Well-posedness of characteristic formulations of GR

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Well-posedness

What: Property of a Partial Differential Equation (PDE) problem.

- There exists a solution
- It is unique
- It depends continuously on the given data, in some norm

Why: A numerical solution can converge to the continuum one only for well-posed PDE problems.

Well-posedness & hyperbolic PDEs

Hyperbolic PDE system: $A^t \, \partial_t \mathbf{V} + A^p \, \partial_p \mathbf{V} = \mathbf{S}\,,$

$$\mathbf{V} = (v_1, v_2, \ldots, v_q)^T\,, \quad A^\mu = \begin{pmatrix} a_{11} & \ldots & a_{1q} \\ \vdots & \ddots & \vdots \\ a_{q1} & \ldots & a_{qq} \end{pmatrix}\,, \quad det(A^t) \neq 0\,.$$

Principal symbol: $P^s = (A^t)^{-1} \, A^p \, s_p$, with s^i arbitrary unit spatial vector.

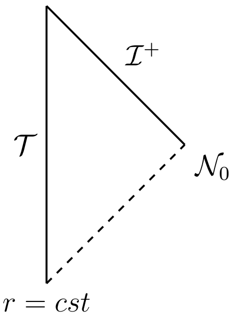
- Weak hyperbolicity: P^s has real eigenvalues $\forall \; s^i$
- Strong hyperbolicity: P^s is also diagonalizable $\forall \; s^i$

No strong hyperbolicity $\rightarrow$ **No** well-posed initial boundary value problem.

Characteristic formulations of GR

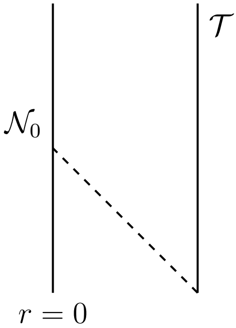
Spacetime foliated with null hypersurfaces.

Asymptotically flat
(Bondi-Sachs)

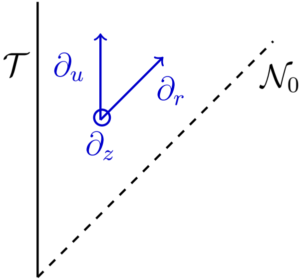


Gravitational waveforms
(CCE & CCM)

Asymptotically AdS
(ingoing Eddington-Finkelstein)

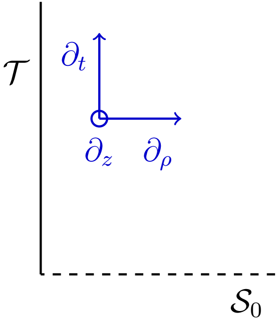


Numerical holography
(heavy ion collisions)



Characteristic Initial Boundary Value Problem (CIBVP)

$\longrightarrow$



Initial Boundary Value Problem (IBVP)

$$A^t\,\partial_t \mathbf{V} + A^\rho\,\partial_\rho \mathbf{V} \simeq 0$$

$$det(A^u) = 0 \qquad\qquad det(A^t) \neq 0$$

Well-posed characteristic initial value problem $\leftrightarrow$ well-posed Cauchy problem. [1]

[1] Alan Rendall, 1990

Analysis & Results

Analysis:

$$A^u\,\partial_u\mathbf{V} + A^r\,\partial_r\mathbf{V} + A^z\,\partial_z\mathbf{V} \simeq 0$$

1. Linearization on a fixed background
2. 1st order reduction
3. Coordinate transformation to IBVP

Result: The PDE system is only weakly hyperbolic $\longrightarrow$ ill-posed in L_2 norm.

- Non-diagonalizable A^z, only block diagonal
- True $\forall$ 1st order reductions within the basis

Toy models

Weakly hyperbolic:

$$\begin{aligned} \partial_r \psi_v - \partial_z \psi &= 0 \\ \partial_u \psi - 2\partial_r \psi - \psi_v &= 0 \end{aligned} \quad \longrightarrow \quad \begin{aligned} A^z &= \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \\ \mathbf{V} &= (\psi_v,\, \psi)^T \end{aligned}$$

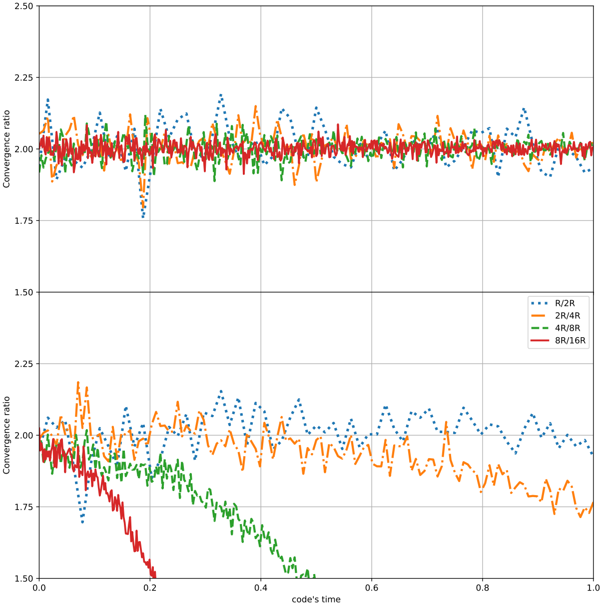
Strongly hyperbolic:

$$\begin{aligned} \partial_r \psi_v - \partial_z \psi &= 0 \\ \partial_u \psi - 2\partial_r \psi - \partial_z \psi_v - \psi_v &= 0 \end{aligned} \quad \longrightarrow \quad \begin{aligned} A^z &= \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, \\ \mathbf{V} &= (\psi_v,\, \psi)^T \end{aligned}$$

Norm convergence

For different resolutions:

- Exact solution $\psi = 0$
- Random noise on given data
- $L_2 = \left[\int \left(\psi_v^2 + \psi^2\right) drdz\right]^{1/2}$
- Coarse resolution: R
- Convergence ratio: $log_2\left(\frac{L_2(2^n R)}{L_2(2^{n+1} R)}\right)$



Summary & Future work

Summary:

- Asymptotically flat: axially symmetric Bondi-Sachs (1)
- Asymptotically AdS: planary symmetric ingoing Eddington-Finkelstein (2)

(1) & (2) $\rightarrow$ weak hyperbolicity $\rightarrow$ ill-posed CIBVP in L_2 norm

Future work: how to fix or evade the problem?

- Modify the gauge choice
- Evolve the curvature

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Thank you!

Bondi-Sachs axially symmetric:

$$ds^2 = \left(\frac{V}{r}e^{2\beta} - U^2 r^2 e^{2\gamma}\right) du^2 + 2e^{2\beta} du\, dr$$
$$+ 2Ur^2 e^{2\gamma}\, du\, d\theta - r^2 \left(e^{2\gamma}\, d\theta^2 + e^{-2\gamma} \sin^2\theta\, d\phi^2\right) . \qquad (1)$$

Metric functions:

$$\beta(u, r, \theta) , \quad \gamma(u, r, \theta) , \quad U(u, r, \theta) , \quad V(u, r, \theta) .$$

Ingoing Eddington-Finkelstein planary symmetric:

$$ds^2 = -A dv^2 + \Sigma^2 \left[e^B dx_\perp^2 + e^{-2B} dz^2\right] + 2drdv + 2Fdvdz . \qquad (2)$$

Metric functions:

$$A(v, r, z) , \quad B(v, r, z) , \quad \Sigma(v, r, z) , \quad F(v, r, z) .$$