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# Rank-optimal approximations of higher-order tensors for low-dimensional space-time Galerkin approximations of parameter dependent dynamical systems

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We interpret a dynamical system of type

$$\dot{y} = f(y;\mu), \quad \text{on } (0,T], \quad y(0) = y_0, \qquad \text{as a map} \qquad \mathbb{G}\colon \mu \mapsto y$$

that maps an input or a parameter $\mu$ onto the state $y$ which is a function of space $x \in \Omega$ and time $t \in [0,T]$. We assume that the state is observed in a time-space tensor space, i.e. $y(t,x) = \tau(t) \cdot \xi(x)$, with $\tau \in S := L^2(0,T)$ and $\xi \in Y := L^2(\Omega)$. Thus, the state observations can be approximated using the tensor product of some bases $\{\nu_1, \dots, \nu_q\} \subset Y$ and $\{\psi_1, \dots, \psi_s\} \subset S$ of finite dimensional spaces of $S$ and $Y$. If we also consider the inputs in a finite dimensional space $\mathsf{span}\{\mu_1, \dots, \mu_r\}$, then an approximation of the Input/Output (I/O) map $\mathbb{G}$ is a mapping $\mathbf{G}\colon \mathbb{R}^r \to \mathbb{R}^q \cdot \mathbb{R}^s$, which can be interpreted as a tensor $\mathbf{H} \in \mathbb{R}^{r \times q \times s}$.

Using a higher-order SVD of $\mathbf{H}$, cf. [2], we propose some low-dimensional bases for subspaces of $Y$ and $S$ optimized with respect to measurements in the space and time domain and sampling in the input space. In [1], we have shown that this approach can be seen as a generalization of the well-known proper orthogonal decomposition (POD) method. Basically, instead of using snapshots of the solution trajectory at discrete time instances, time information is sampled by testing against test functions in $L^2(0,T)$. The space dimension $Y$ of the I/O map is readily defined by the expansion of the state in a *Finite Element* basis. The benefits of this approach are discussed and illustrated in [1].

In this contribution, we extend the I/O map based approach to control or parameter dependent setups in two steps. Firstly, we also sample the parameter space which adds to the space and time sampling and, thus, leads to a three dimensional input-to-output tensor. Secondly, we employ the higher-order SVD [2] to define new space and time bases that are optimized with respect to inputs and time or inputs and space, respectively. After a space-time Galerkin projection, the solution of the reduced systems can be obtained as the solution of a small system of algebraic equations.

By means of *Burgers'* equation for varying viscosity parameters, we compare this space-time-parameter Galerkin approach with the standard POD method and discuss it's relation to *Proper Generalized Decomposition* [3] and computational issues such as online-offline decompositions and stability of the approximations with respect to changes in the reduced bases.

## References

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