{-i\omega x} F\left(\lambda, \bar{\lambda}, 1 - i\omega/\alpha, \left(1 + e^{-2\alpha(x-x_m)}\right)^{-1}\right)$$

$$\tilde{G}(x,\bar{x},\omega) = \frac{1}{2i\omega}\psi_\infty^-(\bar{x},\omega)\psi_\infty^+(x,\omega) + \frac{A^+}{2i\omega A^-}\psi_\infty^+(\bar{x},\omega)\psi_\infty^+(x,\omega)$$

$$\omega\tilde{G}(x,\bar{x},\omega)e^{-i\omega t} \underset{|\omega|\to\infty}{\simeq} e^{i\omega(x-\bar{x}-t)} + e^{i\omega(x+\bar{x}-2x_m-t)}e^{-\mathrm{Sgn}(\mathrm{Re}\,\omega)\pi\omega/\alpha}$$

Inverse Laplace transform

Second case: $x > x_m$



Less trivial limit for $|\omega| \to \infty$!

$$\psi_H^-(x,\omega) = e^{-i\omega x} F\left(\lambda, \bar{\lambda}, 1 - i\omega/\alpha, \left(1 + e^{-2\alpha(x - x_m)}\right)^{-1}\right)$$

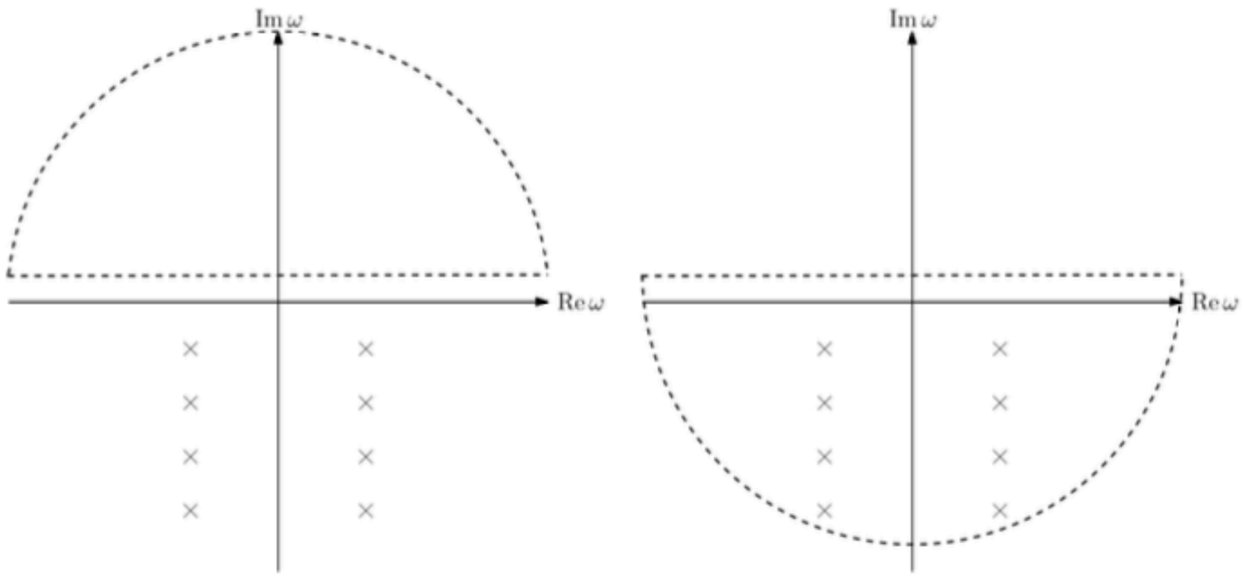
$$\tilde{G}(x,\bar{x},\omega) = \frac{1}{2i\omega}\psi_\infty^-(\bar{x},\omega)\psi_\infty^+(x,\omega) + \frac{A^+}{2i\omega A^-}\psi_\infty^+(\bar{x},\omega)\psi_\infty^+(x,\omega)$$

Poles $\omega = i\alpha(k+1) \quad (k \geq 0)$ Poles $\omega = -i\alpha(k+1) \quad (k \geq 0)$

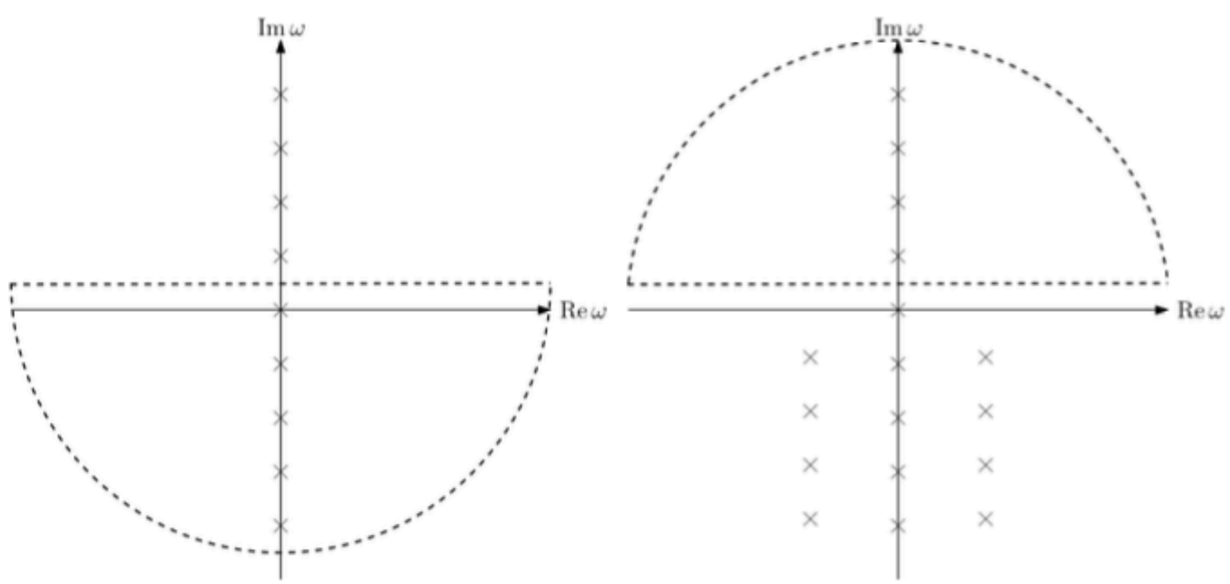
They cancel against these ones!

INVERSE LAPLACE TRANSFORM

$$\omega \tilde{G}(x, \bar{x}, \omega) e^{-i\omega t} \simeq e^{i\omega(x-\bar{x}-t)} + e^{i\omega(x+\bar{x}-2x_m-t)} e^{-\mathrm{Sgn}(\mathrm{Re}\omega)\pi\omega/\alpha}$$



(1) $t < x - \bar{x}$ **(2)** $t > x + \bar{x} - 2x_m$



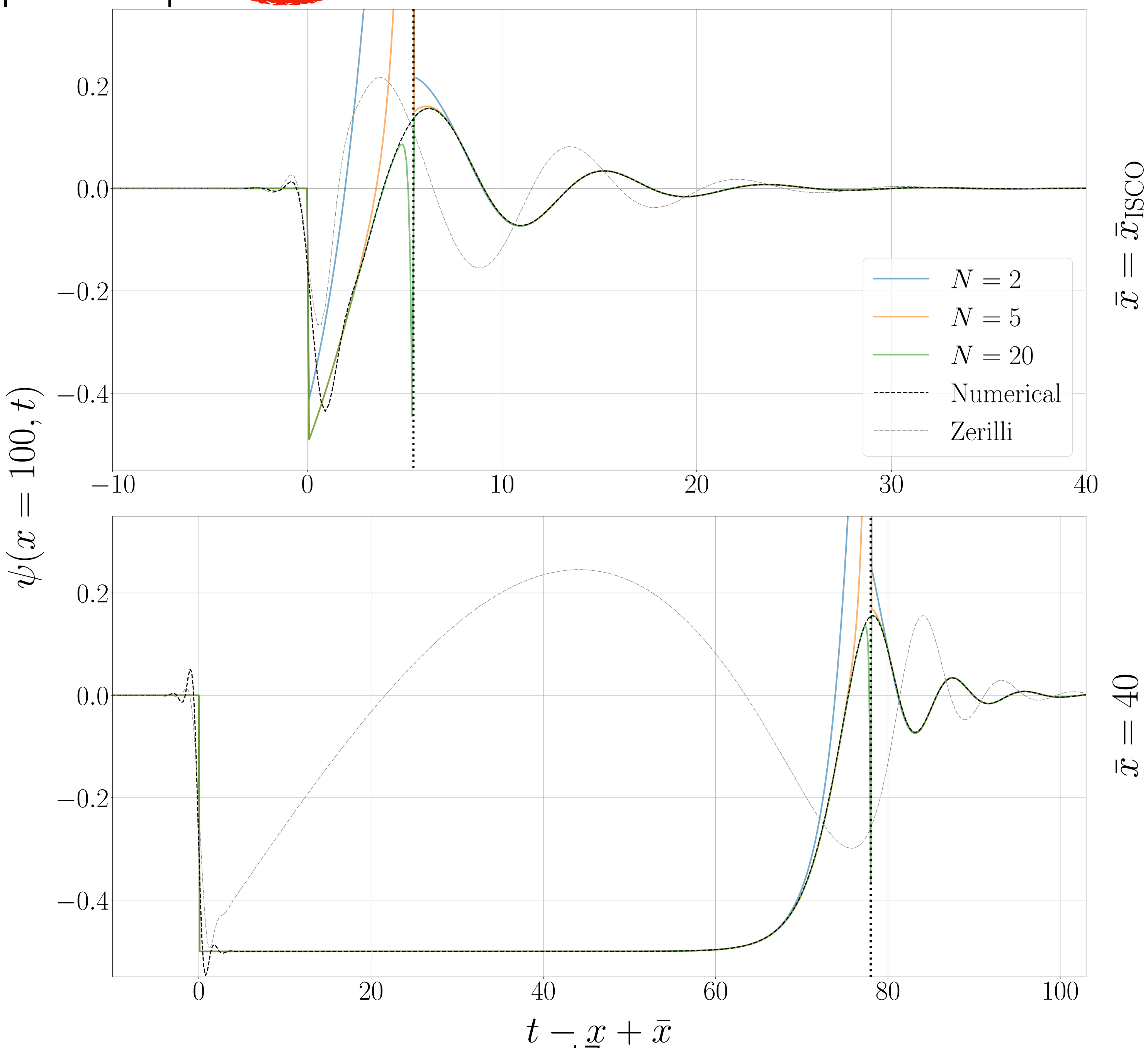
(3) $x - \bar{x} < t < x + \bar{x} - 2x_m$

Kuntz (2025)

THE PROMPT RESPONSE

$$G(x,\bar{x},t) = G_{\mathrm{E}}(x,\bar{x},t)\,, \qquad x-\bar{x} < t < x+\bar{x}-2x_m$$

$$G_{\mathrm{E}}(x,\bar{x},t) = \frac{1}{2}\sum_{k\geq 0} s_k \frac{(-1)^{k+1}}{(k!)^2}\left|\frac{\Gamma(\lambda+k)}{\Gamma(\lambda)}\right|^2 \left(e^{-\alpha k(x+\bar{x}+t)} + e^{\alpha k(t-x-\bar{x})}\right) F\left(\lambda,\bar{\lambda},k+1,e^{-2\alpha x}\left(1+e^{-2\alpha x}\right)^{-1}\right) F\left(\lambda,\bar{\lambda},k+1,e^{-2\alpha\bar{x}}\left(1+e^{-2\alpha\bar{x}}\right)^{-1}\right)$$



Kuntz (2025)

SCHWARZSCHILD GF

It's more complicated but very similar in spirit

$$\psi_H^-(r,\omega) = r^3 e^{-i\omega(r+\log(r-1))}\mathrm{HeunC}(\ell(\ell+1)-6+6i\omega,6i\omega,1-2i\omega,5,2i\omega;1-r)$$

$$\psi_\infty^-(r,\omega) = r^3 e^{-i\omega(r+\log(r-1))}\mathrm{HeunC}_\infty(\ell(\ell+1)-6,-6i\omega,5,1-2i\omega,-2i\omega;1-r)$$

$$\psi_\infty^-(r,\omega) = r^3 e^{i\omega(r+\log(r-1))}\mathrm{HeunC}_\infty(\ell(\ell+1)-6,6i\omega,5,1+2i\omega,2i\omega;1-r)$$

Connection coefficients: $\psi_H^-(x,\omega) = A^-(\omega)\psi_\infty^-(x,\omega) + A^+(\omega)\psi_\infty^+(x,\omega)$

(Known only perturbatively in an "instanton expansion") 2201.04491

Schwarzschild GF

Analytic structure:

- ψ_H^- has simple poles at the Matsubaras $\omega = -i(k+1)/2 \quad (k \geq 0)$
- ψ_∞^+ has a pole at $\omega = 0$ and a branch cut along the negative imaginary axis
- $\psi_\infty^-(r,\omega) = \psi_\infty^+(r,-\omega)$ has a pole at $\omega = 0$ and a branch cut along the positive imaginary axis
- A^- has zeros at the QNMs ω_n, simple poles at the Matsubaras, a pole at $\omega = 0$ and a branch cut along the negative imaginary axis

So the full GF has:

$$\tilde{G}(x,\bar{x},\omega) = \frac{\psi_H^-(\bar{x},\omega)\psi_\infty^+(x,\omega)}{2i\omega A^-}$$

Poles at the QNM frequencies and branch cut along negative imaginary

Kuntz (2025)

Schwarzschild GF

If $x > x_{\text{bounce}}$ (to be computed later on)

$$\tilde{G}(x,\bar{x},\omega) = \frac{1}{2i\omega}\psi^-_\infty(\bar{x},\omega)\psi^+_\infty(x,\omega) + \frac{A^+}{2i\omega A^-}\psi^+_\infty(\bar{x},\omega)\psi^+_\infty(x,\omega)$$

pole at $\omega = 0$

branch cut along the positive imaginary axis

branch cut along the negative imaginary axis

Same

How to find x_{bounce}?

Convergence of $\tilde{G}e^{-i\omega t} \sim \dfrac{\psi^-_H(\bar{x},\omega)e^{-i\omega(t-x)}}{\omega A^-}$

~ 1

gr-qc/9607064

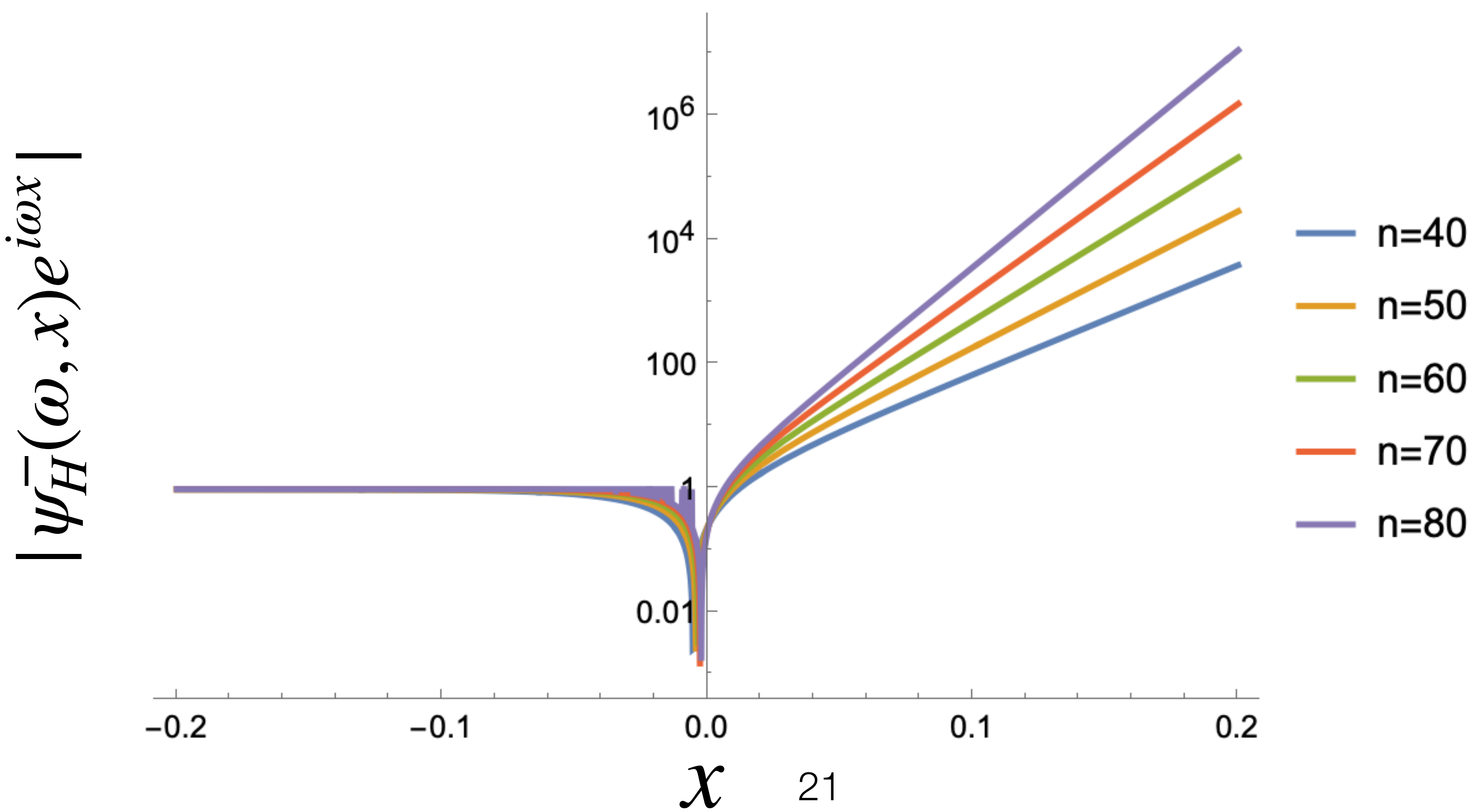
Bounce radius

Numerically:

$$\psi_H^- \underset{|\omega|\to\infty}{\sim} e^{-i\omega x} \text{ for } x < x_{\text{bounce}}$$

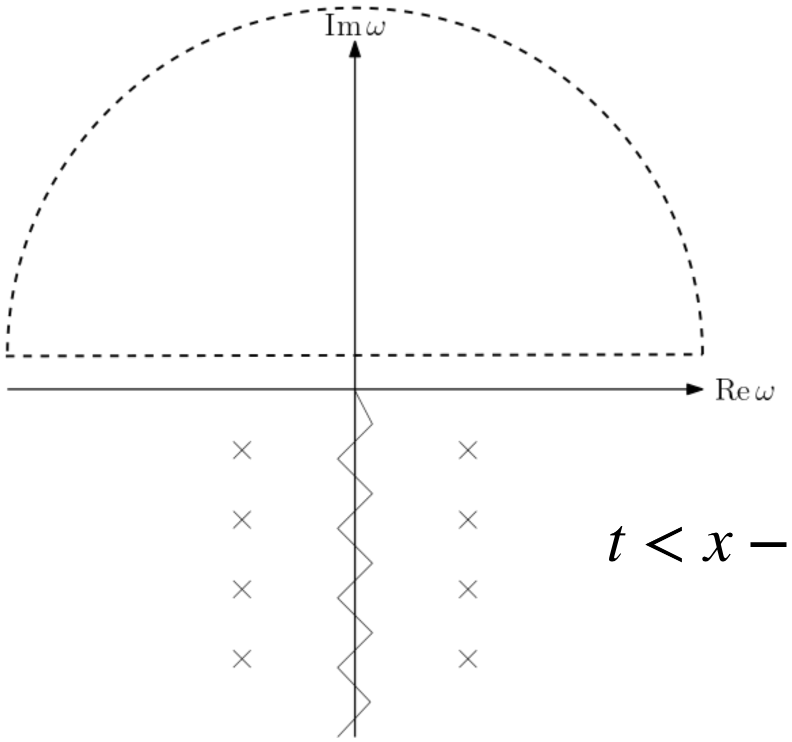
$$\psi_H^- \underset{|\omega|\to\infty}{\sim} (\ldots)e^{-i\omega x} + (\ldots)e^{i\omega x} \text{ for } x > x_{\text{bounce}}$$

$x_{\text{bounce}} = 0$!! ($r_{\text{bounce}} \simeq 2.55GM$ from $x = r + \log(r-1)$)

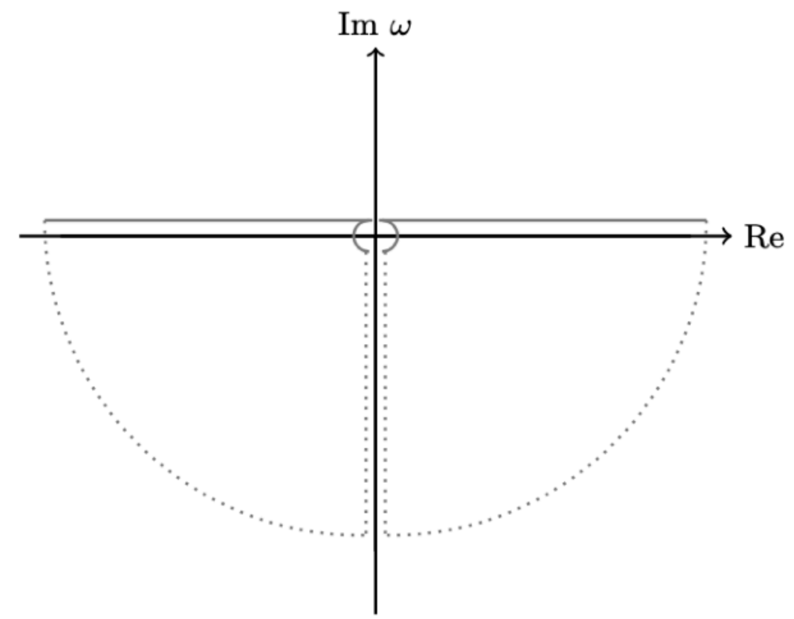


CONTOUR FOR SCHWARZSCHILD

$$\tilde{G}(x,\bar{x},\omega) = \frac{1}{2i\omega}\psi^-_\infty(\bar{x},\omega)\psi^+_\infty(x,\omega) + \frac{A^+}{2i\omega A^-}\psi^+_\infty(\bar{x},\omega)\psi^+_\infty(x,\omega)$$

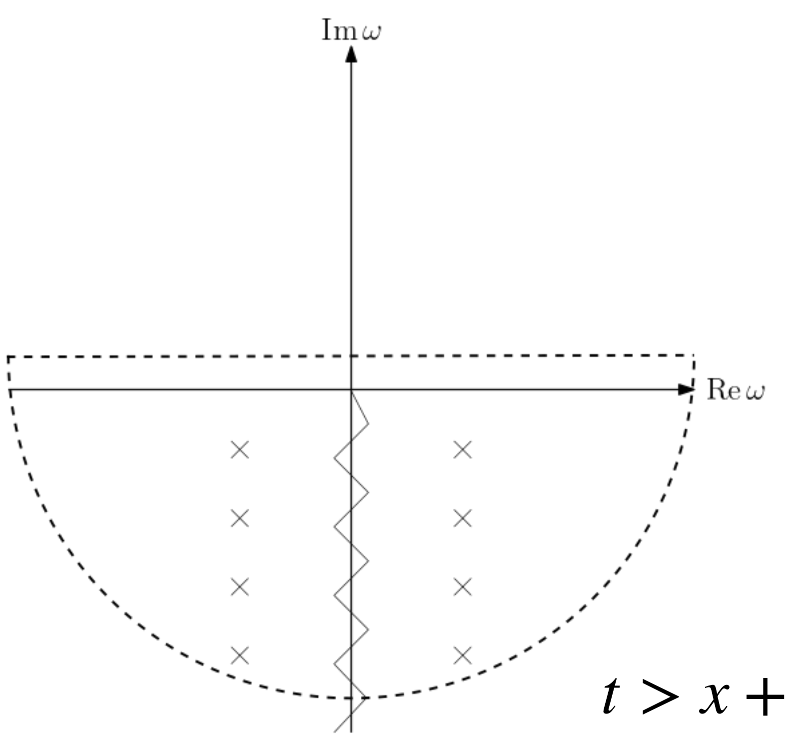


$t < x - \bar{x}$

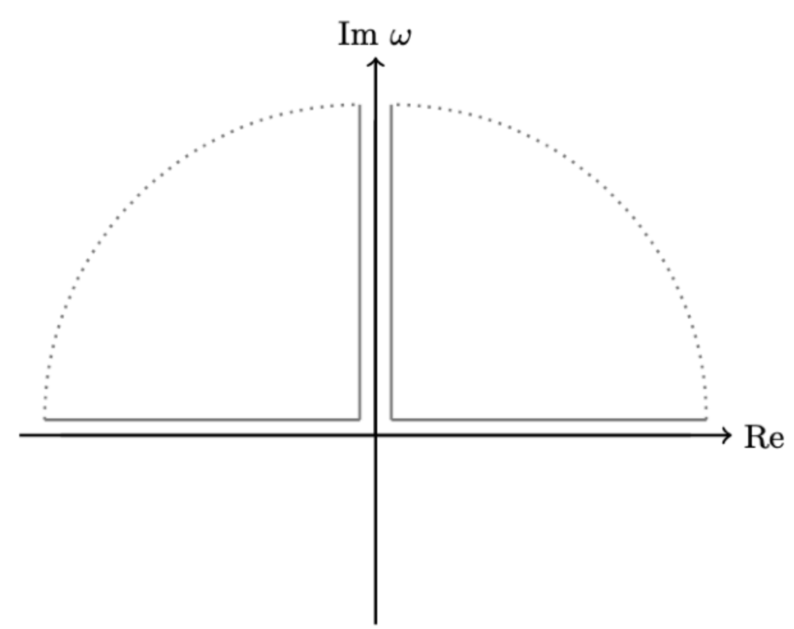


(a) G^-

$x - \bar{x} < t < x + |\bar{x}|$



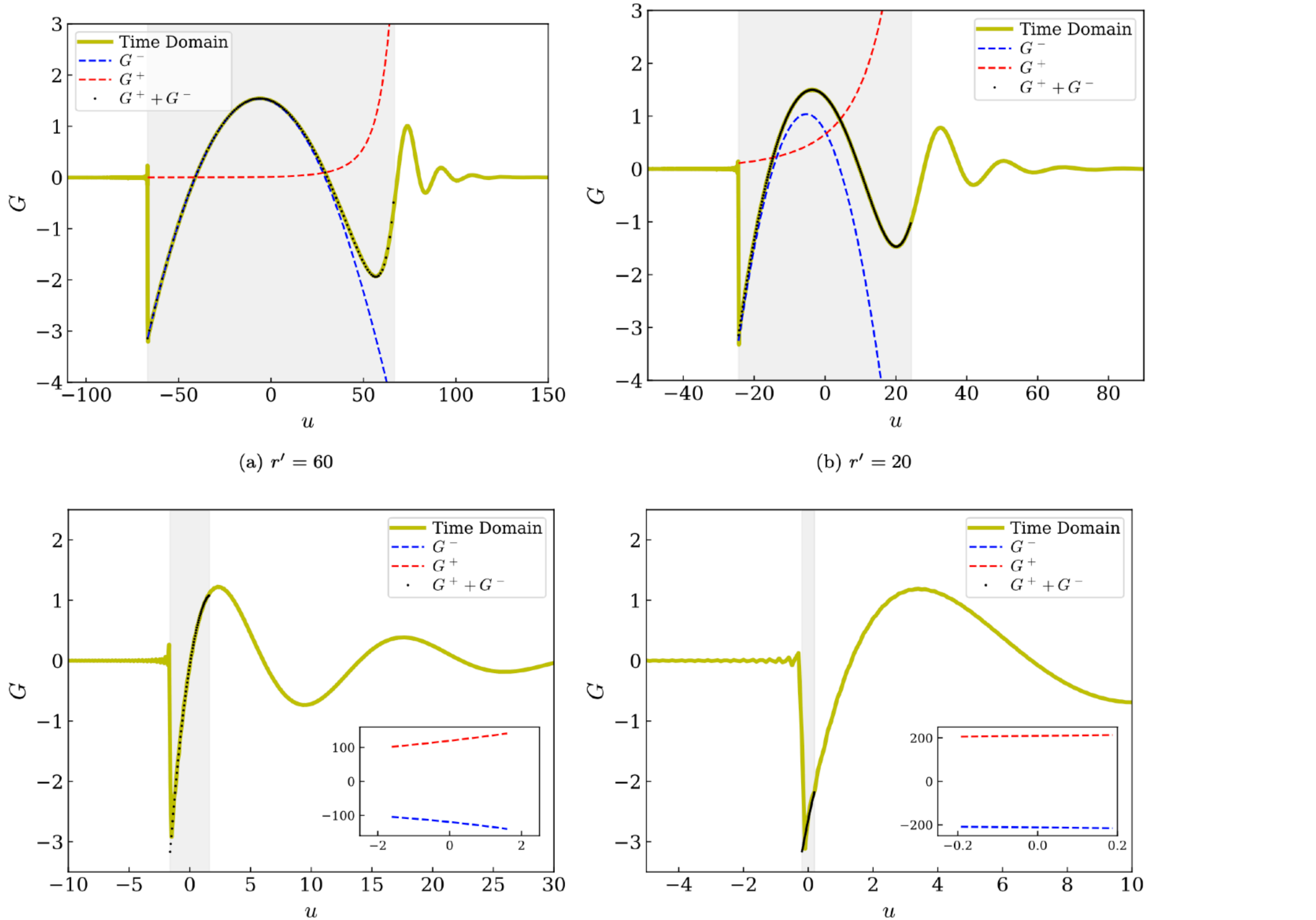
$t > x + |\bar{x}|$



(b) G^+

Ma (2026)

Contour for Schwarzschild



(a) $r' = 60$

(b) $r' = 20$

(c) $r' = 3$

(d) $r' = 2.6$

Ma (2026)

Contour for Schwarzschild

Conclusion:

$$G(x,\bar{x},t) = \Theta(t-x+\bar{x})\Theta(x+\bar{x}-t)G_{\rm P}(x,\bar{x},t) + \Theta(t-x-|\bar{x}|)G_{\rm QNM}(x,\bar{x},t)$$

Prompt, exist only for $\bar{x} > 0$

Usual QNM+Tail from Leaver

Contour for Schwarzschild

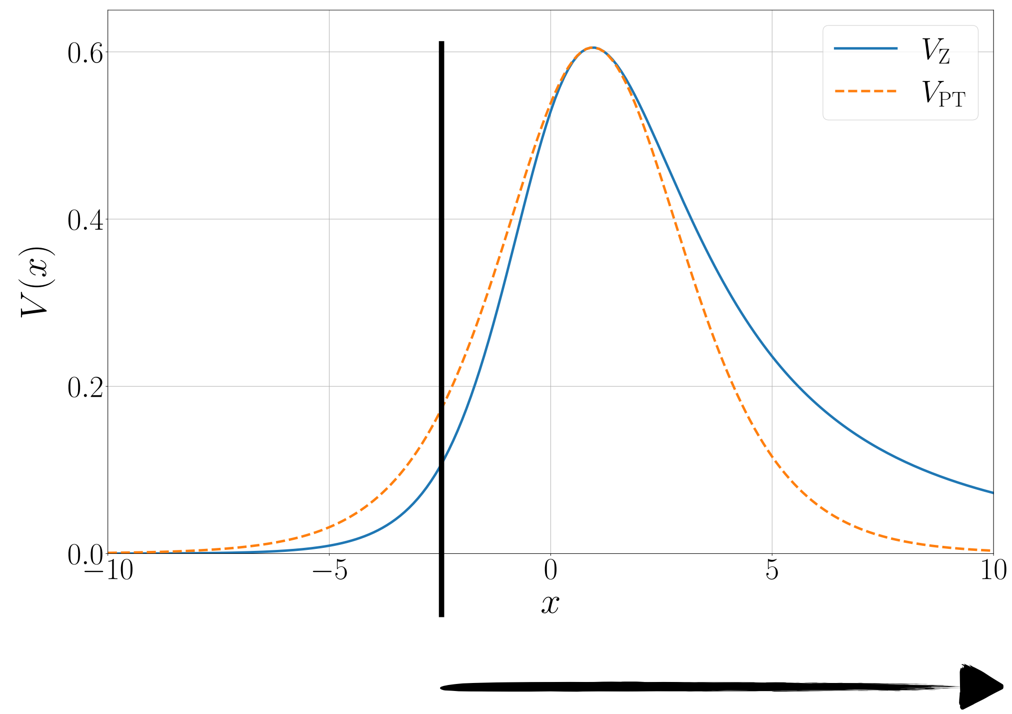
Conclusion:

$$G(x,\bar{x},t) = \Theta(t-x+\bar{x})\Theta(x+\bar{x}-t)G_{\mathrm{P}}(x,\bar{x},t) + \Theta(t-x-|\bar{x}|)G_{\mathrm{QNM}}(x,\bar{x},t)$$

Prompt, exist only for $\bar{x} > 0$

Usual QNM+Tail from Leaver

$\bar{x} < 0$



Bounce time $t = x + |\bar{x}|$

Contour for Schwarzschild

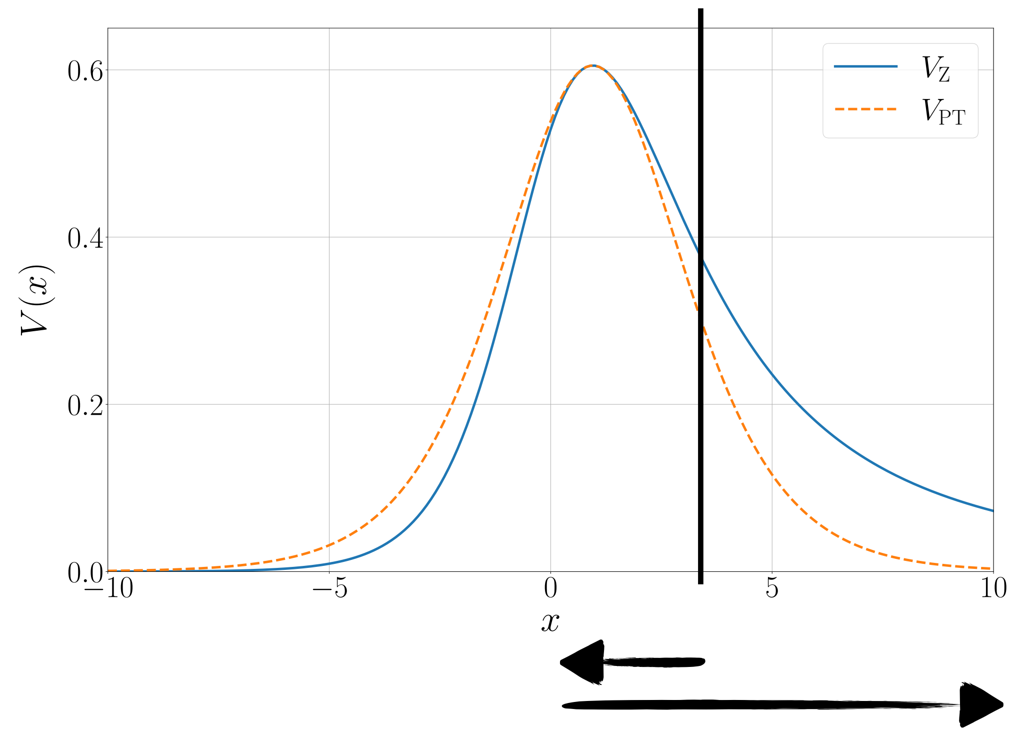
Conclusion:

$$G(x,\bar{x},t) = \Theta(t-x+\bar{x})\Theta(x+\bar{x}-t)G_{\mathrm{P}}(x,\bar{x},t) + \Theta(t-x-|\bar{x}|)G_{\mathrm{QNM}}(x,\bar{x},t)$$

Prompt, exist only for $\bar{x} > 0$

Usual QNM+Tail from Leaver

$\bar{x} > 0$



Bounce time $t = x + |\bar{x}|$

Plunging particle

If we neglect initial conditions:

$$\psi(x,t) = \int_{-\infty}^{\infty} \mathrm{d}\bar{x} \int_{t_0}^{\infty} \mathrm{d}\bar{t}\; G(x,\bar{x},t-\bar{t})S(\bar{x},\bar{t}) \qquad S(\bar{x},\bar{t}) = f(\bar{x}_{\mathrm{P}}(\bar{t}))\delta(\bar{x}-\bar{x}_{\mathrm{P}}(\bar{t})) + g(\bar{x}_{\mathrm{P}}(\bar{t}))\delta'(\bar{x}-\bar{x}_{\mathrm{P}}(\bar{t}))$$

$$\longrightarrow \qquad \psi = \psi_I + \psi_A$$

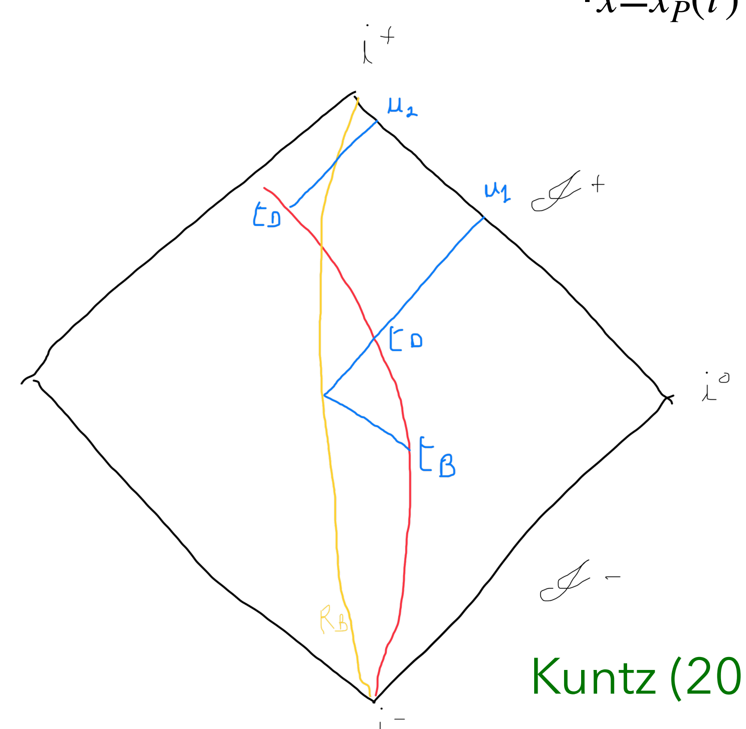
$$\psi_{\mathrm{I}}(t,x) = -\frac{g(\bar{x}_{\mathrm{P}}(\bar{t}))}{1-\bar{v}(\bar{t})}\Theta(\bar{x})G_P\bigg|_{\bar{t}=\bar{t}_D(u),\bar{x}=\bar{x}_D(u)} - \frac{g(\bar{x}_{\mathrm{P}}(\bar{t}))}{1-\bar{v}(\bar{t})}\Theta(-\bar{x})G_{\mathrm{QNM}}\bigg|_{\bar{t}=\bar{t}_{\mathrm{D}}(u),\bar{x}=\bar{x}_{\mathrm{D}}(u)}$$

$$\psi_{\mathrm{A}}(x,t) = \int_{t_0}^{t_{\mathrm{B}}(u)} \left[f(\bar{x}_{\mathrm{P}}(\bar{t}))G_{\mathrm{QNM}} - g(\bar{x}_{\mathrm{P}}(\bar{t}))\partial_{\bar{x}}G_{\mathrm{QNM}}\right]\Big|_{\bar{x}=\bar{x}_P(\bar{t})} \mathrm{d}\bar{t} + \int_{t_{\mathrm{B}}(u)}^{t_{\mathrm{D}}(u)} \left[f(\bar{x}_{\mathrm{P}}(\bar{t}))G_{\mathrm{P}} - g(\bar{x}_{\mathrm{P}}(\bar{t}))\partial_{\bar{x}}G_{\mathrm{P}}\right]\Big|_{\bar{x}=\bar{x}_P(\bar{t})} \mathrm{d}\bar{t}$$

"Bounce time" $\quad \bar{t}_{\mathrm{B}} + |\bar{x}_P(\bar{t}_{\mathrm{B}})| = t - x = u$

"Direct time" $\quad \bar{t}_{\mathrm{D}} - \bar{x}_P(\bar{t}_{\mathrm{D}}) = t - x = u$

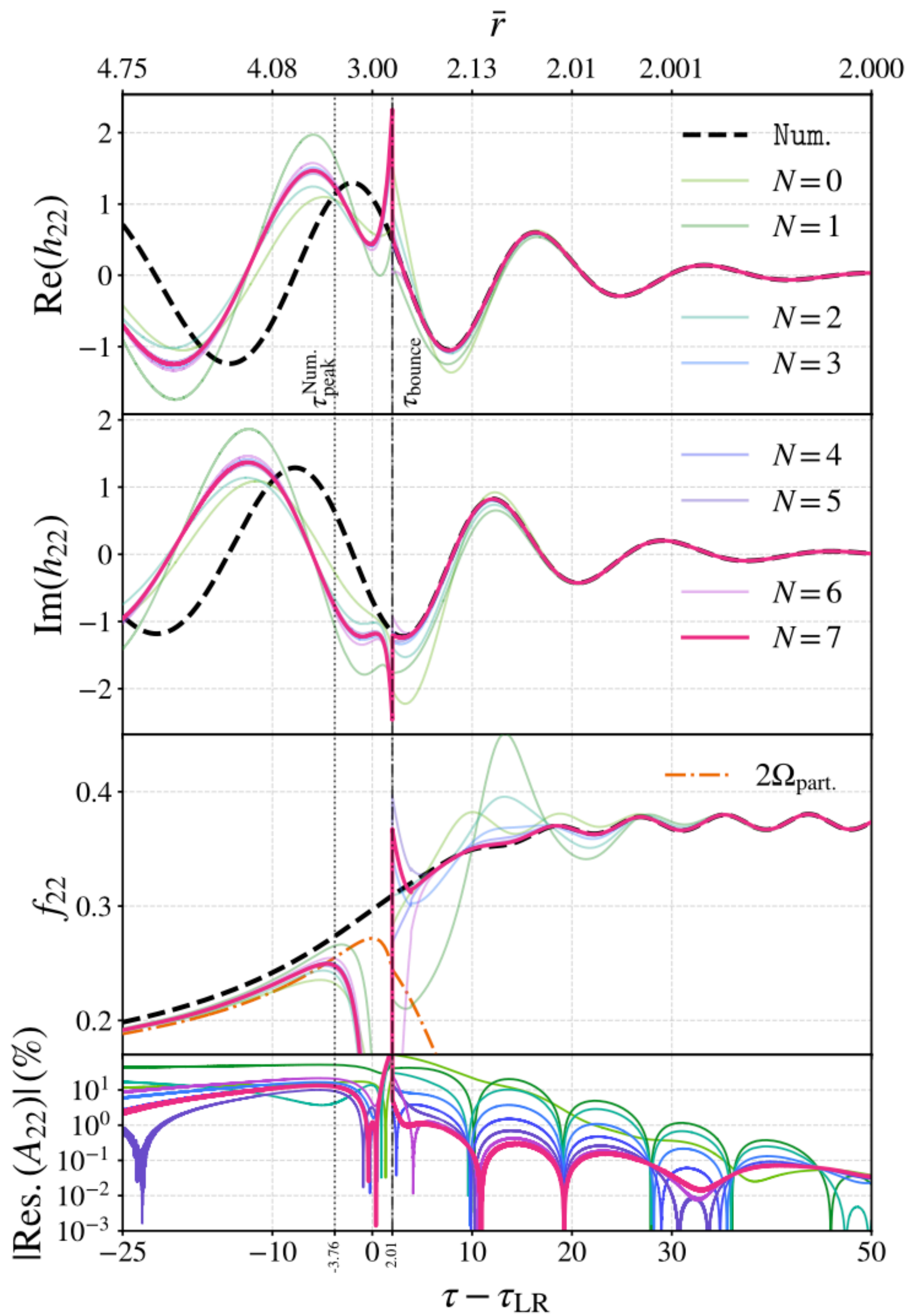
($t_{\mathrm{B}} = t_{\mathrm{D}}$ once the particle has crossed the bounce)



Kuntz (2025)

PLUNGING PARTICLE

Considering G_{QNM} only:



See Marina's talk!

Redshift modes

What happens when the particle falls onto the horizon?

$\bar{x}_\mathrm{P} \sim -\bar{t} \to -\infty$

$\bar{t}_\mathrm{D} - \bar{x}_P(\bar{t}_\mathrm{D}) = u$

$\longrightarrow$ $\bar{t}_D \simeq u/2$ "Direct wave"

Let's consider e.g. $$\psi_\mathrm{I}(t,x) = -\frac{g(\bar{x}_\mathrm{P}(\bar{t}))}{1-\bar{v}(\bar{t})} G_\mathrm{QNM}\Big|_{\bar{t}=\bar{t}_\mathrm{D}(u), \bar{x}=\bar{x}_\mathrm{D}(u)}$$

When $\bar{x}_P \to -\infty$: $$G_\mathrm{QNM}(x, \bar{x}_\mathrm{P}(\bar{t}_D), t-\bar{t}_D) = i\int_{i\gamma-\infty}^{i\gamma+\infty} \frac{\mathrm{d}\omega}{2\pi}\frac{1}{2\omega A_\mathrm{in}(\omega)} \simeq -0.55$$

$$\longrightarrow \quad \psi_\mathrm{I}(t,x) = \mathrm{Const} \times e^{x_P} + \mathrm{O}(e^{2x_P}) = \mathrm{Const} \times e^{-u/2} + \mathrm{O}(e^{-u})$$

Redshift/Horizon/Matsubara modes?

Kuntz & Della Rocca (2026)

REDSHIFT MODES

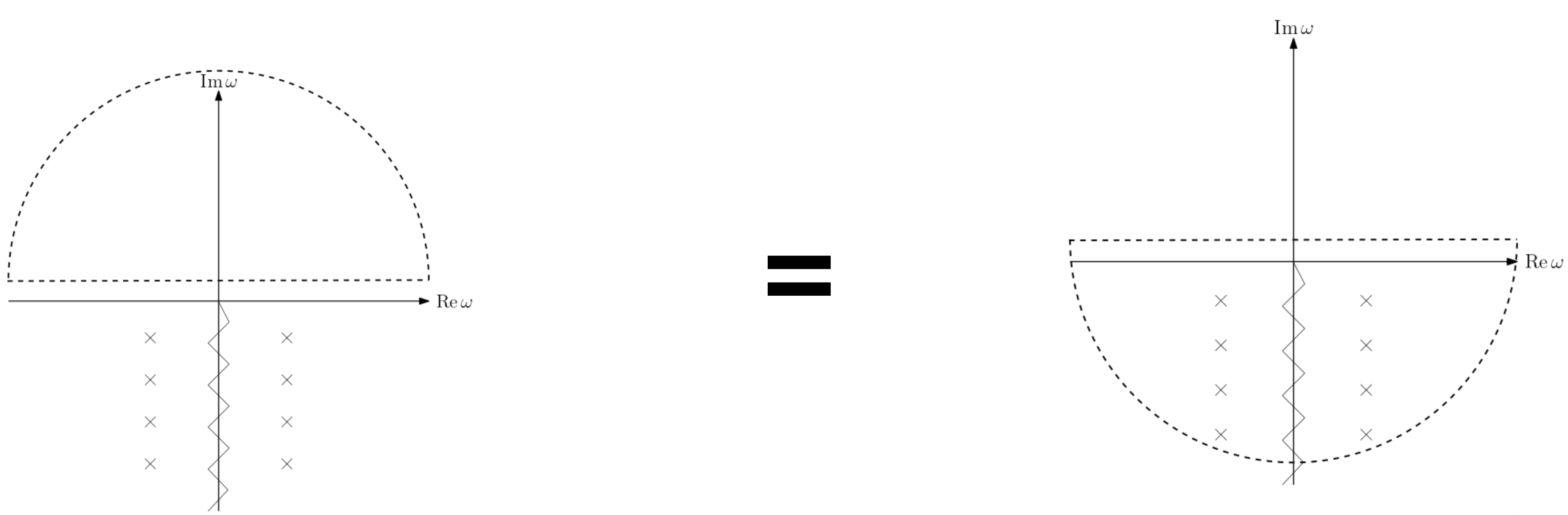
One should also take into account the activation piece

$$\psi_{\mathrm{A}}(t,x) \simeq \mathrm{Const} \times e^{-u/2} + \mathrm{QNM\ piece}$$

$$\psi(x,t) \simeq i e^{-u/2}\left(f_0 + \frac{g_0}{2}\right)\int \frac{\mathrm{d}\omega}{2\pi}\frac{1}{2\omega A_{\mathrm{in}}(2i\omega - 1)}$$

This vanishes exactly!!

Reason: can close either in lower or upper half plane



Kuntz & Della Rocca (2026)

Redshift modes

The proof can be extended to any order $e^{-u}, e^{-3u/2}, \ldots$

Ingredients:

- Redshift modes involve the Green function computed on the light-cone $\bar{t} - \bar{x} = t - x$ for $t - x \to \infty$
- They also involve an integral of the source in time, which brings a factor ω^{-1}
- Although the Green function itself is discontinuous on the light-cone, the discontinuity being given by the integral $\int d\omega (\omega A_{\text{in}})^{-1}$ (see Eq. (9)), the redshift mode amplitude is continuous on the light-cone because of the supplementary ω^{-1} factor which allows to close the contour either on the upper or on the lower half-plane
- By causality, this amplitude vanish

$$\psi(x,t) = \int_{-\infty}^{\infty} d\bar{x} \int_{t_0}^{\infty} d\bar{t}\; G(x, \bar{x}, t - \bar{t}) S(\bar{x}, \bar{t})$$

Kuntz & Della Rocca (2026)

Conclusions

- We now have the full Schwarzschild Green function
- A new "bounce radius" $r_* = 0$ appears to be a fundamental quantity for QNM
- The prompt is a pole in 0 and a branch cut, appearing before bounce crossing
- Direct waves/redshift modes/horizon modes/Matsubara modes are vanishing