Static Quadratic Love Numbers

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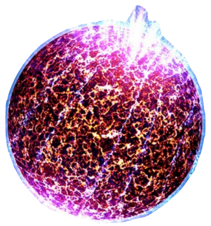
9 June 2026

IFPU, Trieste - Perspective on Gravity: From Theory to Observations

Static Quadratic Love Numbers of neutron stars



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With Paolo Pani, Massimiliano Maria Riva, Luca Santoni and Nikola Savic 2512.14663

Previous work: 2312.05065, 2410.03542 on Schwarzschild Black Holes

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Newtonian case

- Poisson equation: $\nabla^2\Phi = 4\pi G\rho$

tidal field

$$\Phi = -G\int d^3x' \frac{\rho(\vec{x})}{|\vec{x}-\vec{x}'|} + \sum_{\ell\geq 2, m} \mathcal{E}_{\ell m} r^{\ell} Y_{\ell m}$$

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- Poisson equation: $\nabla^2 \Phi = 4\pi G \rho$

tidal field

$$\Phi = -G \underbrace{\int d^3x' \, \frac{\rho(\vec{x})}{|\vec{x} - \vec{x}'|}}_{\displaystyle \frac{M}{r} + \sum_{\ell \geq 2, m} \frac{Q_{\ell m}}{r^{\ell+1}} Y_{\ell m}} + \sum_{\ell \geq 2, m} \mathcal{E}_{\ell m} r^{\ell} Y_{\ell m}$$

induced mass multipoles $\qquad Q_\ell \sim M R^\ell$

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induced mass multipoles $\qquad Q_\ell \sim MR^\ell$

- Linear response $\qquad Q_{\ell m} = \lambda_\ell \mathcal{E}_{\ell m} = k_\ell \frac{R^{2\ell+1}}{G}\mathcal{E}_{\ell m}$ $\qquad$ linear tidal Love number

$$\Phi = -\frac{GM}{r} - \sum_{\ell\geq 2,m} r^\ell \mathcal{E}_{\ell m} Y_{\ell m}\left[1 + k_\ell \left(\frac{R}{r}\right)^{2\ell+1}\right]$$

Newtonian case

M

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induced mass quadrupole $\quad Q_2 \sim M R^2$

- Linear response $\quad Q_2 = \lambda_2 \mathcal{E}_2 = k_2 \frac{R^5}{G} \mathcal{E}_2 \quad$ linear tidal Love number

$$\Phi = -\frac{GM}{r} - r^2 \mathcal{E}_2 Y_{20} \left[1 + k_2 \left(\frac{R}{r} \right)^5 \right]$$

Newtonian case

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tidal field

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induced mass quadrupole $\quad Q_2 \sim MR^2$

- Quadratic response $\quad Q_2 = \lambda_2\mathcal{E}_2 + \lambda_{222}\mathcal{E}_2\mathcal{E}_2 = k_2\frac{R^5}{G}\mathcal{E}_2 + p_2\frac{R^8}{G^2M}\mathcal{E}_2\mathcal{E}_2$

$$\Phi = -\frac{GM}{r} - r^2\mathcal{E}_2Y_{20}\left[1 + k_2\left(\frac{R}{r}\right)^5 + \mathcal{E}_2\frac{p_2}{GM}\frac{R^8}{r^5}\right]$$

$$\lambda_{222} = p_2\frac{R^8}{G^2M}$$

quadratic tidal Love number

Relativistic case

- Metric and general relativity:

$$\Phi \quad \Rightarrow \quad g_{\mu\nu} : \qquad \delta g_{tt} \ \text{polar \&} \ \ \delta g_{t\phi} \ \text{axial perts}$$

$$\nabla^2 \Phi = 4\pi G \rho \quad \Rightarrow \quad G_{\mu\nu}[g_{\mu\nu}] = R_{\mu\nu}[g_{\mu\nu}] - \frac{1}{2} g_{\mu\nu} R[g_{\mu\nu}] = 8\pi G T_{\mu\nu}$$

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- Nonlinearities:

r_s/r non-linearities

$$\delta g_{tt} = \frac{r_s}{r} + \underbrace{r^2\mathcal{E}_2 Y_{20}\Big[\Big(1 + a_1\frac{r_s}{r} + \ldots\Big)}_{\delta\bar{g}_{tt}} + k_2\frac{R^5}{r^5}\Big(1 + b_1\frac{r_s}{r} + \ldots\Big)\Big]$$

tidal field $\delta\bar{g}_{tt}$ solution of $\bar{G}_{\mu\nu}[\bar{g}_{\mu\nu}] = 0$ regular at the origin

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- Nonlinearities:

[Poisson '20 & Poisson & Pitre '25]

r_s/r non-linearities

$$\delta g_{tt} = \frac{r_s}{r} + \underbrace{r^2\mathcal{E}_2 Y_{20}\Bigg[\left(1 + a_1\frac{r_s}{r} + \dots\right)}_{} + k_2\frac{R^5}{r^5}\left(1 + b_1\frac{r_s}{r} + \dots\right)\Bigg]$$

tidal field $\quad \delta\bar{g}_{tt}$ solution of $\bar{G}_{\mu\nu}[\bar{g}_{\mu\nu}] = 0$ regular at the origin

$$-\underbrace{r^4\mathcal{E}_2\mathcal{E}_2 Y_{20}\Big[(1+\cdots)}_{\text{quadratic tidal field}} + k_2\frac{R^5}{r^5}(1+\cdots) + k_2^2\frac{R^{10}}{r^{10}}(1+\cdots) + \underbrace{\frac{p_2}{GM}\frac{R^8}{r^7}(1+\cdots)}_{\text{quadratic Love number}}\Big]$$

- Controlled expansion parameters: $\quad r_s\mathcal{E}^{1/\ell} \ll \frac{r_s}{r} \ll \frac{r_s}{R} \leq 1$

- Expression not-gauge invariant. Ambiguity in definition of Love numbers.

[e.g. Gralla '17]

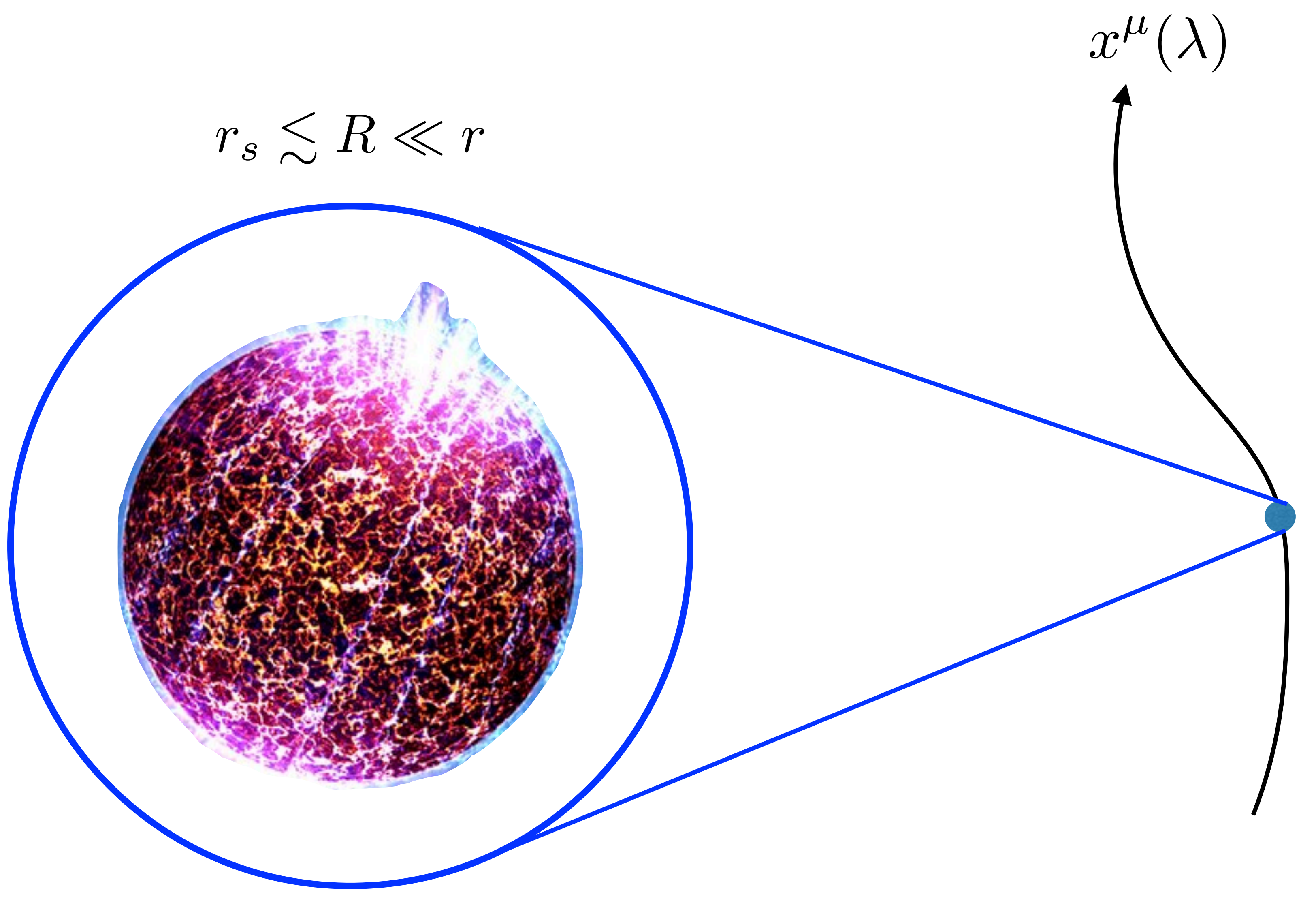
Worldline EFT

[Goldberger & Rothstein '04 + many others!]

- EFT of self-interacting gravitons coupled to classical worldline sources

$$S = S_{\rm EH}[g_{\mu\nu}] + S_{\rm WL}[g_{\mu\nu}, x^\mu(\lambda)]$$

- From afar, everything looks point-like and localized on a center-of-mass worldline $x^\mu(\lambda)$



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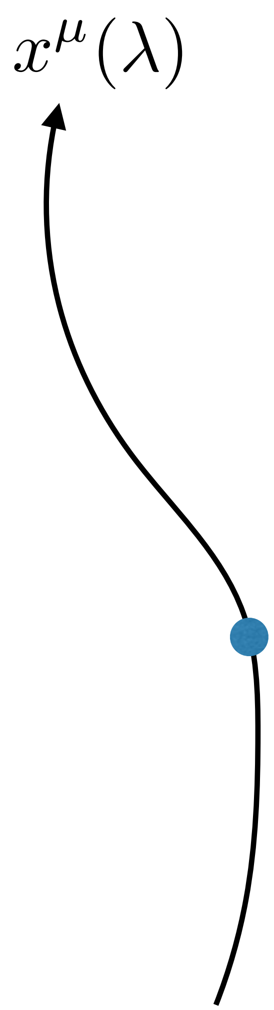

- Micro-structure modelled as series of multipole moments

quadrupole

$$S_{\mathrm{WL}} = \int d\tau \left[M + Q_{ij} E^{ij} + Q_{ijk} \nabla^{\langle i} E^{jk\rangle} + \mathrm{magnetic} + \cdots \right]$$

mass

octupole

$$E_{ij} = C_{i0j0} \ , \qquad B_{ij} = \frac{1}{2} \epsilon_{i0kl} C^{kl}{}_{j0} \qquad C_{\mu\nu\rho\sigma}$$ Weyl tensor

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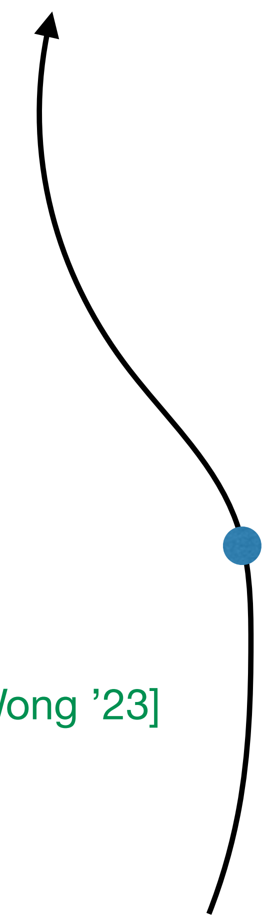

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$$S_{\mathrm{WL}} = \int d\tau \left[\underbrace{M}_{\text{mass}} + \overbrace{Q_{ij}E^{ij}}^{\text{quadrupole}} + \underbrace{Q_{ijk}\nabla^{\langle i}E^{jk\rangle}}_{\text{octupole}} + \text{magnetic} + \cdots \right]$$

- Integrating out multipoles $\quad Q_{ij} = \lambda_2 E_{ij} + \lambda_{222} E_{i\langle j}E^{i}{}_{k\rangle} + \ldots$

linear response + quadratic + … [De Luca, Khoury, Wong '23]

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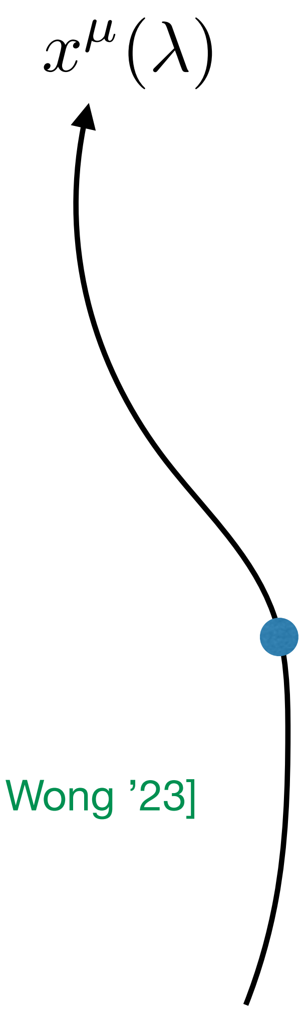

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linear LN　　　　quadratic LN

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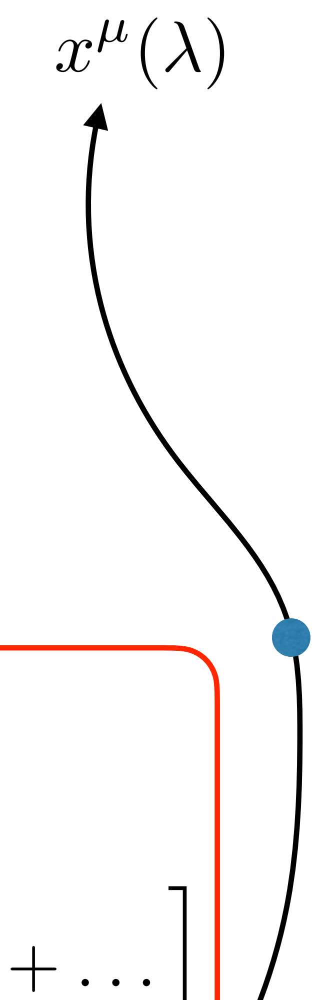

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mass octupole

$$S_{\rm WL} = \int d\tau \Big[M + \sum_\ell \left(\lambda_\ell^{(E)} (\nabla E)_\ell^2 + \lambda_\ell^{(B)} (\nabla B)_\ell^2 \right)$$

(partity invariant operators)

$$+ \sum_{\vec\ell} \left(\lambda_{\vec\ell}^{(E^3)} (\nabla E)_{\ell_1} (\nabla E)_{\ell_2} (\nabla E)_{\ell_3} + \lambda_{\vec\ell}^{(EB^2)} (\nabla E)_{\ell_1} (\nabla B)_{\ell_2} (\nabla B)_{\ell_3} \right) + \ldots \Big]$$

Coordinate-independent definition of Love numbers

Effect on the waveform

$$r_s \equiv 2GM$$

$$C \equiv \frac{GM}{R} \quad \Rightarrow$$

compactness

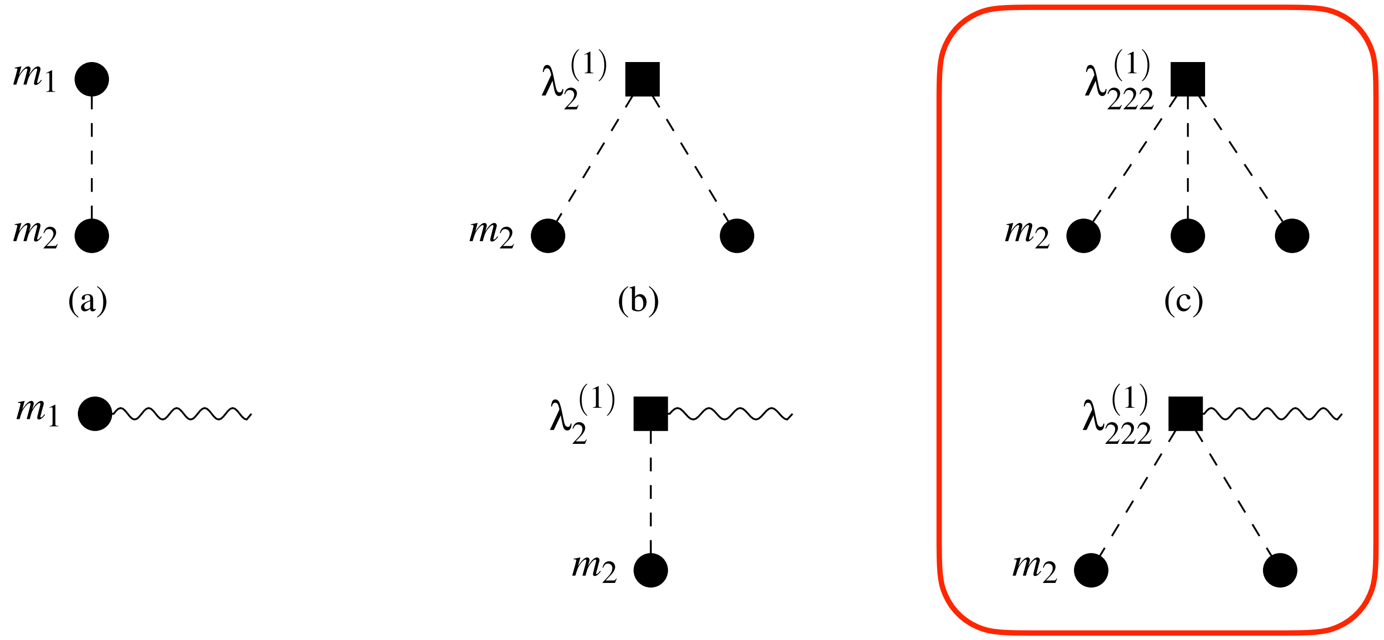
$$\int d\tau \left[M + \underbrace{k_2 \frac{R^5}{G} E_{ij} E^{ij}}_{M \frac{k_2}{C^5} \cdot \left(\frac{r_s}{r}\right)^6} + \underbrace{p_2 \frac{R^8}{G^2 M} E_{ij} E^j{}_k E^{ki}}_{M \frac{p_2}{C^8} \cdot \left(\frac{r_s}{r}\right)^9} + \cdots \right]$$

5PN effect

8PN effect

Effect on the waveform

- Balance law for circular orbits: $\dot{E} = -\mathcal{F}$



$\Rightarrow$ binding energy E

$\Rightarrow$ pseudo stress-energy tensor

$\Rightarrow$ binary quadrupole moment

$\Rightarrow$ energy flux $\mathscr{F}$

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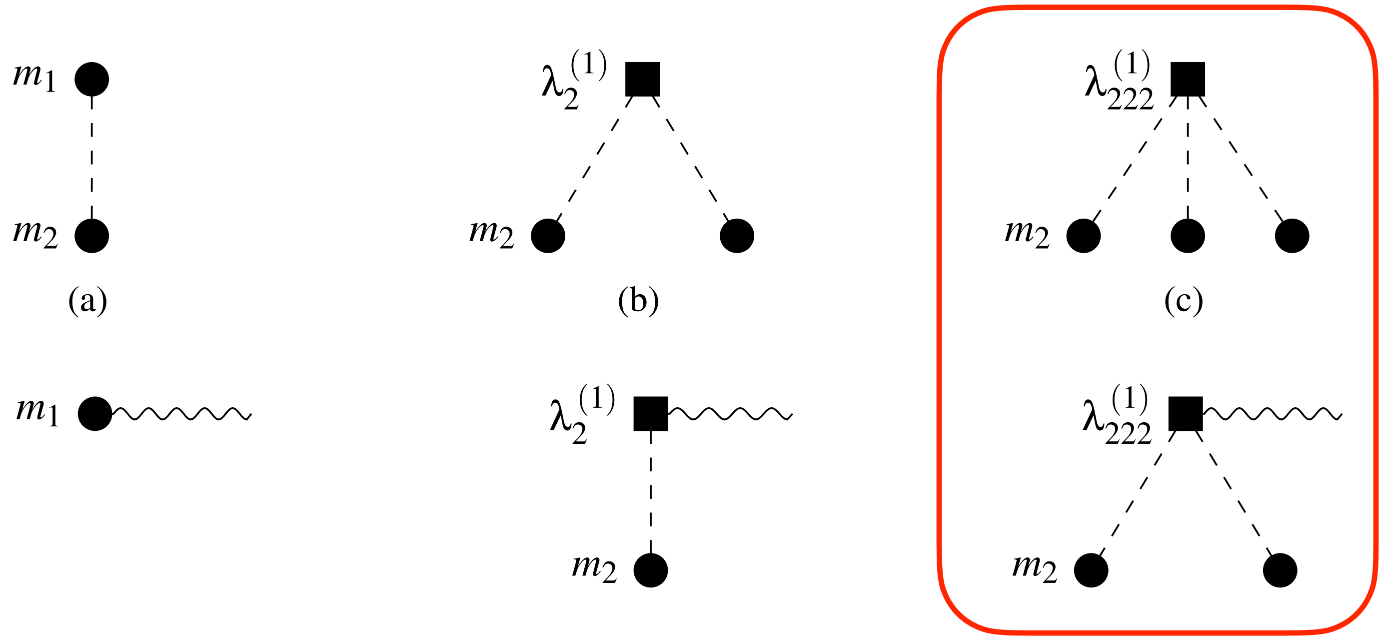

⇒ binding energy E

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- Waveform phase: $v(f) = (\pi G M f)^{1/3}$

[Henry, Faye, Blanchet ’20;
Mandal, Mastrolia, et al. ’24]

$$\psi(f) = 2\pi f t_c + \frac{3}{128\eta v^5}\Big[\underbrace{1 + \#v^2 + \ldots \#v^8}_{\text{point particle}} + \underbrace{\#\Lambda_L v^{10}(1 + \#v^2 + \#v^4)}_{\text{linear Love}} - \underbrace{\#\Lambda_Q v^{16}}_{\text{quadratic Love}}\Big]$$

$$\Lambda_L = \frac{(M_1 + 12M_2)M_1^4[k_2/C^5]_1 + (1 \leftrightarrow 2)}{(M_1 + M_2)^5}$$

Flanagan & Hinderer ’07

$$\Lambda_Q = \frac{(M_1 + 35M_2)M_1^5[p_2/C^8]_1 + (1 \leftrightarrow 2)}{(M_1 + M_2)^6}$$

Pani et al. ‘25

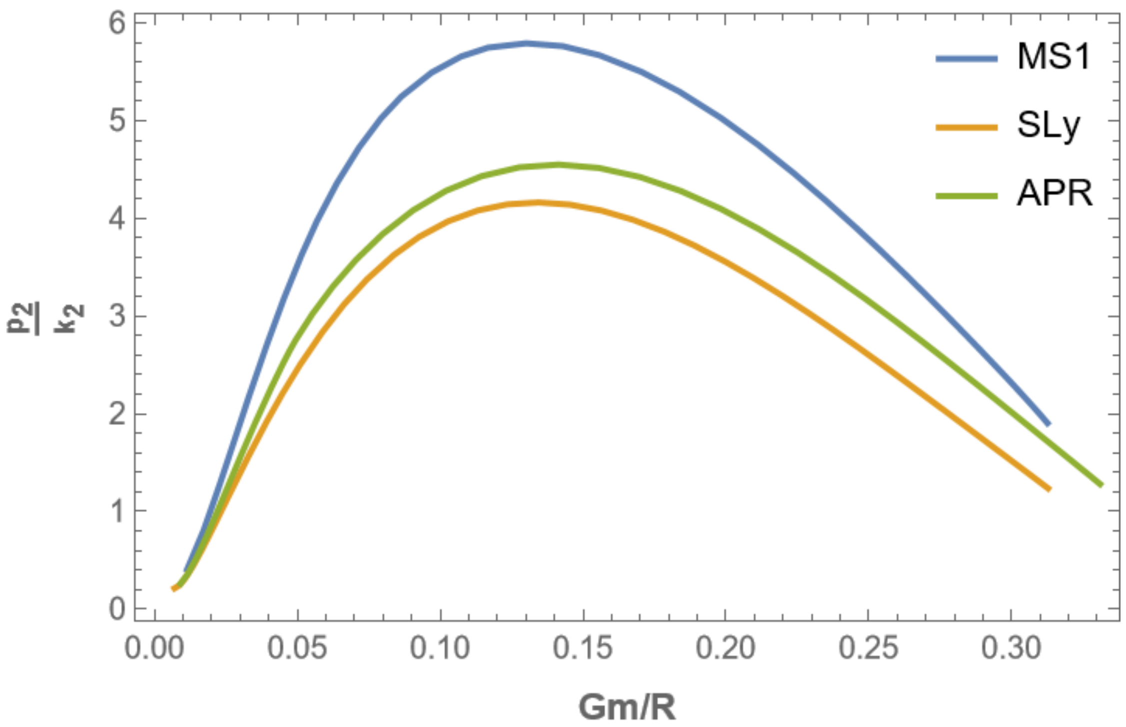
Effect on the waveform

- Equal stars (for simplicity): $M_1 = M_2$

$$\psi_{\rm QL} \sim \frac{p_2}{C^8} v^{16}$$

$$\psi_{\rm LL} \sim \frac{k_2}{C^5} v^{10}$$

$$\frac{\psi_{\rm QL}}{\psi_{\rm LL}} \approx 0.1 \cdot \frac{p_2/k_2}{5} \left(\frac{f}{f_{\rm merger}} \right)^2$$



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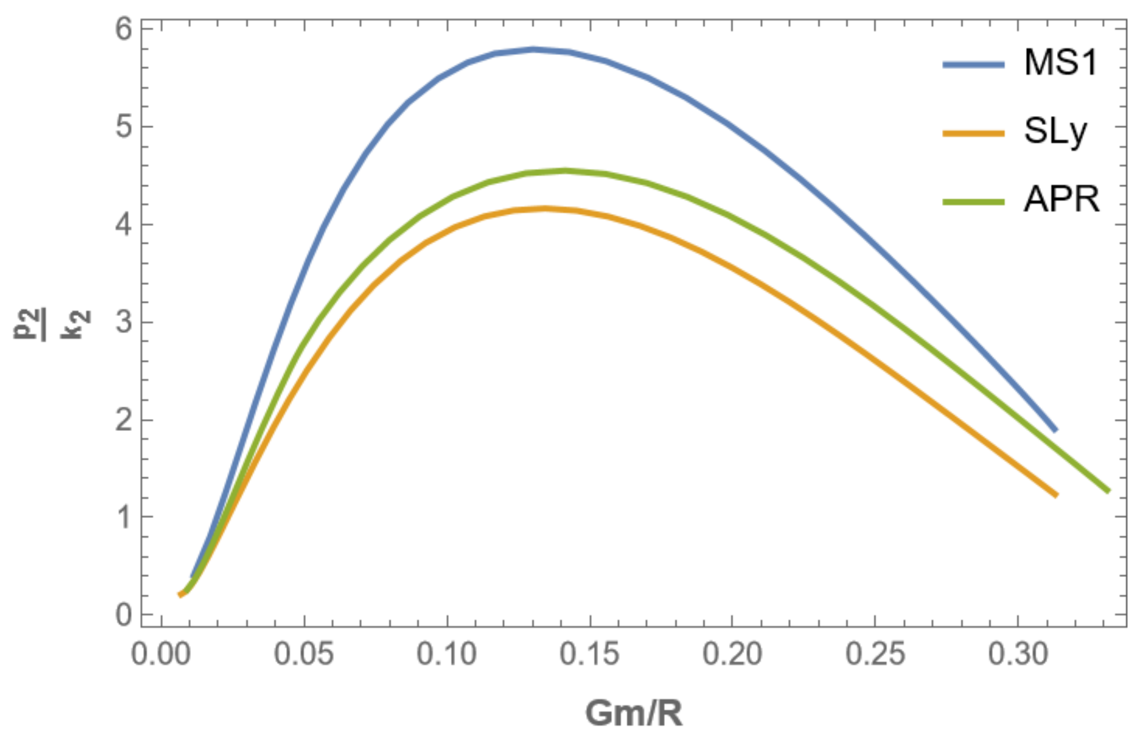
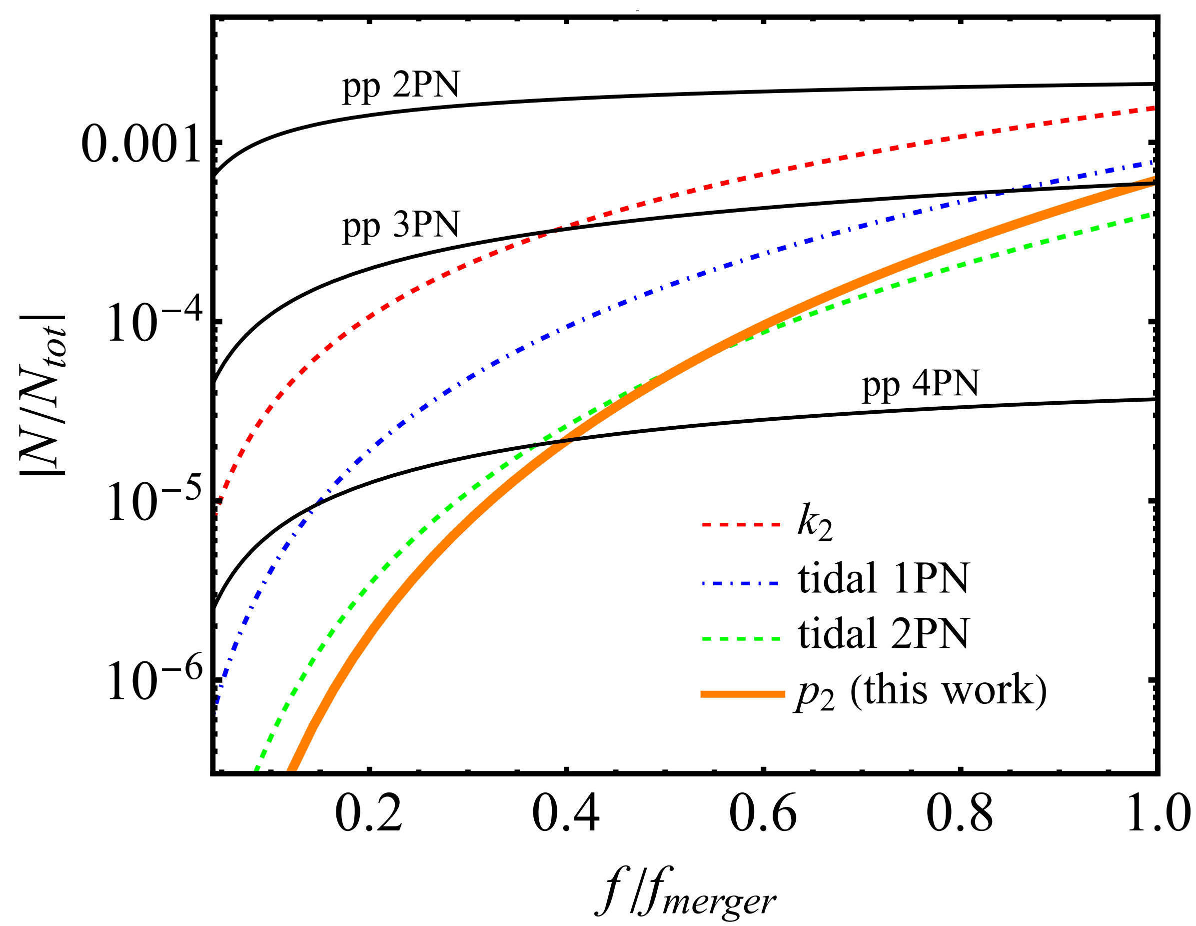
$$M_1 = M_2 = 1.4 M_\odot \,, \qquad C = 0.1 \,, \qquad p_2/k_2 = 5$$





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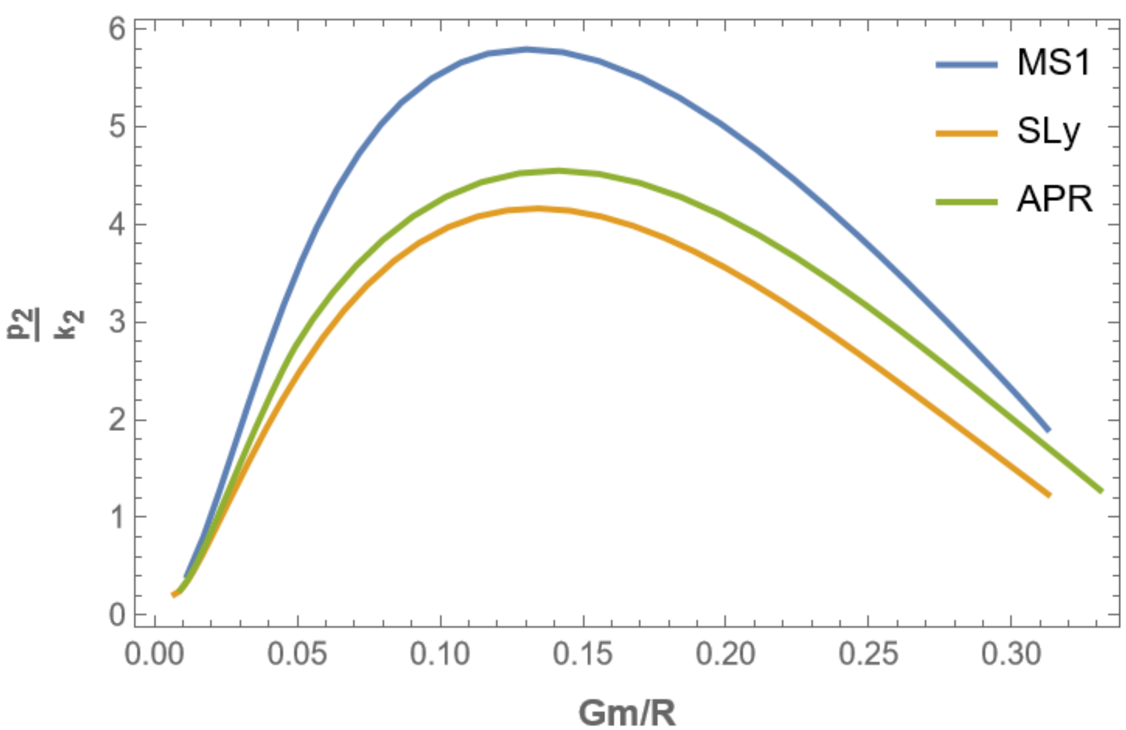
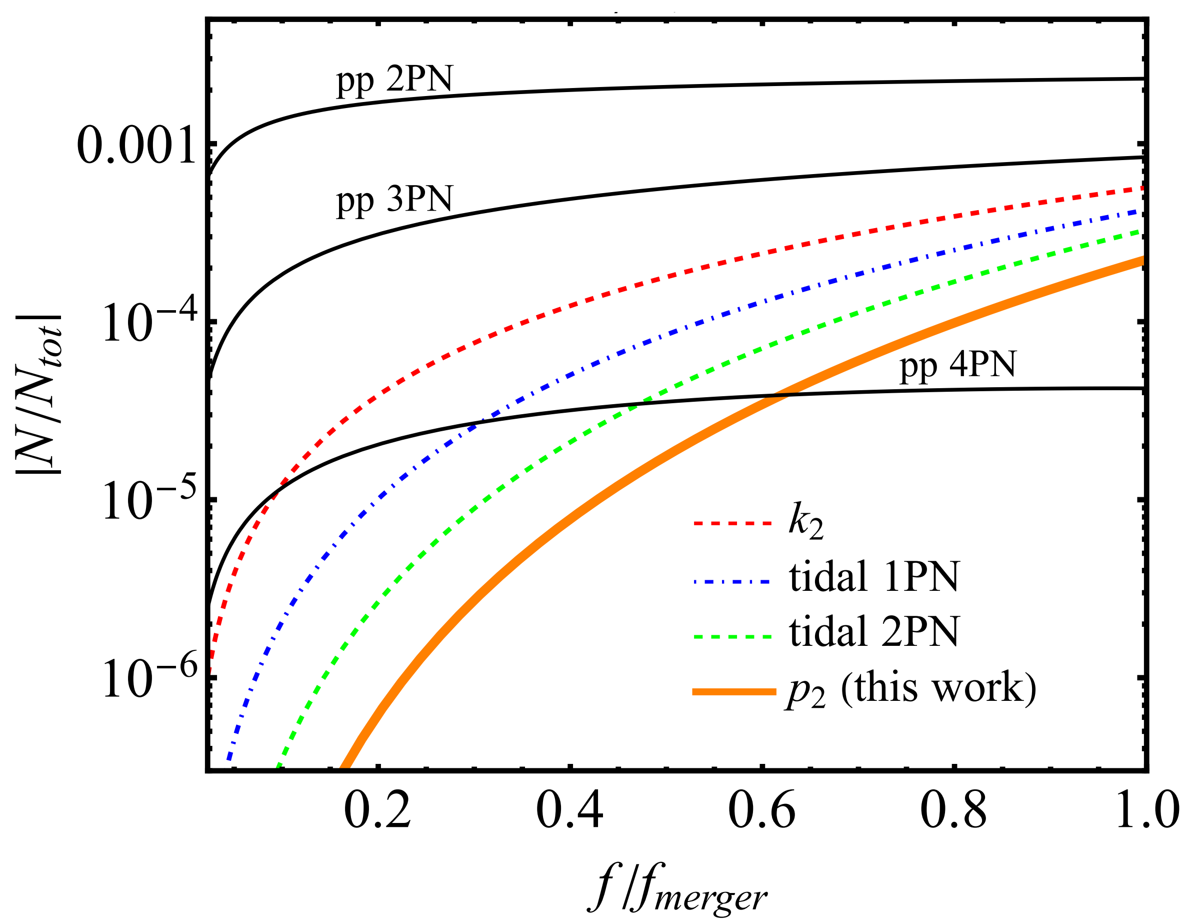




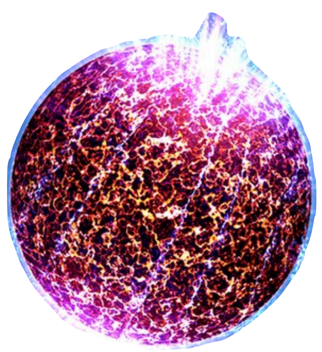
- Dedicated analysis (including other sub-leading effects, such as dynamical tide) needed to asses observability of effect

Computing relativistic Love Numbers

- Compute $\lambda_i' s$ by matching stellar perturbation theory in GR to the EFT

UV: Full General Relativity (TOV eqs.)

$$G^{(0)}_{\mu\nu} + \delta G^{(1)}_{\mu\nu} + \delta G^{(2)}_{\mu\nu} = 8\pi G \left(T^{(0)}_{\mu\nu} + \delta T^{(1)}_{\mu\nu} + \delta T^{(2)}_{\mu\nu} \right)$$



IR: Worldline Effective Field Theory

$$\int d^4x \frac{R}{16\pi G} + \int d\tau \left[M + \lambda_2 E_{ij} E^{ij} + \lambda_{222} E_{ij} E^{jk} E_k{}^i \right]$$

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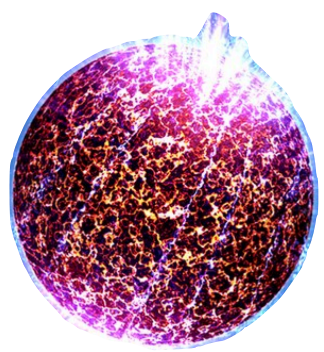

$$\Downarrow$$

$$\delta g^{(1)}_{\mu\nu}$$

$$\delta g^{(2)}_{\mu\nu}$$

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$$\Downarrow$$

$$\langle \delta g^{(1)}_{\mu\nu}(\lambda_2) \rangle_{\text{EFT}}$$

$$\langle \delta g^{(2)}_{\mu\nu}(\lambda_2, \lambda_{222}) \rangle_{\text{EFT}}$$

$$r \gg R$$

$$=$$

$$\Downarrow$$

$$\lambda_i$$

λ is gauge independent, but $\delta g_{\mu\nu}$ must be matched choosing a gauge

See Ivanov, Parra-Martinez, Caron-Huot and many others for amplitude-based approaches

EFT calculation

- Action $$S = S_{\mathrm{EH}}[g_{\mu\nu}] + S_{\mathrm{WL}}[g_{\mu\nu}, x^{\mu}(\lambda)]$$

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- Background-field method

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu} + \kappa h_{\mu\nu}$$

Background tidal field
(satisfies vacuum Einstein eqs.)

$$\bar{G}_{\mu\nu}[\bar{g}_{\mu\nu}] = 0$$

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- Gauge-fixing term (background de Donder)

$$S_{\mathrm{GF}} = -\int d^4x \sqrt{-\bar{g}}\, \bar{g}^{\mu\nu} \bar{\Gamma}_{\mu} \bar{\Gamma}_{\nu} \,, \qquad \bar{\Gamma}_{\nu} = \bar{g}^{\rho\sigma} \left(\bar{\nabla}_{\rho} h_{\sigma\nu} - \frac{1}{2} \bar{\nabla}_{\nu} h_{\rho\sigma} \right)$$

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- One-point expectation value $$\langle h_{\mu\nu}(x)\rangle = \int \mathcal{D}[h]\, h_{\mu\nu}\, e^{iS[H_{\alpha\beta}, h_{\alpha\beta}, x^\rho]}$$

= all Feynman diagrams with one external leg of $h_{\mu\nu}$

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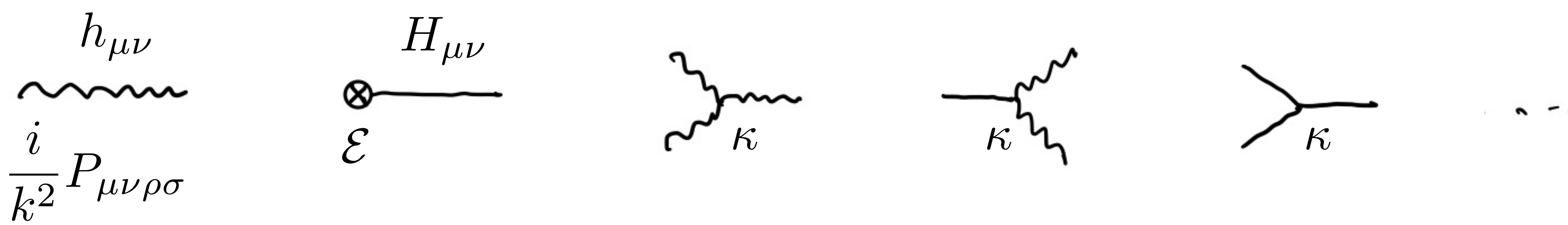

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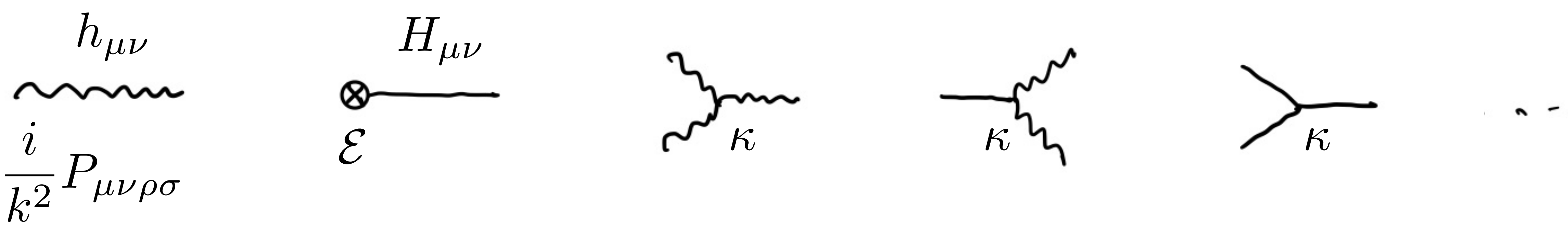

$$\delta g_{tt}^{(\ell=2)} = r^2 \mathcal{E}_2 Y_{20}\left(1 + \quad \ldots \quad\right) - r^4 \mathcal{E}_2 \mathcal{E}_2 Y_{20}\left(1 + \quad \ldots \quad\right)$$

EFT calculation

- Action $$S = S_{\rm EH}[H_{\mu\nu}, h_{\mu\nu}] + S_{\rm GF}[H_{\mu\nu}, h_{\mu\nu}] + \underbrace{S_{\rm WL}[H_{\mu\nu}, h_{\mu\nu}, x^\mu]}_{\text{worldline couplings}}$$

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M $\qquad$ λ_ℓ $\qquad$ λ_ℓ , $\lambda_{\ell\ell_1\ell_2}$

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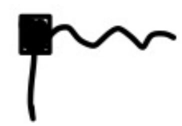
M

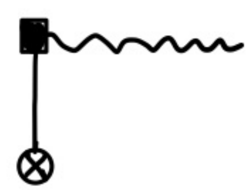
λ_ℓ

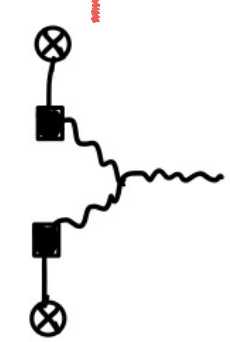


λ_ℓ , $\lambda_{\ell\ell_1\ell_2}$

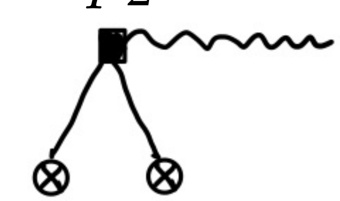
$\cdots$

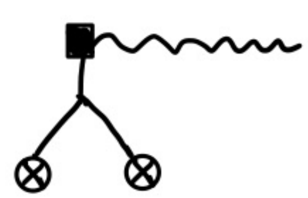
$$\delta g_{tt}^{(\ell=2)} = r^2 \mathcal{E}_2 Y_{20}\left(1 + k_2\frac{R^5}{r^5}\right) - r^4\mathcal{E}_2\mathcal{E}_2 Y_{20}\left(1 + k_2\frac{R^5}{r^5} + k_2^2\frac{R^{10}}{r^{10}} + \frac{p_2}{GM}\frac{R^8}{r^7}\right)$$

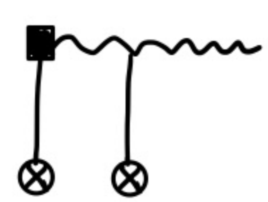




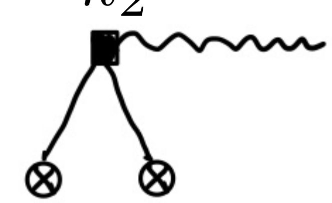
p_2







k_2



EFT calculation

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Classical limit = no diagrams
with close loops of $h_{\mu\nu}$

EFT calculation

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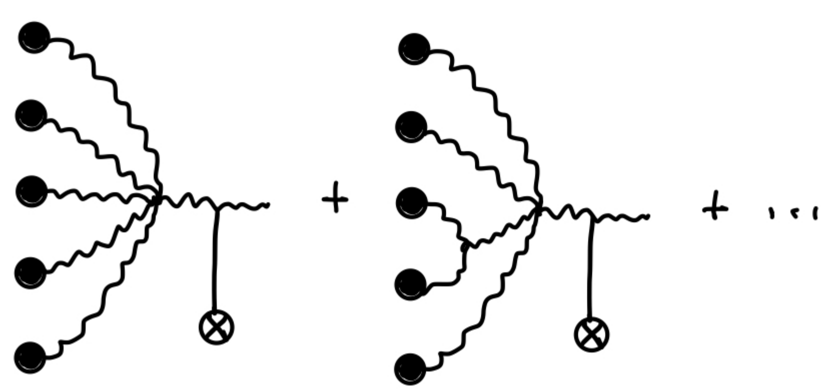

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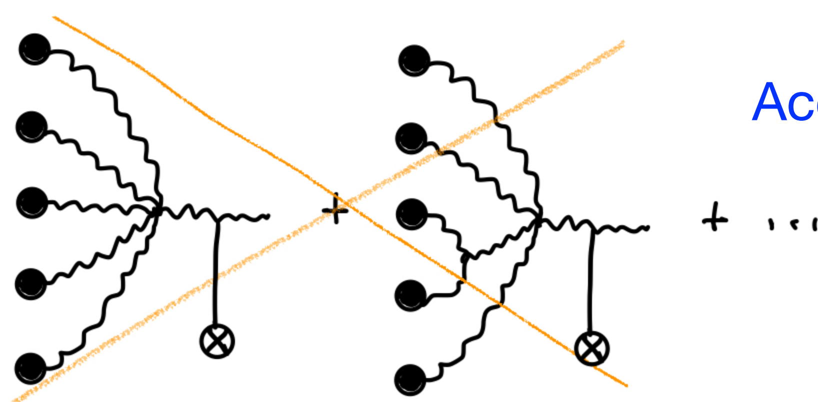

Accidental symmetry of static GR

Parra-Martinez, Podo, '25

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Accidental symmetry of static GR

Parra-Martinez, Podo, '25

Full GR calculation

- Tolman–Oppenheimer–Volkoff equations inside the star and vacuum Einstein equations outside

$$G_{\mu\nu} = T_{\mu\nu} = 8\pi G\left[(\rho + p)u_\mu u_\nu + p g_{\mu\nu}\right]$$

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- Expand metric and fluid variables around background <u>in the static limit</u> (no time dependence)

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- Regge-Wheeler approach $\delta g_{\mu\nu} = \delta g_{\mu\nu}^{+} + \delta g_{\mu\nu}^{-}$ [Regge & Wheeler '57]

$$\delta g_{\mu\nu}^{+} dx^\mu dx^\nu = \left[f(r)H_0(r)dt^2 + 2H_1(r)dtdr + H_2(r)/f(r)dr^2 + r^2K(r)d\Omega^2\right]Y_{\ell m}$$

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- Solve Einstein equations perturbatively at linear and quadratic order, imposing regularity at the star center and “continuity” at the star surface

$$G_{\mu\nu} = \bar{G}_{\mu\nu} + {}^{(1)}G_{\mu\nu} + {}^{(2)}G_{\mu\nu} \qquad T_{\mu\nu} = \bar{T}_{\mu\nu} + {}^{(1)}T_{\mu\nu} + {}^{(2)}T_{\mu\nu}$$

$${}^{(1)}H_0'' + C_1 {}^{(1)}H_0' + C_0 {}^{(1)}H_0 = 0$$ Linear [Damour & Nagar ’09]

$${}^{(2)}H_0'' + C_1 {}^{(2)}H_0' + C_0 {}^{(2)}H_0 = S({}^{(1)}H_0 \star {}^{(1)}H_0)$$ Quadratic

Full GR calculation

- Linear master equation:

$$e^{-\Lambda(r)} \equiv 1 - \frac{M(r)}{r} \qquad \lambda = \ell(\ell+1)$$

$$
\begin{aligned}
&{}^{(1)}H_0'' + {}^{(1)}H_0' \frac{2\left[r - M + 2\pi r^3(\bar{p}-\bar{\rho})\right]}{r(r-2M)} \\
&+ {}^{(1)}H_0 \left\{ \frac{4\pi r(\bar{p}+\bar{\rho})\bar{\rho}'}{(r-2M)\bar{p}'} - \frac{2M\left(-2r\lambda + 52\pi r^3\bar{p} + 20\pi r^3\bar{\rho}\right) + 2r^2\lambda + 4M^2 + 4\pi r^4\left[\bar{p}\left(16\pi r^2\bar{p} - 9\right) - 5\bar{\rho}\right]}{r^2(r-2M)^2} \right\} = 0
\end{aligned}
$$

- Quadratic master equation:

$$
\begin{aligned}
&{}^{(2)}H_0'' + {}^{(2)}H_0' \frac{2\left[r - M + 2\pi r^3(\bar{p}-\bar{\rho})\right]}{r(r-2M)} \\
&+ {}^{(2)}H_0 \left\{ \frac{4\pi r(\bar{p}+\bar{\rho})\bar{\rho}'}{(r-2M)\bar{p}'} - \frac{2M\left(-2r\lambda + 52\pi r^3\bar{p} + 20\pi r^3\bar{\rho}\right) + 2r^2\lambda + 4M^2 + 4\pi r^4\left[\bar{p}\left(16\pi r^2\bar{p} - 9\right) - 5\bar{\rho}\right]}{r^2(r-2M)^2} \right\} = S_{H_0}
\end{aligned}
$$

Example: Source for $\ell = 2$ sourced by $\ell_1 = 2$, $\ell_2 = 2$

$$
\begin{aligned}
-S_{H_0} = &\left({}^{(1)}H_0'\right)^2 + {}^{(1)}H_0\,{}^{(1)}H_0' \frac{3\left(M + 4\pi r^3\bar{p}\right)}{r(r-2M)} + \left({}^{(1)}H_0\right)^2 \left\{ \frac{\pi r^3(\bar{p}+\bar{\rho})}{r^2 c_s^4}\left(\frac{2r c_s c_s'}{M + 4\pi r^3\bar{p}} + \frac{1}{r-2M}\right) \right. \\
&\left. + \frac{6\pi r(\bar{p}+\bar{\rho})}{c_s^2(r-2M)} + \frac{rM\left[2\pi r^2(71\bar{p} + 15\bar{\rho}) + 3\right] - 7M^2 + \pi r^4\left[\bar{p}\left(176\pi r^2\bar{p} - 27\right) - 15\bar{\rho}\right] + 3r^2}{r^2(r-2M)^2} \right\}
\end{aligned}
$$

Matching at $\mathcal{O}(\mathcal{E})$

[Pani, Riva, Santoni, Savic, FV '25]

Full GR: $$\delta g_{tt}^{(\ell=2)} = \tilde{\mathcal{E}}_2 Y_{20}\left(1-\frac{r_s}{r}\right)\left[r(r-r_s)+\frac{a}{r_s^3}\,Q_2^2(2r/r_s-1)\right]$$

$$a = a(C,y) \qquad\qquad y = \left.\frac{r\,{}^{(1)}H_0'(r)}{{}^{(1)}H_0(r)}\right|_{r=R}$$

EFT: $$\delta g_{tt}^{(\ell=2)} = r^2\mathcal{E}_2 Y_{20}\left(1+k_2\frac{R^5}{r^5}\right)$$

- Matching at $r \gg r_s$: $\quad \mathcal{E}_2 = \tilde{\mathcal{E}}_2\,, \qquad k_2 = \#\dfrac{a}{R^5} \qquad \Rightarrow \qquad k_2 = k_2(C,y)$

$$k_2 = \frac{8C^5(1-2C)^2[1+(1-2C)(1-y)]}{10C\,[4C^4-4C^3+26C^2-24C+(4C^4+6C^3-22C^2+15C-3)\,y+6]+15(1-2C)^2\log(1-2C)[1+(1-2C)(1-y)]}$$

[Hinderer '06;
Damour & Nagar '09]

Matching at $\mathcal{O}(\mathcal{E}^2)$

[Pani, Riva, Santoni, Savic, FV ’25]

Full GR: $$\delta g_{tt}^{(\ell=2)} = \tilde{\mathcal{E}}_2 Y_{20}\left(1-\frac{r_s}{r}\right)\left[r(r-r_s)+\frac{a}{r_s^3}\,Q_2^2(2r/r_s-1)\right]$$

$$+\tilde{\mathcal{E}}_2\tilde{\mathcal{E}}_2 Y_{20}\left(1-\frac{r_s}{r}\right)\left[c_1 r(r-r_s)+c_2\,Q_2^2(2r/r_s-1)+H_0^{\rm part}(r)\right]$$

$$c_{1,2}=c_{1,2}(C,y,z) \qquad y=\left.\frac{r\,{}^{(1)}H_0'(r)}{{}^{(1)}H_0(r)}\right|_{r=R} \qquad z=\frac{r_s}{2\,{}^{(1)}H_0(r)}\left.\left(\frac{{}^{(2)}H_0(r)}{{}^{(1)}H_0(r)}\right)'\right|_{r=R}$$

EFT: $$\delta g_{tt}^{(\ell=2)} = r^2\mathcal{E}_2 Y_{20}\left(1+k_2\frac{R^5}{r^5}\right)-r^4\mathcal{E}_2\mathcal{E}_2 Y_{20}\left(1+k_2\frac{R^5}{r^5}+k_2^2\frac{R^{10}}{r^{10}}+\frac{p_2}{GM}\frac{R^8}{r^7}\right)$$

- Matching at $r \gg r_s$: $\Rightarrow \quad p_2 = p_2(C,y,z)$

Quadratic Love Number

[Pani, Riva, Santoni, Savic, FV ’25]

$$p_2 = -\frac{8(1-2C)C^8}{245D(C,y)^3}\left[zA(C) + B(C)\log^2(1-2C) + L(C,y)\log(1-2C) + 2C\sum_{i=0}^{3} K_i(C)y^i\right]$$

$$D(C,y) = 2C\{C[2C(C(2C(y+1)+3y-2)-11y+13)+3(5y-8)]-3y+6\}+3(1-2C)^2\log(1-2C)[2C(y-1)-y+2]\,,$$

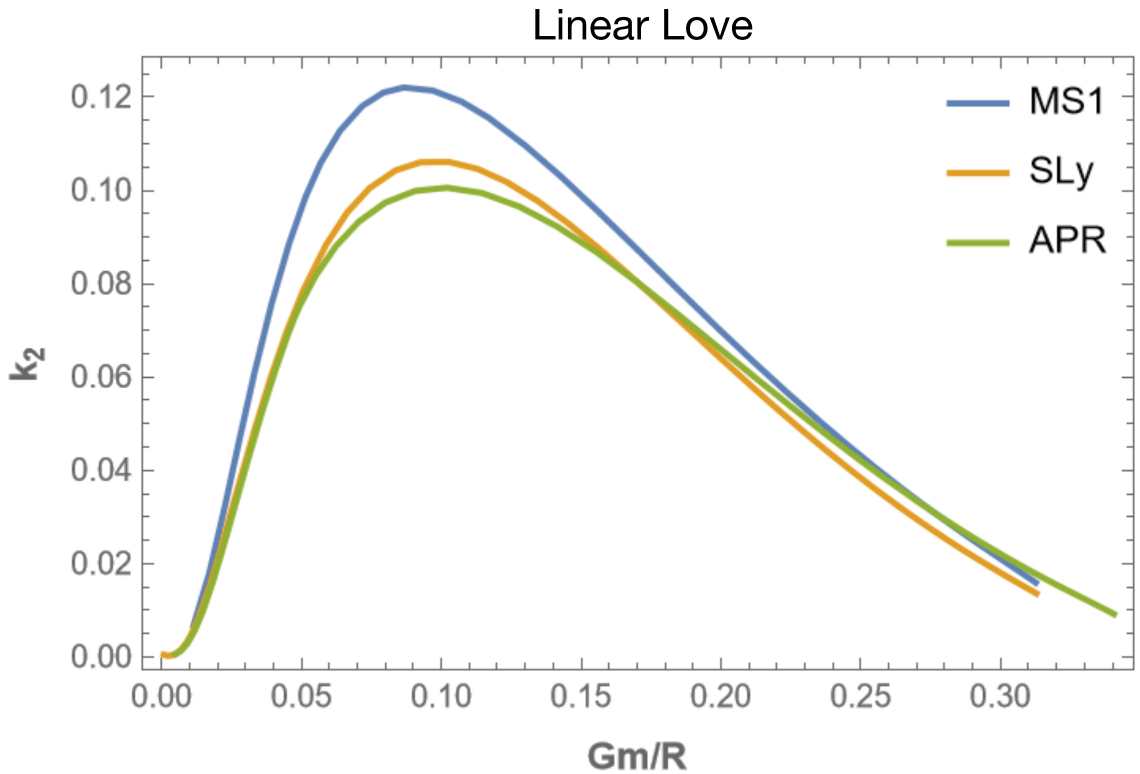
$$\begin{aligned}
A(C) &= -7168(1-2C)^2C^7\,,\\
B(C,y) &= 216(1-2C)^4(2C(y-1)-y+2)^3\,,\\
L(C,y) &= 9(1-2C)^2(2C(y-1)-y+2)^2\\
&\quad\times\{2C[2C(2C(8C(2C(y+1)+3y-2)+157y-141)-615y+788)+687y-1129]-245(y-2)\}\,,
\end{aligned}$$

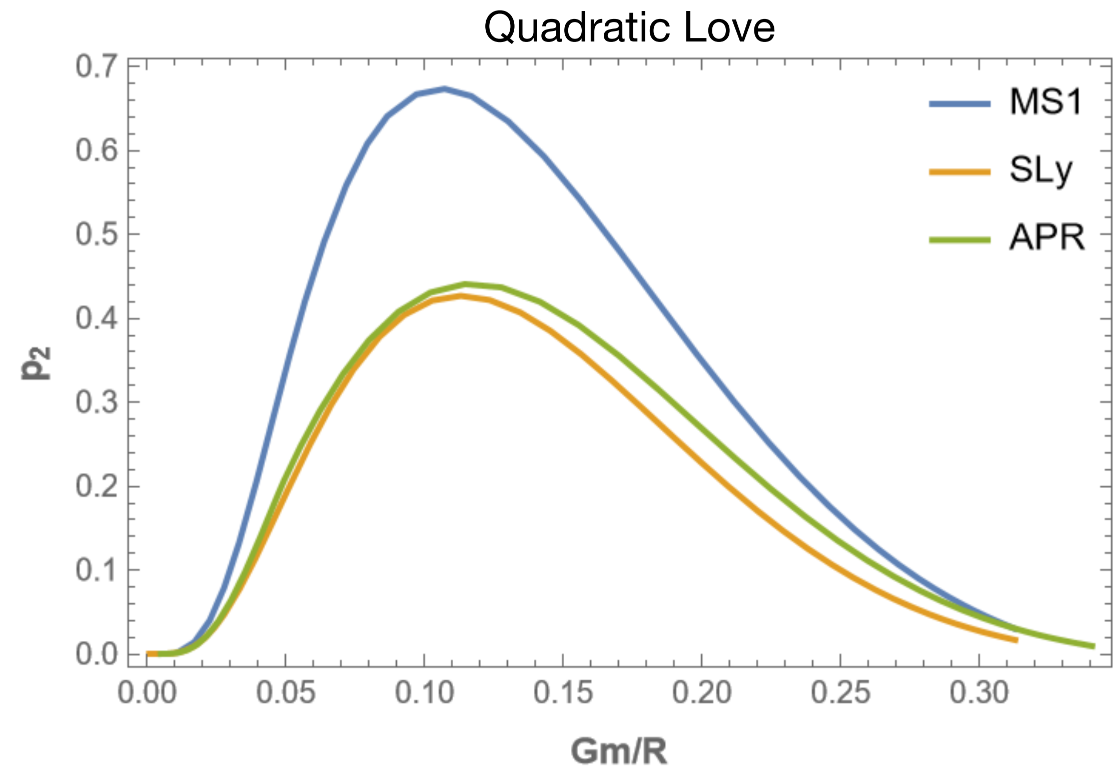
$$\begin{aligned}
K_0(C) &= -9600C^{10}+35968C^9-44288C^8+35712C^7+291312C^6-1125936C^5\\
&\quad+1769544C^4-1486464C^3+698016C^2-172944C+17640\,,\\
K_1(C) &= -9600C^{10}+10688C^9+41344C^8-11776C^7-914832C^6+2932200C^5\\
&\quad-4069944C^4+3043764C^3-1279980C^2+285876C-26460\,,\\
K_2(C) &= 9600C^{10}-19200C^9+1632C^8-49632C^7+916272C^6-2491776C^5\\
&\quad+3046572C^4-2032164C^3+769698C^2-156168C+13230\,,\\
K_3(C) &= 9600C^{10}+2496C^9-7008C^8+28176C^7-297552C^6+689208C^5\\
&\quad-741996C^4+443154C^3-152106C^2+28233C-2205\,.
\end{aligned}$$

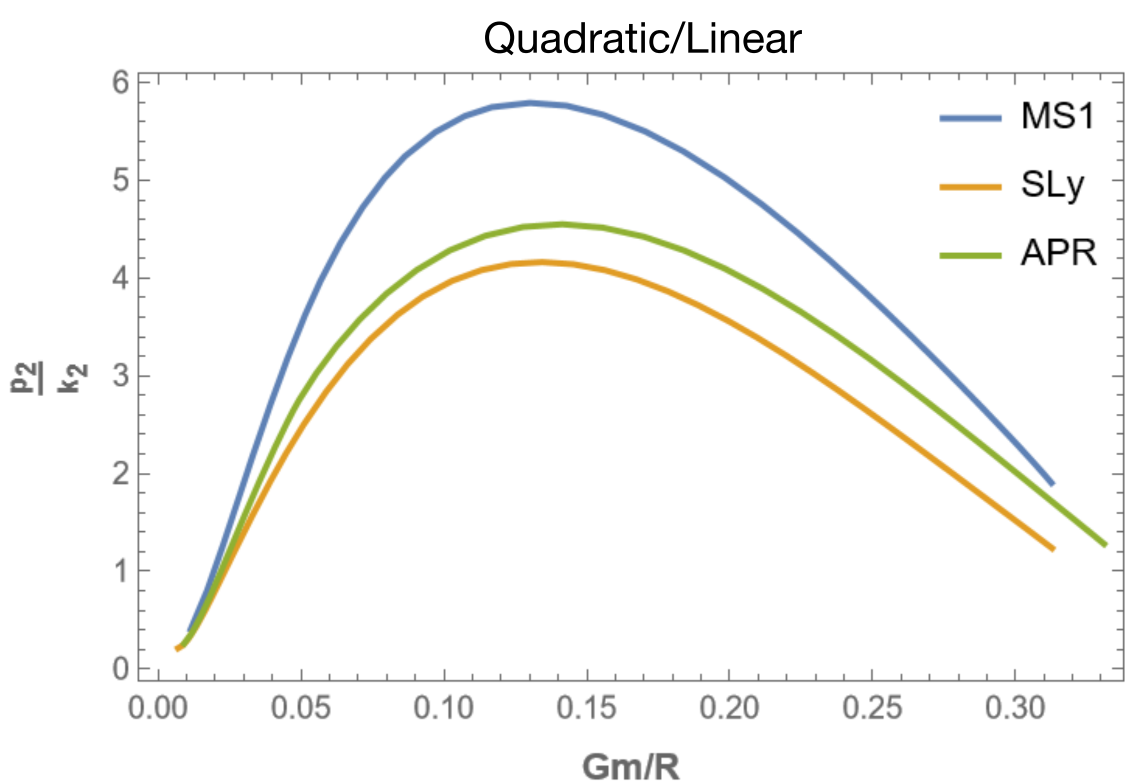
Results

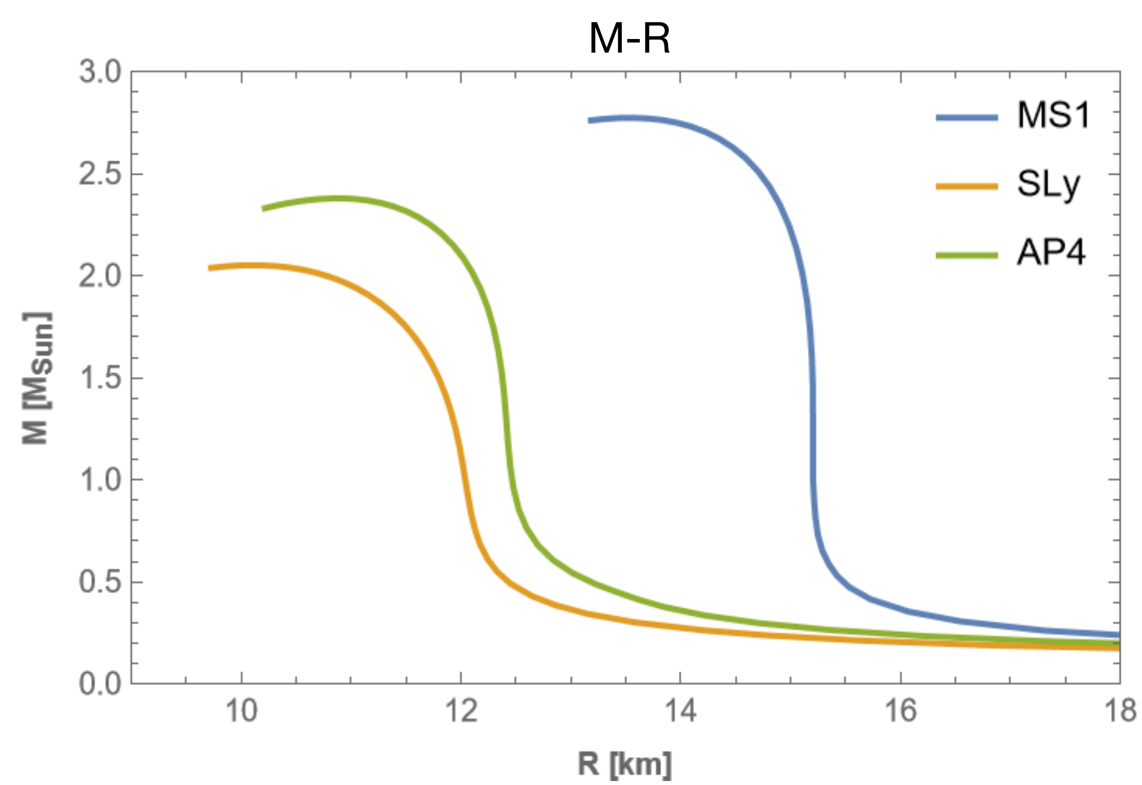
[Pani, Riva, Santoni, Savic, FV ’25]

- “Realistic” equations of state





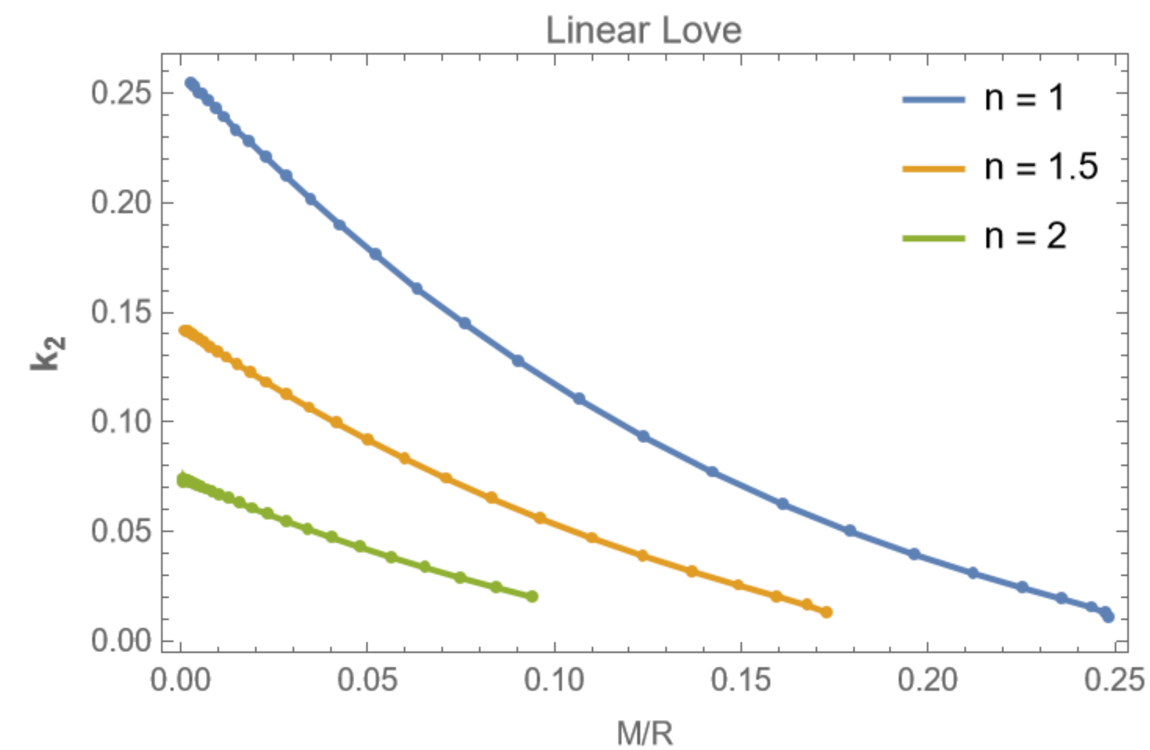


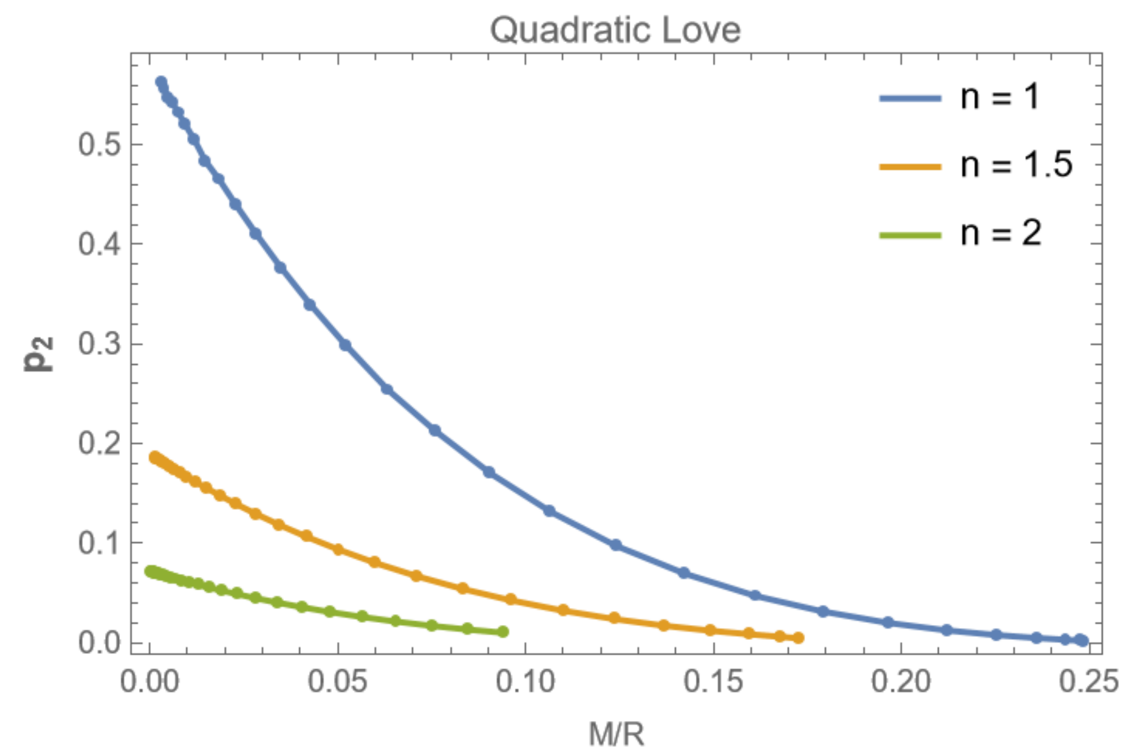


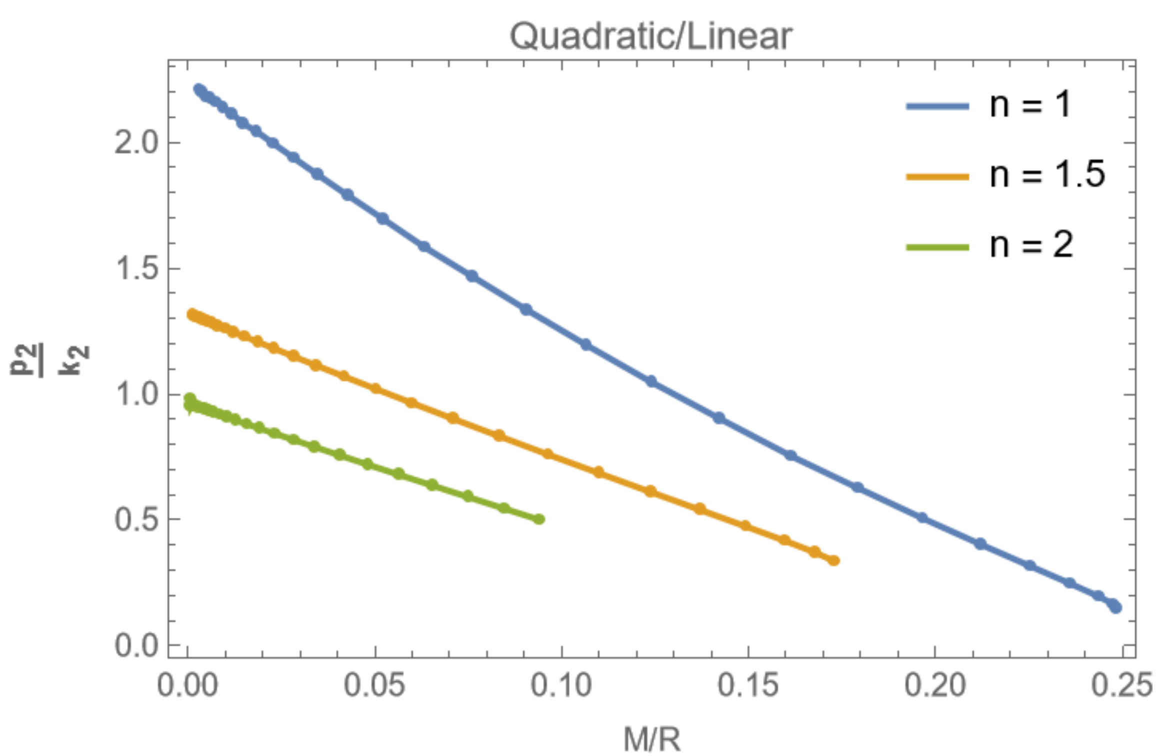
Results

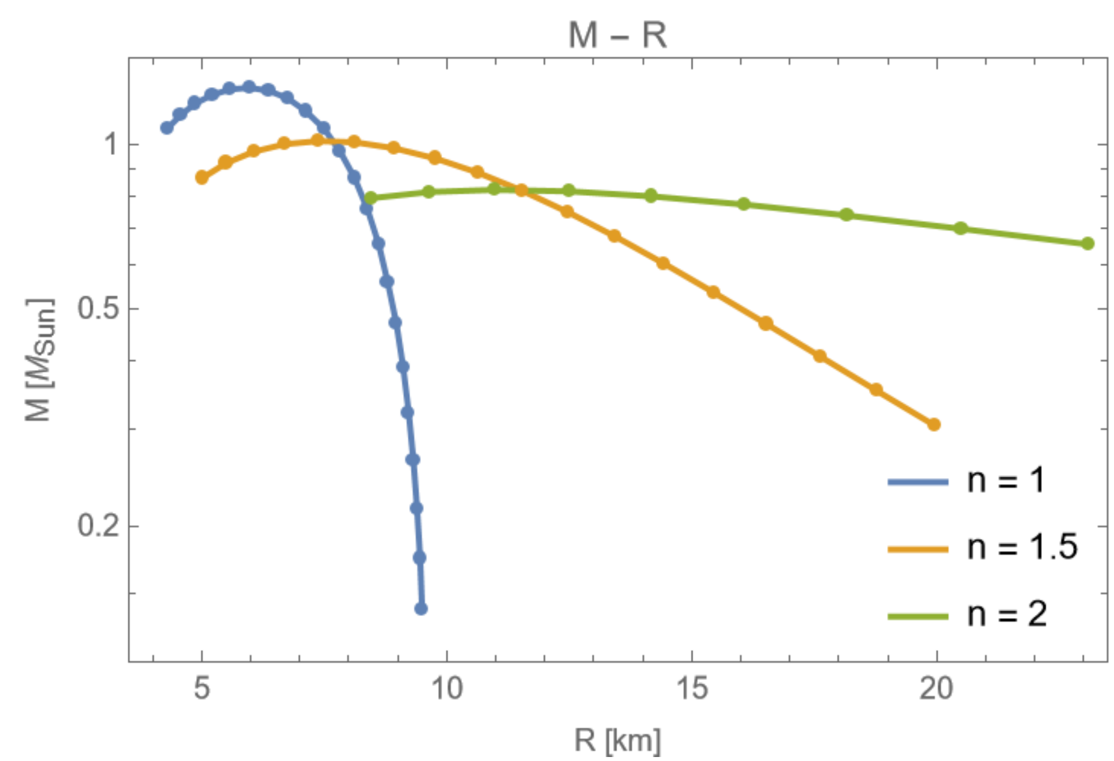
[Pani et al. '25
Agreement with Pitre & Poisson '25]

- Polytropic equation of state: $p = C\rho^{1+1/n}$



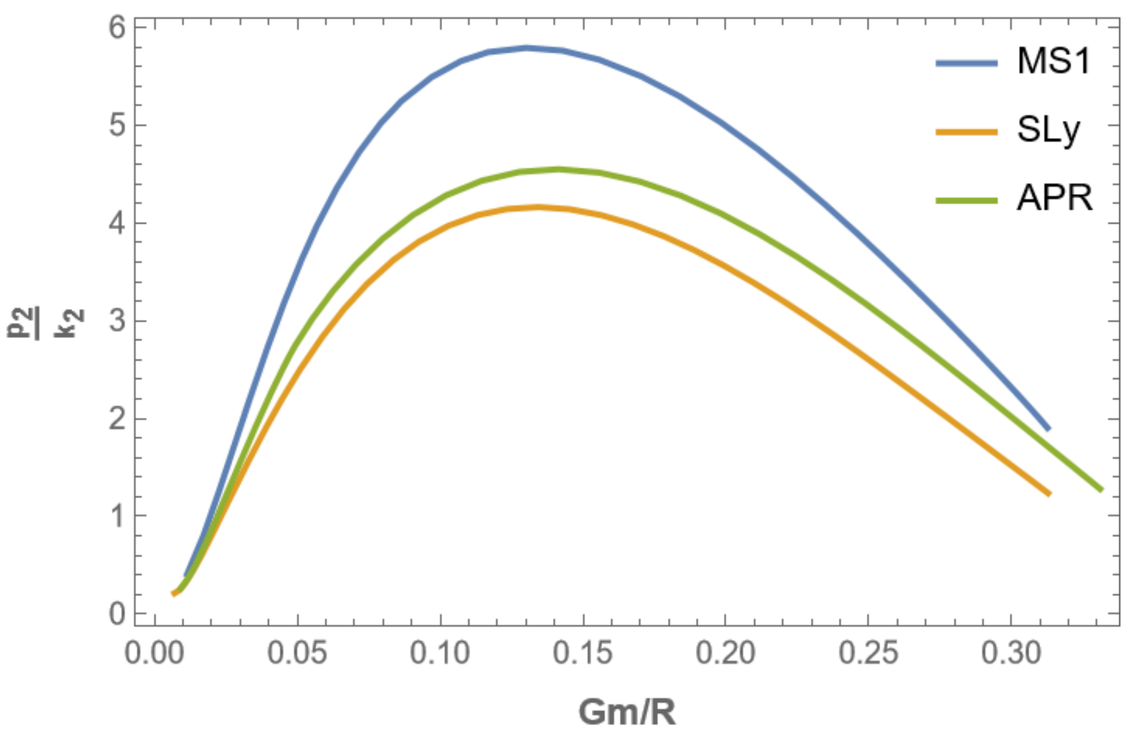




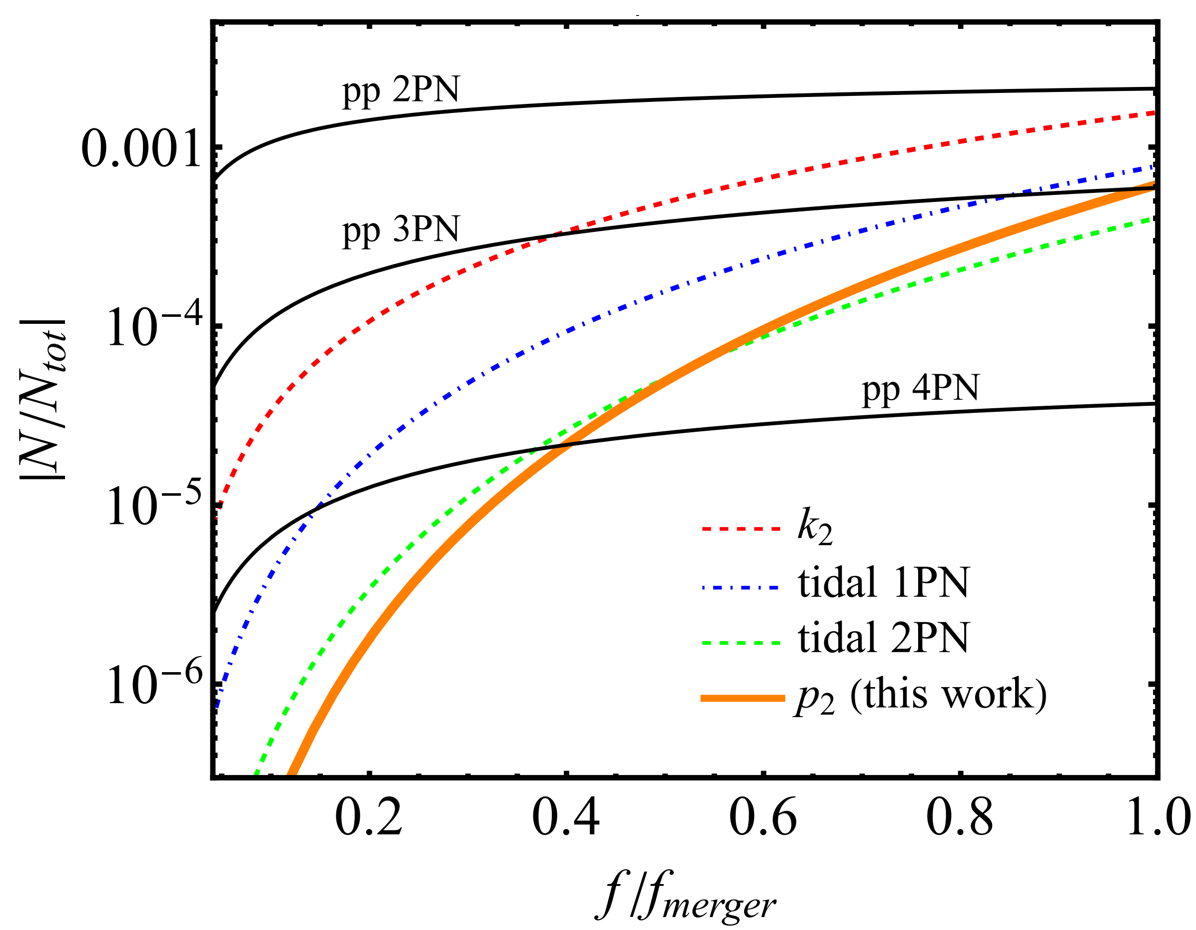


Conclusion

- Obtained static quadratic tidal Love number of neutron stars by matching the effective field theory with stellar perturbation theory at second-order



$$M_1 = M_2 = 1.4 M_\odot \ , \qquad C = 0.1 \ , \qquad p_2/k_2 = 5$$



- Future: Extend to odd modes and all multipoles, study observability including other subdominant effects, going beyond static…