Isomonodromic deformations: confluence, reduction & quantisation

Marta Mazzocco

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Isomonodromic deformations of systems of rank m with poles of any order on $\mathbb{P}^1$

- Algebraic approach to “irregular times”
- Hamiltonian formulation in terms of flows on co-adjoint orbits
- Confluence
- Quantisation

Fuchsian case

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Phase space:

$$\mathcal{O}_1^* \times \mathcal{O}_2^* \times \ldots \mathcal{O}_n^* \times \mathcal{O}_\infty^*, \qquad \left\{ A^{(i)} \underset{,}{\otimes} A^{(j)} \right\} = \delta_{ij} \left[A^{(i)} \otimes \mathbb{I}, \Pi \right]$$

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Quantise to KZ for $\mathfrak{g} = \mathfrak{gl}_m$

$$d\psi = \sum_{i \neq j} \Pi^{ij} \frac{du_i - du_j}{u_i - u_j} \psi, \qquad B \times U(\mathfrak{gl}_\mathfrak{m}(\mathbb{C}))^{\otimes n} \to B,$$

$\Pi^{ij} \in \operatorname{End}(U(\mathfrak{gl}_m(\mathbb{C})^{\otimes n})$

[Reshetikhin '92]

Irregular case

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Confluent KZ?

State of the art

- Quantisation of Gaudin

$$dA^{(i)} = \sum_{i \neq j} [A^{(i)}, A^{(j)}] \frac{du_i - du_j}{\widetilde{u}_i - \widetilde{u}_i},$$

  - simple poles: Mukhin, Tarasov, Varchenko '06 (see also Talalaev '11)
  - non semisimple order two pole at infinity: Feigin, Frenkel, Rybnikov '10
  - generalisation of Gaudin corresponding to non-highest weight reps of any simple Lie algebra: Feigin, Frenkel, Toledano–Laredo '10
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  - any lie algebra: Millson, Toledano–Laredo '05, Felder, Markov, Tarasov, '00
- KZ equations with irregular singularities
  - irregular singular points of arbitrary Poincaré rank for $\mathfrak{sl}_2$: Jimbo, Nagoya and Sun '08
  - one irregular singularity of Poincaré rank 2 and several other simple poles: Rembado '19
  - Feigin, Frenkel, Toledano–Laredo quantum integrable systems are expected to give rise to *confluent KZ equations*- not been explicitly written.

Phase space

$$A(\lambda) = \sum_{i=1}^{n} \sum_{k=0}^{r_i} \frac{A_k^{(i)}}{(\lambda - u_i)^{k+1}} + \sum_{k=1}^{r_\infty} A_k^{(\infty)} z^{k-1},$$

$$\sum_{k=0}^{r_i} \frac{A_k^{(i)}}{(\lambda - u_i)^{k+1}} \in \widehat{\mathcal{O}}_{r_i}^{*} \subset \hat{\mathfrak{g}}_{r_i}^{\star}, \qquad \sum_{k=1}^{r_\infty} A_k^{(\infty)} z^{k-1} \in \widehat{\mathcal{O}}_{r_\infty}^{*} \subset \hat{\mathfrak{g}}_{r_\infty}^{\star}$$

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for $z = \lambda - u_i$ or $z = \frac{1}{\lambda}$:

$$\hat{\mathfrak{g}}_r = \mathfrak{g}[z]^+ / z^{r+1}\mathfrak{g}[z]^+, \quad \mathfrak{g}[z]^+ \simeq \mathfrak{g}[[z]] \Big/ \mathfrak{g}[z]^-, \quad \mathfrak{g}[z]^- = \left\{ f \in \mathfrak{g}[[z]] : \lim_{z\to\infty} f(z) = 0 \right\}$$

Example

$$\frac{dY}{d\lambda} = \left(U + \frac{V}{\lambda} \right) Y, \qquad U \text{ diagonal}, V \in \mathfrak{so}_n.$$

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$$\frac{dY}{d\lambda} = \left(u\Lambda + \frac{V}{\lambda} \right) Y, \qquad \Lambda \text{ constant}.$$

Isomonodromic deformations in $\widehat{\mathcal{O}}^*_{r_1} \times \widehat{\mathcal{O}}^*_{r_2} \times \dots \widehat{\mathcal{O}}^*_{r_n} \times \widehat{\mathcal{O}}^*_{r_\infty}$

Theorem

$$\sum_{i=1}^{n}\sum_{k=0}^{r_i}\frac{A_k^{(i)}}{(\lambda-u_i)^{k+1}}+\sum_{k=1}^{r_\infty}A_k^{(\infty)}\lambda^{k-1},\quad \sum_{i=1}^{n}\sum_{k=0}^{r_i}\frac{B_k^{(i)}}{(\lambda-u_i)^{k+1}}+\sum_{k=1}^{r_\infty}B_k^{(\infty)}\lambda^{k-1}.$$

are related by a linear Poisson automorphism iff

$$B_k^{(i)}=\sum_{j=k}^{r_i}A_j^{(i)}\mathcal{M}_{k,j}^{(r_i)}(t_1^{(i)},t_2^{(i)},\dots t_{r_i}^{(i)}),$$

$$\mathcal{M}_{k,j}^{(r)}=\frac{1}{j!}\frac{d^j}{d\varepsilon^j}P_r^{(i)}(t^{(i)},\varepsilon)^k\Big|_{\varepsilon=0},\quad P_r^{(i)}(t^{(i)},\varepsilon)=\sum_{k=1}^{r}\varepsilon^k t_k^{(i)}.$$

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Example

$$\hat{\mathcal{O}}_0^\star\times\hat{\mathcal{O}}_3^\star:\quad \frac{A_0^{(1)}}{\lambda-u_1}+t_1^3A_3^{(\infty)}\lambda^2+(t_1^2A_2^{(\infty)}+2t_1t_2A_3^{(\infty)})\lambda+t_1A_1^{(\infty)}+t_2A_2^{(\infty)}+t_3A_3^{(\infty)}$$

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$$A(\lambda) = \sum_{i=1}^{n} \sum_{k=0}^{r_i} \frac{\sum_{j=k}^{r_i} A_j^{(i)} \mathcal{M}_{k,j}^{(r_i)}(t_1^{(i)}, \dots t_{r_i}^{(i)})}{(\lambda - u_i)^{k+1}} + \sum_{k=1}^{r_\infty} \sum_{j=k}^{r_\infty} A_j^{(\infty)} \mathcal{M}_{k,j}^{(r_\infty)}(t_1^{(\infty)}, \dots t_{r_i}^{(\infty)}) \lambda^{k-1}$$

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Theorem

The isomonodromic deformation equation is given by

$$\Omega_j^{(i)} = \int \frac{\partial A}{\partial t_j^{(i)}} d\lambda, \qquad \Omega_{u_j} = \int \frac{\partial A}{\partial u_j} d\lambda.$$

Examples

Example

$A(\lambda) \in \hat{\mathcal{O}}_1^\star \times \hat{\mathcal{O}}_1^\star$:

$$A = \frac{A_0^{(0)}}{\lambda} + \frac{t_1^{(0)} A_1^{(0)}}{\lambda^2} + t_1^\infty A_1^{(\infty)}$$

Fix $t_1^{(\infty)} = 1 \Rightarrow \Omega_1^{(0)} = -\frac{A_1^{(0)}}{\lambda}$.

Example

$A(\lambda) \in \mathcal{O}_0^\star \times \hat{\mathcal{O}}_3^\star$:

$$A = \frac{A_0^{(1)}}{\lambda - u} + \frac{t_1^3 A_3^{(0)}}{\lambda^4} + \frac{t_1^2 A_2^{(0)} + 2 t_1 t_2 A_3^{(0)}}{\lambda^3} + \frac{t_1 A_1^{(0)} + t_2 A_2^{(0)} + t_3 A_3^{(0)}}{\lambda^2} + \frac{A_0^{(0)}}{\lambda}$$

Fix $t_1 = 1$ and $t_2 = 0$, $\Omega_3^{(\infty)} = A_3^{(\infty)} \lambda$ and $\Omega_0^{(1)} = -\frac{A_0^{(1)}}{\lambda - u_1}$. $\Rightarrow$ PDE in u_1, t_3.

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Confluence u_{n-1} and u_n:

$$u_n = w, \qquad u_{n-1} = w + \varepsilon t_1 .$$

$$A(\lambda) = \sum_{i=1}^{n-2} \frac{A^{(i)}}{\lambda - u_i} + \frac{B}{\lambda - w} + \frac{C}{\lambda - w - \varepsilon t_1}, \quad B = A^{(n-1)}, \quad C = A^{(n)}$$

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$$C = \frac{1}{\varepsilon} A_1^{(n-1)} + C_0 + O(\varepsilon), \quad B = -\frac{1}{\varepsilon} A_1^{(n-1)} + B_0 + O(\varepsilon), \quad C_0 + B_0 = A_0^{(n-1)}.$$

$$\tilde{A}(\lambda) := \lim_{\varepsilon \to 0} A(\lambda) = \sum_{i=1}^{n-2} \frac{A^{(i)}}{\lambda - u_i} + t_1 \frac{A_1^{(n-1)}}{(\lambda - w)^2} + \frac{A_0^{(n-1)}}{\lambda - w}$$

Confluence: $1+1$

Theorem

The 1+1 confluence procedure gives a Poisson morphism:

$$\mathcal{O}_1^\star \times \mathcal{O}_2^\star \times \ldots \mathcal{O}_n^\star \times \mathcal{O}_\infty^\star \xrightarrow{\text{confluence}} \mathcal{O}_1^\star \times \mathcal{O}_2^\star \times \ldots \mathcal{O}_{n-2}^\star \times \hat{\mathcal{O}}_{2,n-1}^\star \times \mathcal{O}_\infty^\star.$$

Namely, if $A^{(i)}$, B, C satisfy the the standard Lie Poisson brackets then

$$\left\{A_{1,\alpha}^{(n-2)}, A_{1,\beta}^{(n-2)}\right\} = 0, \quad \left\{A_{1,\alpha}^{(n-2)}, A_{0,\beta}^{(n-2)}\right\} = -\chi_{\alpha\beta}^{\gamma} A_{1,\gamma}^{(n-2)},$$
$$\left\{A_{0,\alpha}^{(n-2)}, A_{0,\beta}^{(n-2)}\right\} = -\chi_{\alpha\beta}^{\gamma} \left(A_{0,\gamma}^{(n-2)}\right).$$

Confluence: $r+1$

Theorem

Consider an r–parameter family of connections of the following form:

$$A = \sum_{k=0}^{r} \frac{B_k(t_1, t_2 \dots t_{r-1})}{(\lambda - u)^{k+1}} + \frac{C}{\lambda - v} + \dots \tag{1}$$

Assume

$$v = u + \sum_{i=1}^{r} t_i \varepsilon^i = u + P_r(t, \varepsilon), \tag{2}$$

then

$$\tilde{A} = \sum_{i=0}^{r+1} \frac{\tilde{B}_i(t_1, t_2 \dots t_r, t_{r+1})}{(\lambda - u)^{i+1}} + \textit{holomorphic terms},$$

where $\tilde{B}_i$'s are given by

$$\tilde{B}_i(t_1 \dots, t_{r+1}) = \sum_{k=i}^{r} \tilde{A}_k \mathcal{M}_{i,k}^{(r+1)}(t_1 \dots t_{r+1}). \tag{3}$$

Hamiltonians

Theorem

The confluent Hamiltonians $H_1^{(i)}, \dots, H_{r_i}^{(i)}$ which correspond to the times $t_1^{(i)}, \dots t_{r_i}^{(i)}$ are defined as follows:

$$\begin{pmatrix} H_1^{(i)} \\ H_2^{(i)} \\ \dots \\ H_{r_i}^{(i)} \end{pmatrix} = \left(\mathcal{M}^{(r)}\right)^{-1} \begin{pmatrix} S_1^{(i)} \\ S_2^{(i)} \\ \dots \\ S_{r_i}^{(i)} \end{pmatrix}, \tag{4}$$

where

$$S_k^{(i)} = \frac{1}{2} \oint_{\Gamma_{u_i}} (\lambda - u_i)^k \mathrm{Tr} A^2 \mathrm{d}\lambda \tag{5}$$

The Hamiltonian H_u corresponding to the time u is instead given by the standard formula

$$H_{u_i} = \frac{1}{2} \operatorname*{Res}_{\lambda = u_i} \mathrm{Tr} A(\lambda)^2.$$

Examples

Example

$A(\lambda) \in \hat{\mathcal{O}}_1^\star \times \hat{\mathcal{O}}_1^\star$ (fix $t_1^{(\infty)} = 1$): $\quad A(\lambda) = \frac{A_0^{(0)}}{\lambda} + \frac{t_1^{(0)} A_1^{(0)}}{\lambda^2} + A_1^{(\infty)}$

$$H = \mathrm{Tr}\left(\frac{(A_0^{(0)})^2}{2t_1^{(0)}} + t_1^{(\infty)} A_0^{(1)} A_1^{(\infty)} \right).$$

Example

$A(\lambda) \in \mathcal{O}_0^\star \times \hat{\mathcal{O}}_3^\star$ (fix $t_1 = 1$ and $t_2 = 0$)

$$A = \frac{A_0^{(1)}}{\lambda - u} + \frac{A_3^{(0)}}{\lambda^4} + \frac{A_2^{(0)} + 2t_2 A_3^{(0)}}{\lambda^3} + \frac{A_1^{(0)} + t_2 A_2^{(0)} + t_3 A_3^{(0)}}{\lambda^2} + \frac{A_0^{(0)}}{\lambda}$$

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Example

$A(\lambda) \in \hat{\mathcal{O}}_1^\star \times \hat{\mathcal{O}}_1^\star$ (fix $t_1^{(\infty)} = 1$): $\quad A(\lambda) = \frac{A_0^{(0)}}{\lambda} + \frac{t_1^{(0)} A_1^{(0)}}{\lambda^2} + A_1^{(\infty)}$

$$H = \mathrm{Tr}\left(\frac{(A_0^{(0)})^2}{2t_1^{(0)}} + t_1^{(\infty)} A_0^{(1)} A_1^{(\infty)} \right).$$

Example

$A(\lambda) \in \mathcal{O}_0^\star \times \hat{\mathcal{O}}_3^\star$ (fix $t_1 = 1$ and $t_2 = 0$)

$$A = \frac{A_0^{(1)}}{\lambda - u} + \frac{A_3^{(0)}}{\lambda^4} + \frac{A_2^{(0)} + 2t_2 A_3^{(0)}}{\lambda^3} + \frac{A_1^{(0)} + t_2 A_2^{(0)} + t_3 A_3^{(0)}}{\lambda^2} + \frac{A_0^{(0)}}{\lambda}$$

$$H_u = \mathrm{Tr} A_0^{(1)} \left(\frac{A_3^{(0)}}{u^4} + \frac{A_2^{(0)}}{u^3} + \frac{A_1^{(0)} + t_3 A_3^{(0)}}{u^2} + \frac{A_0^{(0)}}{u} \right)$$

$$H_3 = \mathrm{Tr} \left(-\frac{A_0^{(1)} A_3^{(0)}}{u} + \frac{2A_0^{(0)} A_2^{(0)} + \left(A_1^{(0)}\right)^2}{2} + t_3 A_1^{(0)} A_3^{(0)} \right)$$

Quantisation

Theorem

The confluent KZ Hamiltonians $\widehat{H}_{u_i}$, and $\widehat{H}_1^{(i)}, \dots, \widehat{H}_r^{(i)}$ are the following elements of the universal enveloping algebra $U(\hat{\mathfrak{g}}_{r_1} \oplus \cdots \oplus \hat{\mathfrak{g}}_{r_\infty})$:

$$\hat{H}_{u_i} = \frac{1}{2} \operatorname*{Res}_{\lambda = u_i} \mathrm{Tr}_0 \hat{A}(\lambda)^2,$$

$$\mathcal{M}^{(r_i)} \begin{pmatrix} \hat{H}_1^{(i)} \\ \hat{H}_2^{(i)} \\ \dots \\ \hat{H}_{r_i}^{(i)} \end{pmatrix} = \begin{pmatrix} \hat{S}_1^{(u_i)} \\ \hat{S}_2^{(u_i)} \\ \dots \\ \hat{S}_{r_i}^{(u_i)} \end{pmatrix}, \quad \hat{S}_k = \frac{1}{2} \oint_{\Gamma_{u_i}} (\lambda - u_i)^k \mathrm{Tr}_0 \hat{A}(\lambda)^2 \mathrm{d}\lambda,$$

where

$$\hat{A}(\lambda) = \sum_i^n \left(\sum_{j=0}^{r_i} \frac{\hat{B}_j^{(i)} \left(t_1^{(i)}, t_2^{(i)} \dots t_{r_i}^{(i)} \right)}{(\lambda - u_i)^{j+1}} \right),$$

$$\hat{B}_j^{(i)}(t_1^{(i)}, \dots t_{r_i}^{(i)}) = \sum_{k=j}^{r} \hat{A}_k^{(i)} \mathcal{M}_{j,k}^{(r_i)}(t_1^{(i)}, t_2^{(i)} \dots t_{r_i}^{(i)}), \quad \hat{A}_k = \sum_\alpha e_\alpha^{(0)} \otimes e_\alpha^{(i)} \otimes z_i^k.$$

Universal phase space

$$A(\lambda) = \sum_{i=1}^{n} \sum_{k=0}^{r_i} \frac{A_k^{(i)}}{(\lambda - u_i)^{k+1}} + \sum_{k=1}^{r_\infty} A_k^{(\infty)} z^{k-1} \in \mathcal{O}_1^* \times \mathcal{O}_2^* \times \ldots \mathcal{O}_n^* \times \mathcal{O}_\infty^*$$

Universal phase space

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Marsden–Weinstein reduction of the standard Poisson structure on

$$\oplus_{i=1}^{n+1} (T^\star \mathfrak{gl}_m)^{r_i+1} = \oplus_{k=1}^{d} T^\star \mathfrak{gl}_m, \qquad d = \sum_{i=1}^{n+1} r_i + n + 1$$

w.r.t. the *inner group action:*

$$\forall (g^{(1)}, \ldots, g^{(\infty)}) \in \otimes_{i=1}^{n+1} G, \quad g^{(i)} \underset{\text{inner}}{\times} (P_k^{(i)}, Q_k^{(i)}) = (g^{(i)} P_k^{(i)}, Q_k^{(i)} (g^{(i)})^{-1}),$$

where we understand $P_k^{(i)}, Q_k^{(i)}$ as lying in the $r_1 + \cdots + r_{i-1} + i + k$ copy of $T^\star \mathfrak{gl}_m$.

Isomonodromic τ-function

Theorem

The semi-classical solution of the confluent KZ system evaluated along solutions of the classical isomonodromic Hamiltonians is given by the isomonodromic τ-function

$$\mathrm{d}\ln(\tau) = \sum_i \left(H^{(i)}_{u_i} \mathrm{d}u_i + \sum_{k=1}^{r_i} H^{(i)}_k \mathrm{d}t^{(i)}_k \right).$$

The explicit form of the semi-classical solution is given by

$$\Psi(Q(t), t) \sim \tau^{\frac{i}{\hbar}}.$$