Ricci curvature & Quantum Geometry

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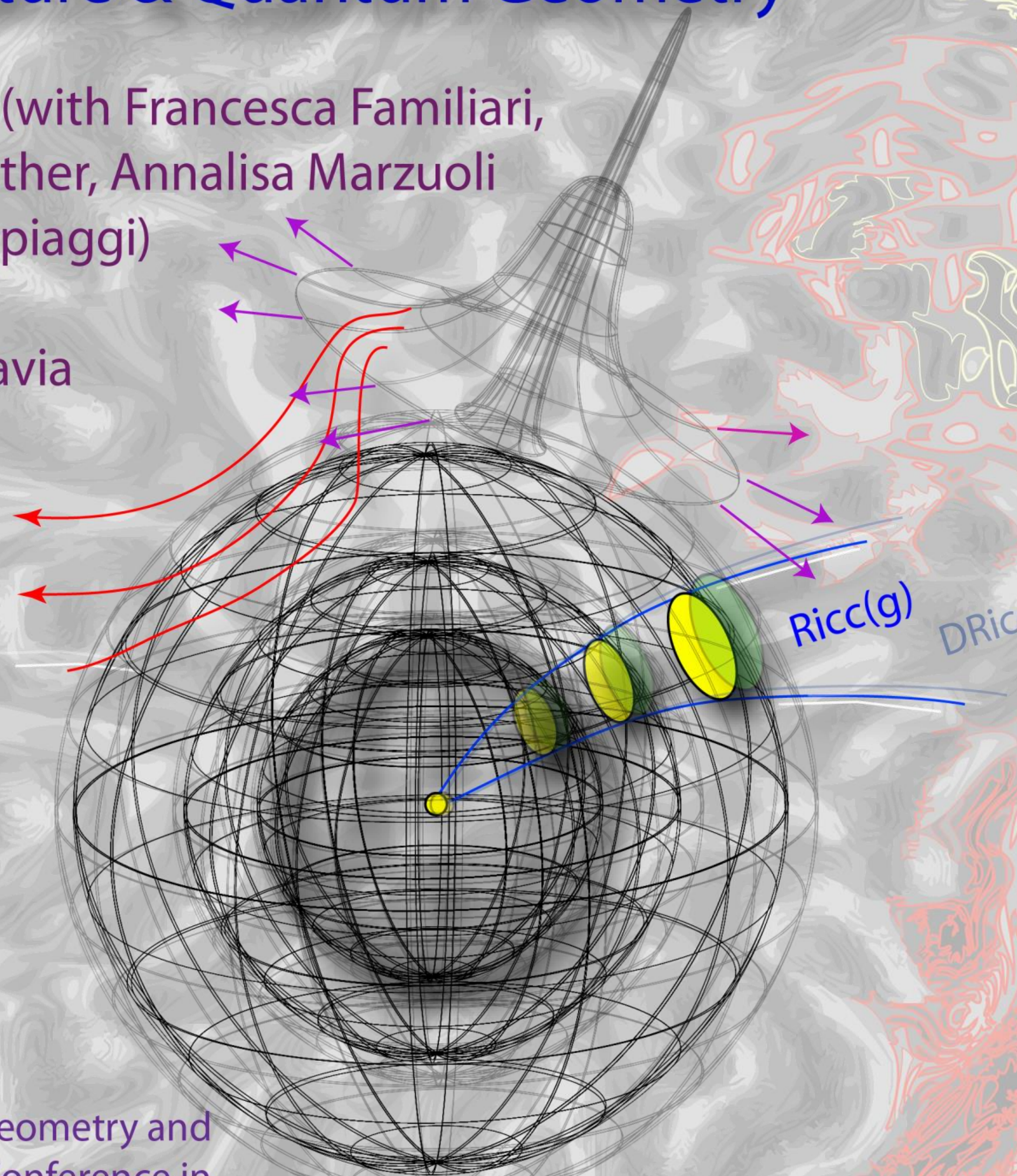

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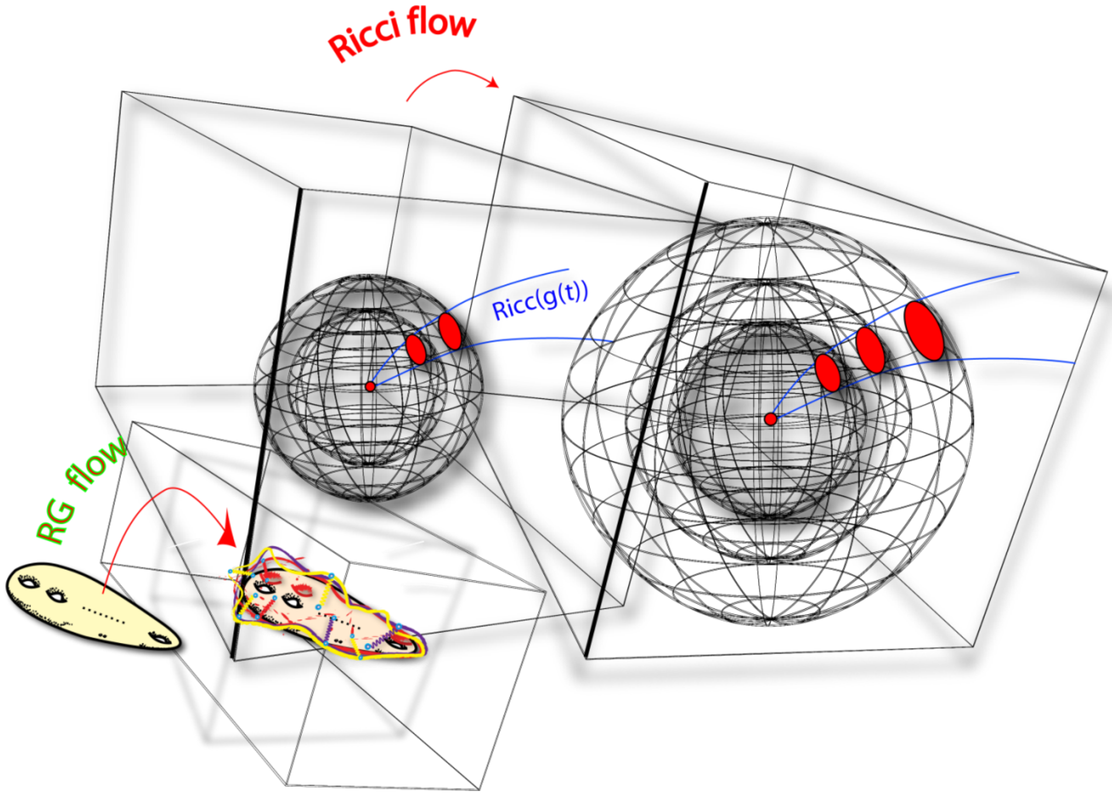
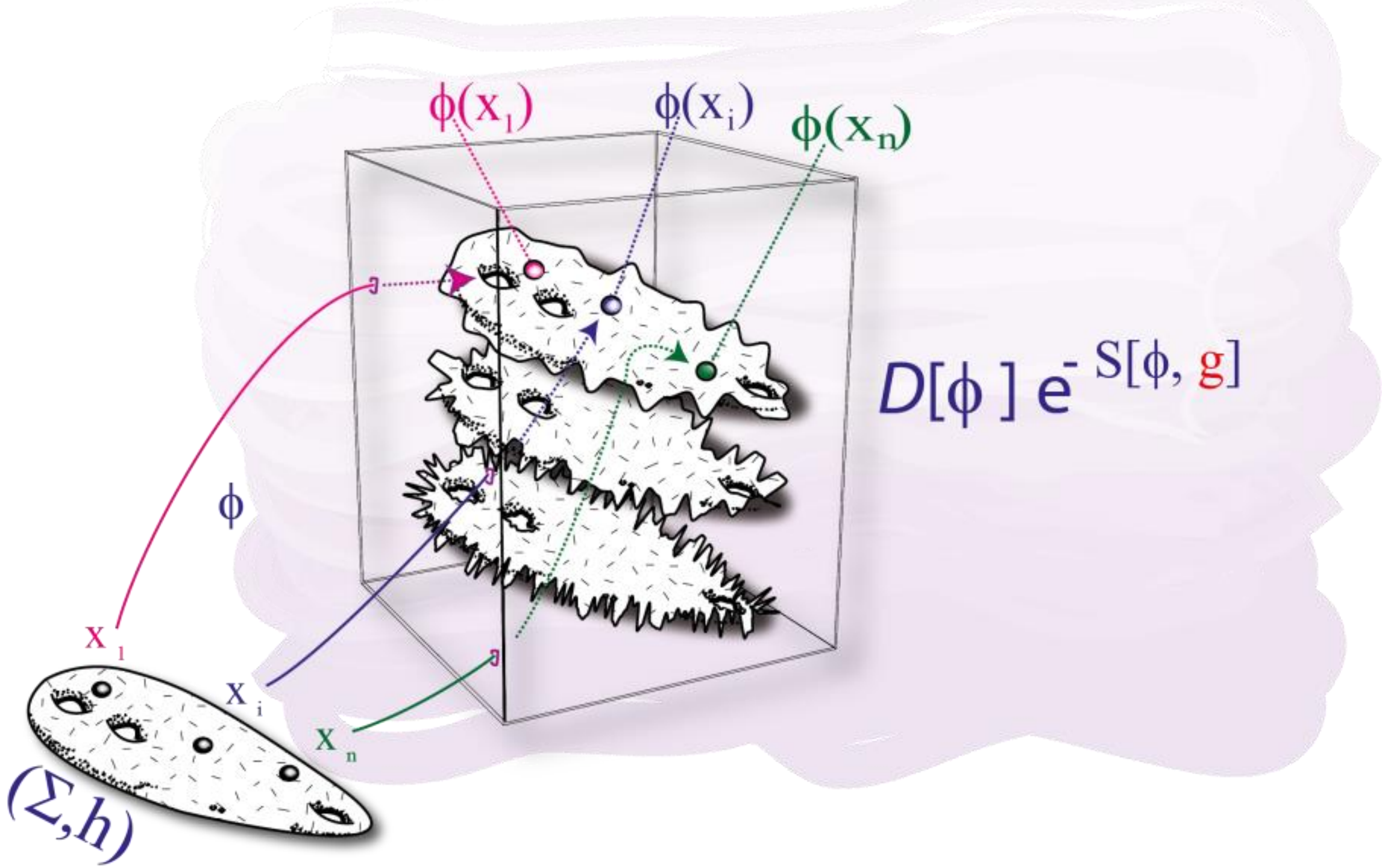
mc021

- Quantum Geometry has many sophisticated realizations. Here we limit ourselves to the (very) conservative circle of ideas associated with a quantum field theory approach to Riemannian Geometry: *viz.* How Riemannian structures are generated out of a suitably controlled spectrum of (random or quantum) fluctuations around a background fiducial geometry.



- Ricci curvature, with its subtle connections to diffusion, optimal transport, (Kantorovich - Rubinstein) - Wasserstein geometry and renormalization group, features prominently in such a scenario.

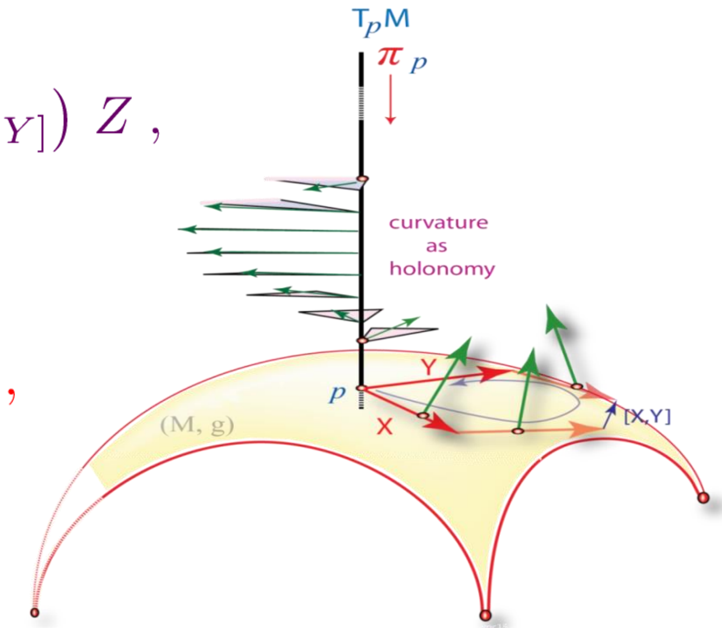
- In this talk, based on work done with Annalisa Marzuoli, Francesca Familiari, Claudio Dappiaggi and Christine Guenther, we touch upon some of these themes as well as on some unconventional aspects that Ricci curvature still holds in store and which stress its basic role in a QFT approach to quantum geometry.



The *CLASSICAL* FRAMEWORK:

(M, g) a C^∞ compact or complete n–dimensional manifold, $(n \geq 3)$, endowed with a Riemannian metric g, the associated Riemannian measure $d\mu_g$, Levi–Civita connection ∇, and its Riemann curvature

- $\mathrm{R}m(g)(X,Y)Z := \left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\right) Z$,



Ricci curvature in the direction u

- $\mathrm{R}ic(g)(u,u) := trace_g \left(\xi \longmapsto \mathrm{R}m(g)(\xi,u)u\right)$,

- Equivariance under the action of $\mathcal{D}iff(M)$:

  $Ric(\phi^* g) = \phi^* Ric(g) \Rightarrow$
  $\Rightarrow \nabla^i \mathcal{R}_{ik} \doteq \frac{1}{2} \nabla_k \mathcal{R}$:
  contracted Bianchi identity
  (D.Hilbert, J. Kazdan)

Basic role of the (normalized) Einstein–Hilbert functional

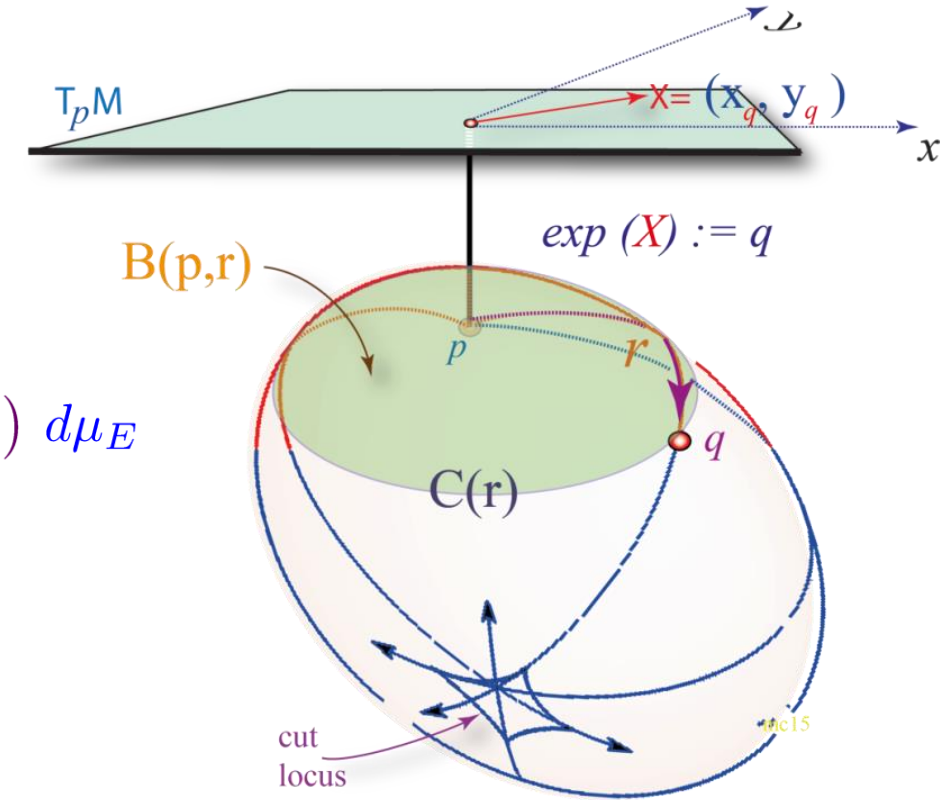
$$\mathcal{S}_{E-H}[g] \doteq \mathrm{Vol}_g(M)^{\frac{2-n}{n}} \int_M \mathrm{R}(g)\, d\mu_g,$$

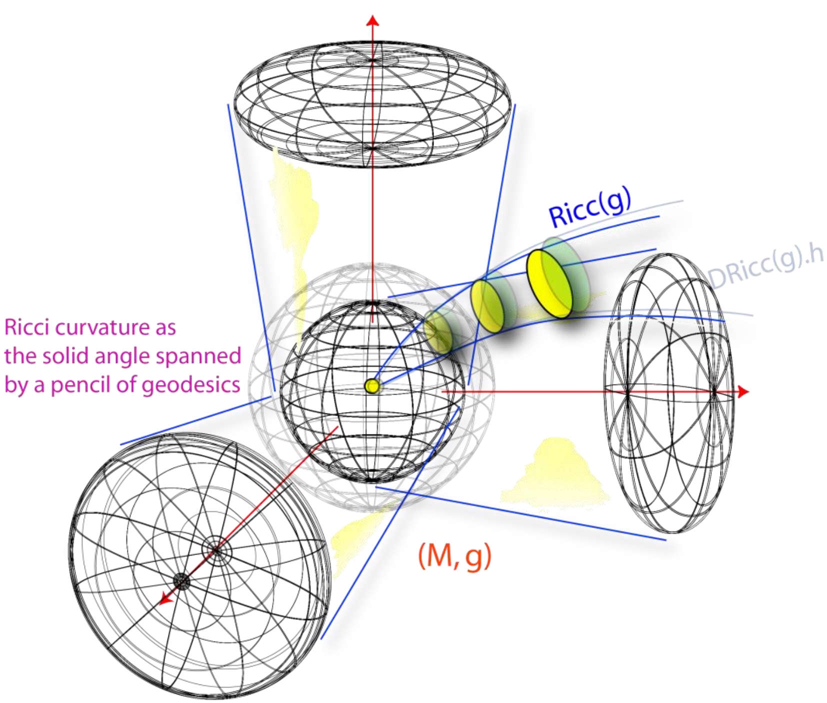
Insight into the Ricci curvature is provided by its expression in *normal geodesic coordinates* (Bertrand-Puiseaux)

- $\exp_p^*(d\mu_g)$

$$= \left(1 - \tfrac{1}{6}\, \mathrm{R}_{ik}(p)\, x^i(q) x^k(q) + \ldots\right) d\mu_E$$

$d\mu_E$: Euclidean measure on T_pM





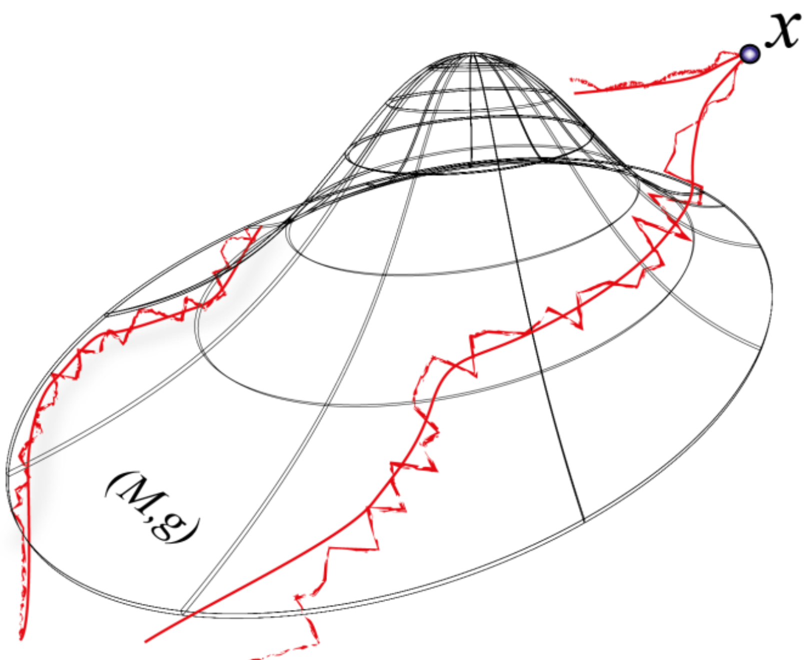
- *Ricci curvature*: distortion (w.r.t. $d\mu_E$) of the solid angle subtended by a small pencil of geodesics issued from p in the direction $X = \exp_p^{-1}(q)$.

$Ric(g) \propto \exp_p^*(d\mu_g)/d\mu_E \implies$ There must be a Laplacian around.

From the point of view of geometric analysis insight on the nature of Ricci curvature is provided by the expression of its components in local harmonic coordinates $(U, \{x^i\}; \triangle_g x^i = 0)$, (C. Lanczos, D. DeTurck, J. Kazdan, ...):

$$\mathcal{R}_{ik} =_{har} -\frac{1}{2}\triangle_{(g)}\left(f_{(ik)}\right) + Q_{ik}\left(g^{-1}, \partial g\right) \;, \quad f_{(ik)} := g_{ik} \;,$$

Hence in harmonic coordinates the Ricci curvature acts as a semi–linear elliptic operator on each scalar function $f_{(ik)} := g_{ik}$: the metric tensor components g_{ik} have maximal regularity in harmonic coordinates.



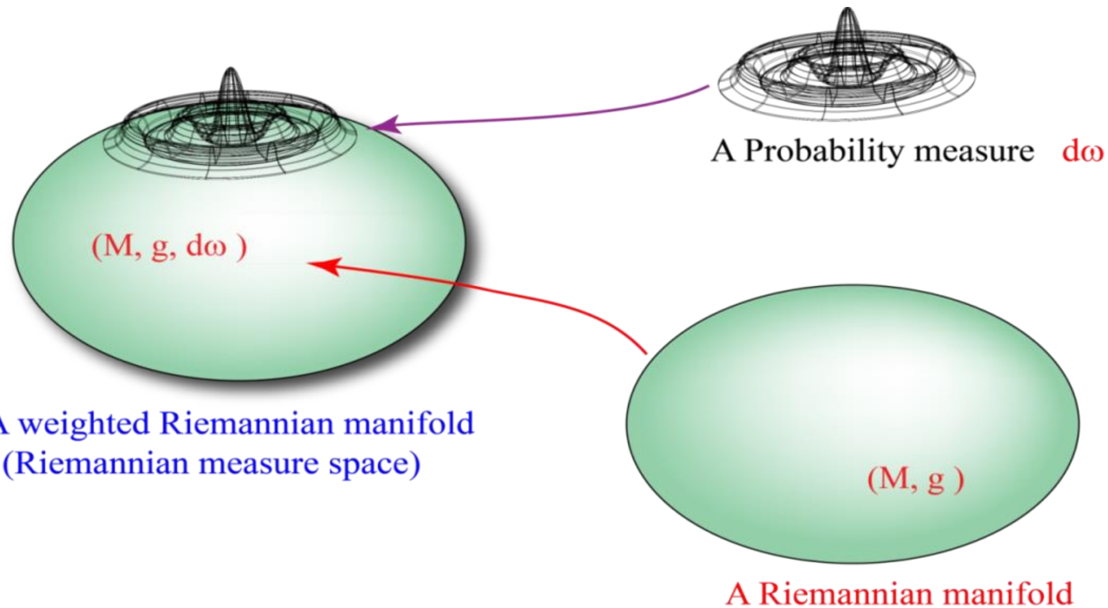
$\frac{1}{2}\triangle_{(g)}$: generator of the Brownian motion diffusion lurks in the background.

When passing from normal geodesic coordinates to harmonic coordinates we gain control on the components of the metric tensor in terms of the Ricci curvature rather than of the full Riemann tensor (DeTurck-Kazdan, Jost-Karcher)

Given the interplay between $Ric(g)$, $d\mu_g$ and diffusion, it is not surprising that the analysis of Ricci curvature often calls for weighted Riemannian manifolds (or Riemannian measure spaces) $(M, g, d\omega)$ *i.e.* smooth n–dim Riemannian manifolds endowed with a probability measure $d\omega$

The relevant operator on $(M, g, d\omega)$ are:
the $d\omega$–weighted divergence
$\nabla_{(\omega)}\circ := e^f\,\nabla\,\left(e^{-f}\circ\right)$

the $d\omega$–weighted Laplacian
$\Delta_g^{(\omega)}\,\psi := \left(\Delta_g - \nabla f\cdot\nabla\right)\,\psi$



- When $d\omega = e^{-f}d\mu_g$ the basic player is the Bakry–Emery Ricci curvature

$$Ric^{BE}(g, d\omega) := \mathrm{Ric}(g) + Hess_g\,f$$

The contracted Bianchi identity $\nabla^i\,\mathrm{R}_{ik} \doteq \frac{1}{2}\nabla_k\,\mathrm{R}$ is replaced by

$$\nabla^i_{(\omega)}\,\mathrm{R}^{BE}_{ik} \doteq \frac{1}{2}\,\nabla_k\,\mathrm{R}^{Per}\,,$$

$$\mathrm{R}^{Per}(g) := \mathrm{R}(g) + 2\,\Delta_g f - |\nabla f|_g^2 = \mathrm{R}(g) + 2\,\Delta_g^{(\omega)} f + |\nabla f|_g^2$$

$\Longrightarrow$ Perelman's modified scalar curvature.

In such a framework, the role of the Einstein-Hilbert functional is played by Perelman's energy

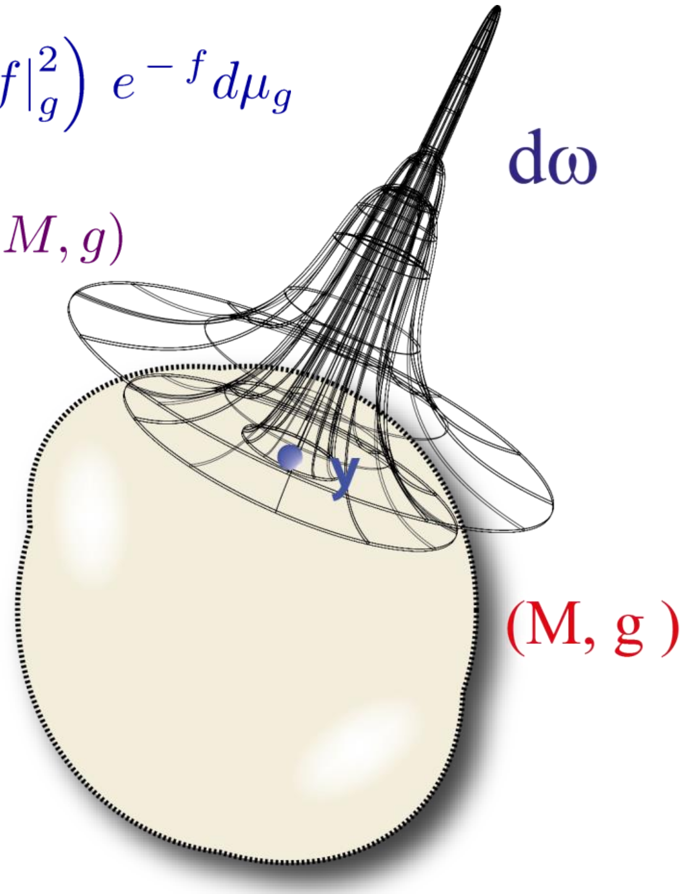
$$\mathcal{F}(g;f) \doteq \int_M \mathrm{R}^{Per}(g)\,d\omega = \int_M \left(\mathrm{R}(g) + |\nabla f|_g^2\right) e^{-f} d\mu_g$$

which induces a subtle geometric functional on (M,g)

$$\mathrm{F}[g] \doteq \inf_{\{f \in W^{1,2}(M),\, \int e^{-f} d\mu_g = 1\}} \int_M \mathrm{R}^{Per}(g)\,d\omega$$

the lowest eigenvalue $\lambda_1[g]$ of the Schrödinger-like operator on (M,g)

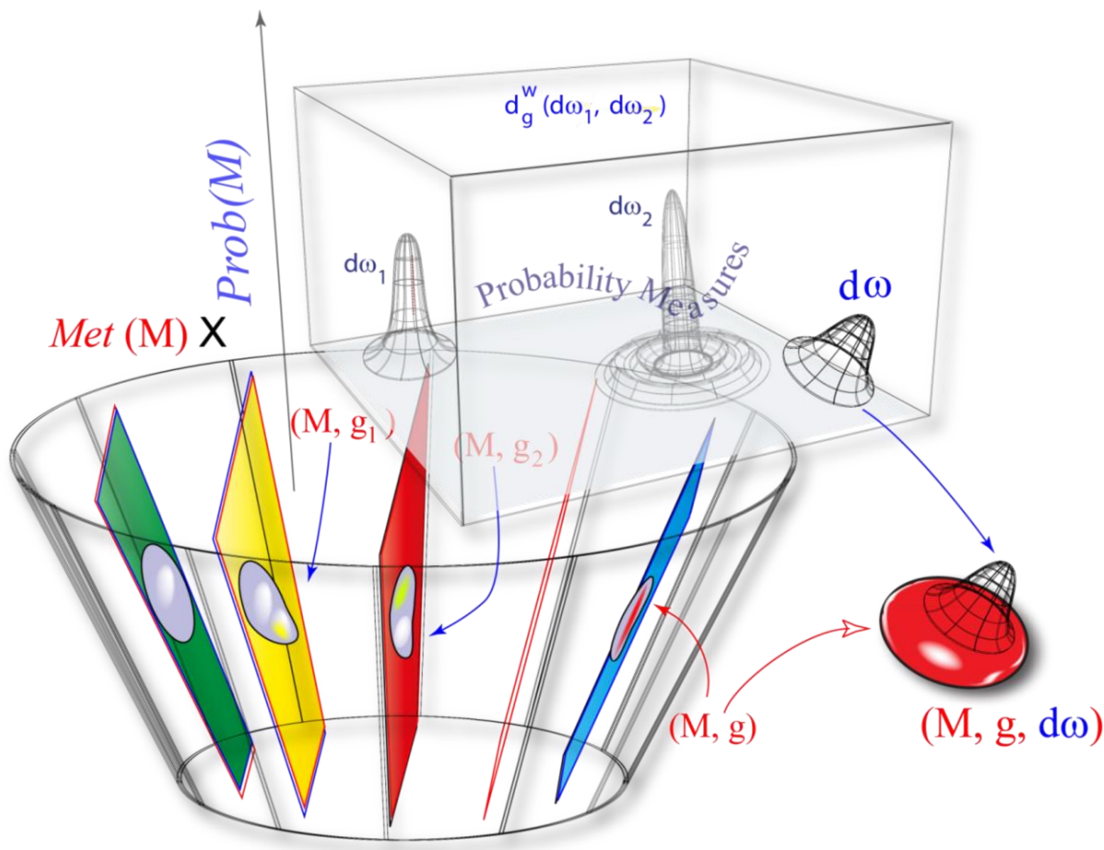
$$-4\,\Delta_g \psi + \mathrm{R}(g)\,\psi = \lambda_1[g]\,\psi, \quad \psi := e^{-f/2}$$



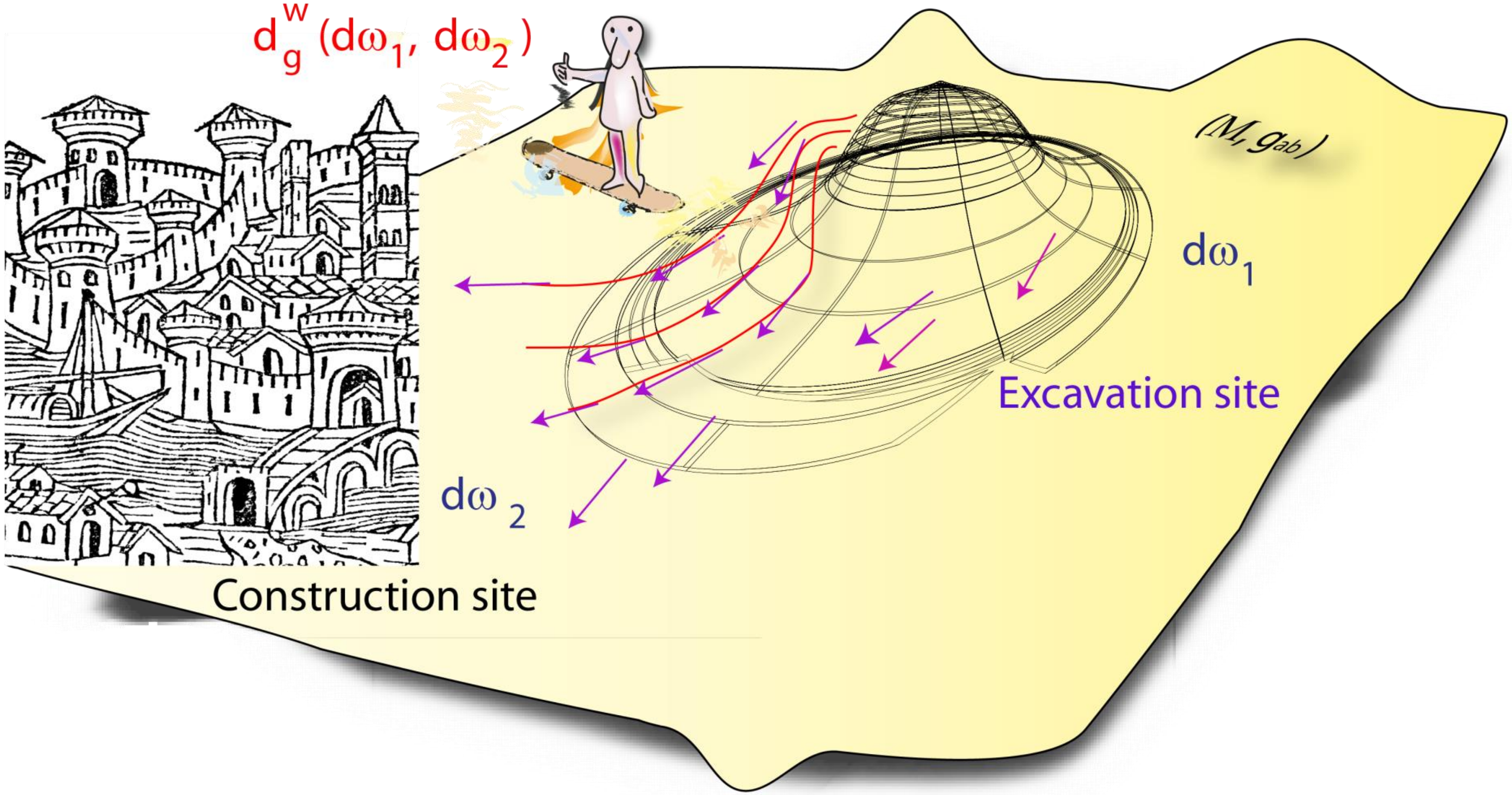
This suggests that the geometry of Ric(g) is more appropriately framed in the Space of weighted Riemannian manifolds $(M, g, d\omega)$, i.e. $\mathcal{M}et(M) \times [Prob(M), d_g^W]$, where $\mathcal{M}et(M)$ is the space of all smooth Riemannian metrics over M, and $Prob(M)$ denotes the space of probability measures $d\omega$ over M endowed with

- The quadratic Kantorovich - Rubinstein - Wasserstein distance $d_g^W(d\omega_1, d\omega_2)$.

Elementary aspects of this framework in discussing Ricci curvature follows from the interplay between scaling and Diff(M) equivariance in Riemannian geometry



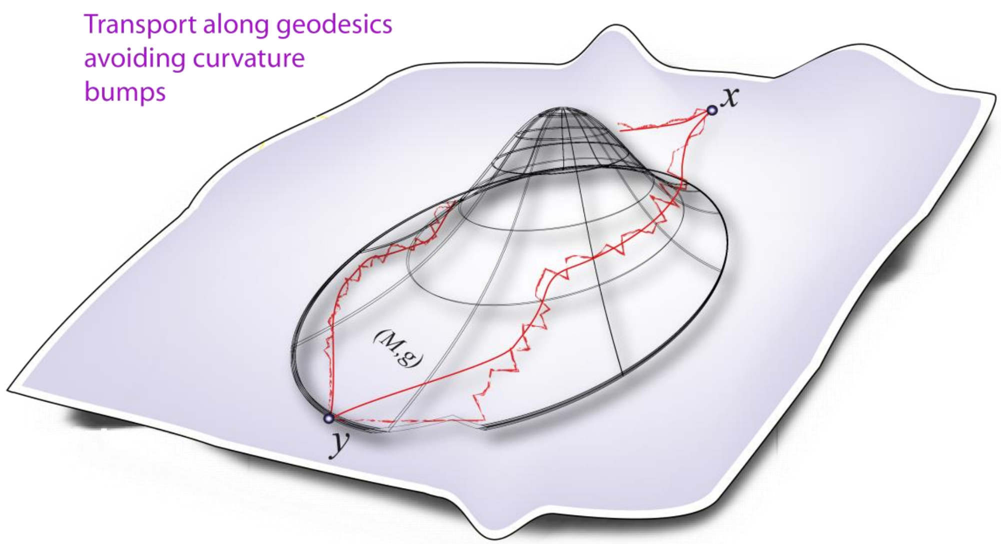
Appendix: The quadratic Wasserstein distance $d_g^W(d\omega_1, d\omega_2)$



The quadratic Wasserstein distance $d_g^W(d\omega_1, d\omega_2)$ plays a basic role in the Monge-Kantorovich problem of optimally transporting one distribution of mass $d\omega_1$, (say from an excavation site on the manifold (M, g)) onto another distribution $d\omega_2$, (realized at the construction site on (M, g)), where optimality is measured against a cost function provided by the squared Riemannian distance function $d_g^2(x, y)$.

$$d^W_{g,s}(d\varpi, d\theta) \doteq \inf_{\pi \in I(d\varpi, d\theta)} \left(\int_{M\times M} d_g(x,y)^s \pi(d\varpi d\theta) \right)^{\frac{1}{s}}$$

$I(d\varpi, d\theta) \subset Prob(M \times M)$: the set of probability measures on $M \times M$ with marginals $d\varpi$ and $d\theta$; ($I(d\varpi, d\theta)$: is called the set of couplings between $d\varpi$ and $d\theta$).



$d^W_{g,s}(d\varpi, d\theta)$ represents, as we consider all possible couplings between the measures $d\varpi$ and $d\theta$, the minimal cost needed to transport $d\varpi$ into $d\theta$ provided that the cost to transport the point x into the point y is given by $d_g(x,y)^s$. The distance $g^W_{g,s}(d\varpi, d\theta)$ metrizes $Prob(M)$ turning it into a geodesic space.

Besides diffeomorphisms, the metric g is naturally acted upon also by overall rescalings according to

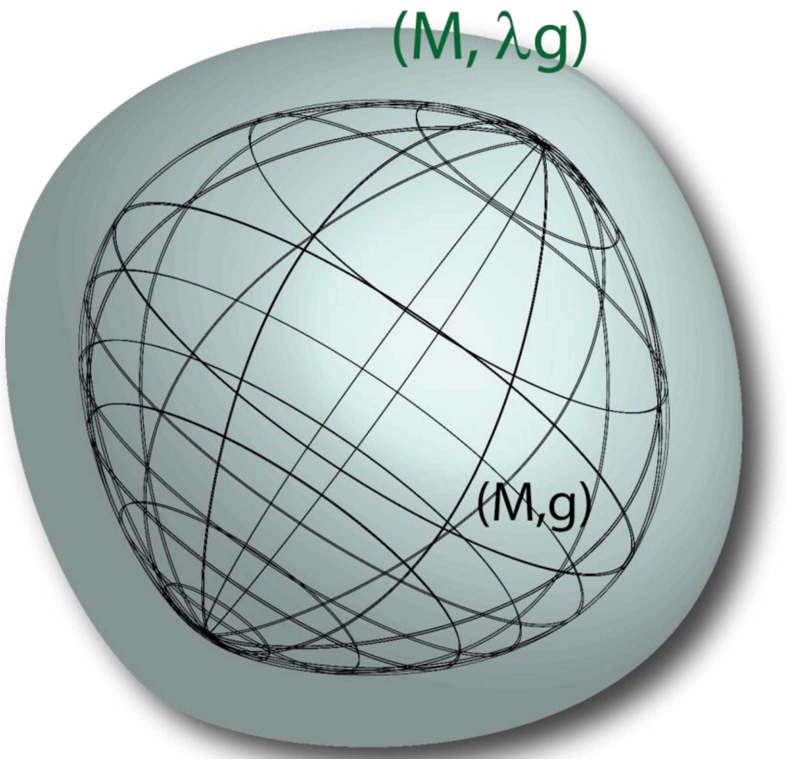
$$g \longmapsto \lambda g \,, \quad \forall \lambda \in R_{>0} \,,$$

(in local coordinates (U, x^i), $g_{ik} \longmapsto \lambda\, g_{ik}$ and $g^{ik} \longmapsto \lambda^{-1}\, g^{ik}$).

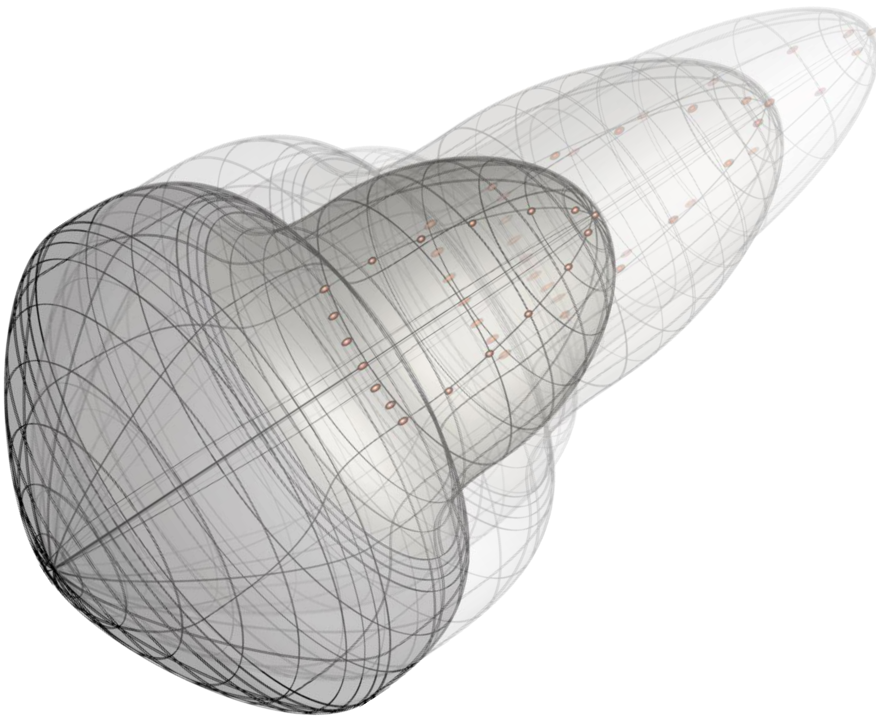
$$\begin{aligned} d_g(p,q) &\longmapsto & d_{\lambda g}(p,q) &= \lambda^{\frac{1}{2}}\, d_g(p,q) \\ Vol_g(\Sigma) &\longmapsto & Vol_{\lambda g}(\Sigma) &= \lambda^{\frac{n}{2}}\, Vol_g(\Sigma) \,, \end{aligned}$$

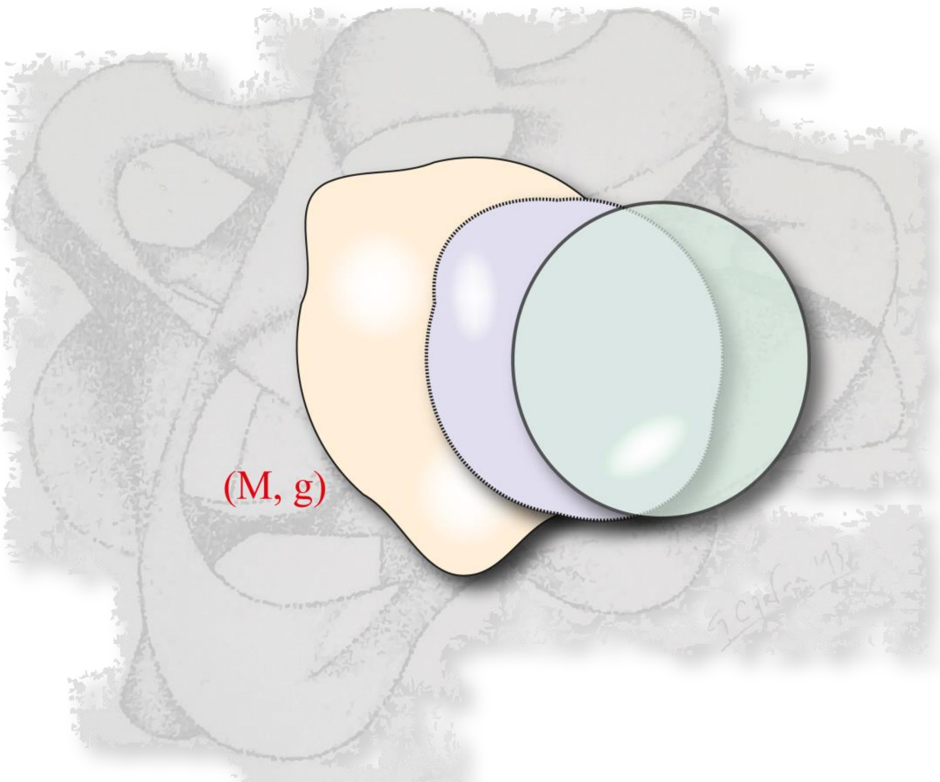
$$\nabla^{(\lambda g)} = \nabla^{(g)},$$

$$\begin{aligned} Hess_{(\lambda g)} &= Hess_{(g)} \,, \\ \triangle_{(\lambda g)} &= \lambda^{-1}\, \triangle_{(g)} \,. \\ \mathcal{R}m(\lambda g) &= \mathcal{R}m(g) \\ \mathcal{S}ec(\lambda g)(X,Y) &= \lambda^{-1}\, \mathcal{S}ec(g)(X,Y) \\ \mathcal{R}ic(\lambda g) &= \mathcal{R}ic(g) \\ \mathcal{R}(\lambda g) &= \lambda^{-1}\, \mathcal{R}(g) \,, \end{aligned}$$



- These scalings relations imply that not only Einstein, but also Quasi-Einstein Metrics do matter in Riemannian geometry.





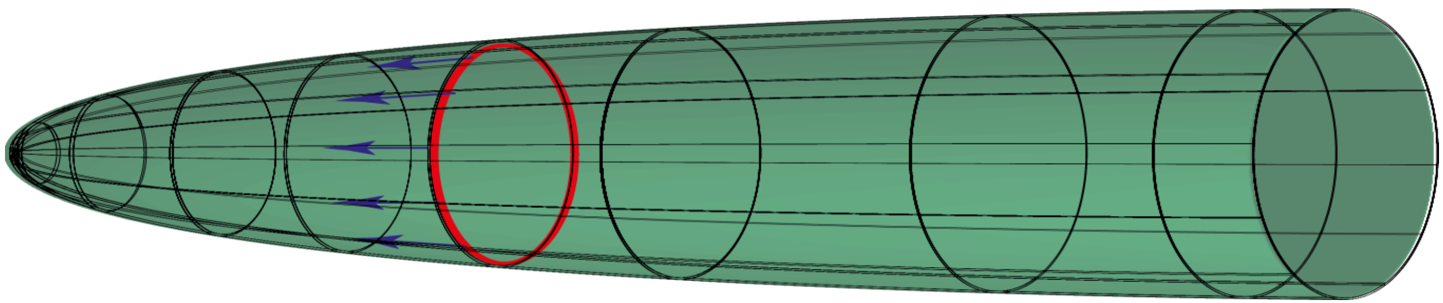
- A Riemannian metric g is Einstein if its Ricci tensor $\mathcal{R}ic(g) = \rho_{(g)}\, g$ for some constant $\rho_{(g)}$.

- The Einstein constant $\rho_{(g)}$ scales non–trivially: Since $\mathcal{R}ic(g)$ is scale–invariant, we must have $\rho_{(\lambda g)} \longmapsto \lambda^{-1}\, \rho_{(g)}$.

Quasi–Einstein metrics are characterized by a Ricci tensor which can be written as

$$\mathcal{R}ic(g) \, = \, \rho_{(g)} \, g \, - \, \frac{1}{2} \, \mathcal{L}_{V_{(g)}} \, g \, = \, \rho_{(g)} \, g \, - \, \frac{1}{2} \, (\nabla_i V_k \, + \, \nabla_k V_i) \ ,$$

for some constant $\rho_{(g)}$ and some complete vector field $V_{(g)} \in C^\infty(M, TM)$.



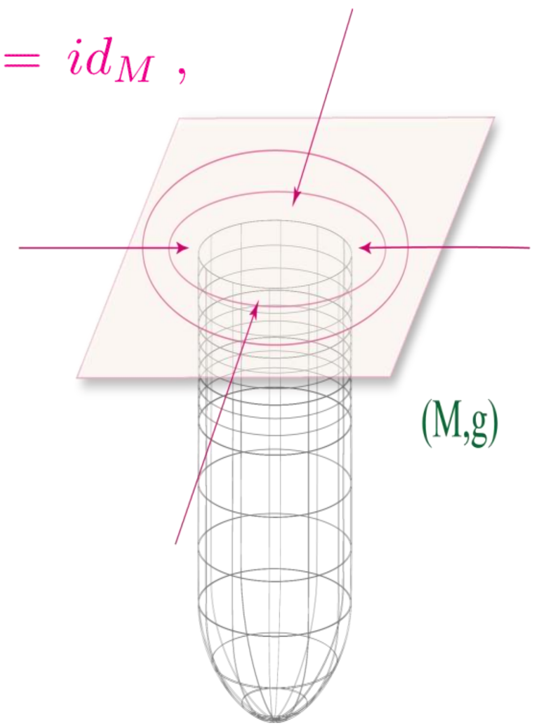
- If V is a gradient, $V^i = g^{ik} \partial_k f$ for some $f \in C^\infty(M, \mathbb{R})$, then the quasi –Einstein condition becomes
- $Ric^{B-E}(g, d\omega) := \mathcal{R}ic(g) \, + \, Hess_g \, f \, = \, \rho_{(g)} \, g$
- *i.e.* the isotropy of the Bakry–Emery Ricci curvature of the Riemannian manifold with density $(M, g, d\omega := e^{-f} d\mu_g))$.

- Quasi–Einstein metrics have a significant dynamical characterization

- For $0 \leq \beta < \epsilon < \frac{1}{2\rho_{(g)}}$, define $\lambda(\beta) := (1 - 2\rho_{(g)}\,\beta)$.

- Consider the one–parameter family of diffeomorphisms $\phi_\beta : M \longrightarrow M$ solution of the non–autonomous ordinary differential equation

$$\frac{\partial}{\partial\beta}\phi_\beta(p) = \frac{1}{\lambda(\beta)}V_{(g)}(\phi_\beta(p)), \quad \phi_{\beta=0} = id_M \;,$$



- and the one–parameter family of metrics defined by

$$g(\beta) := \lambda(\beta)\,\phi^*_\beta g \;.$$

with $g(\beta = 0) = g$.

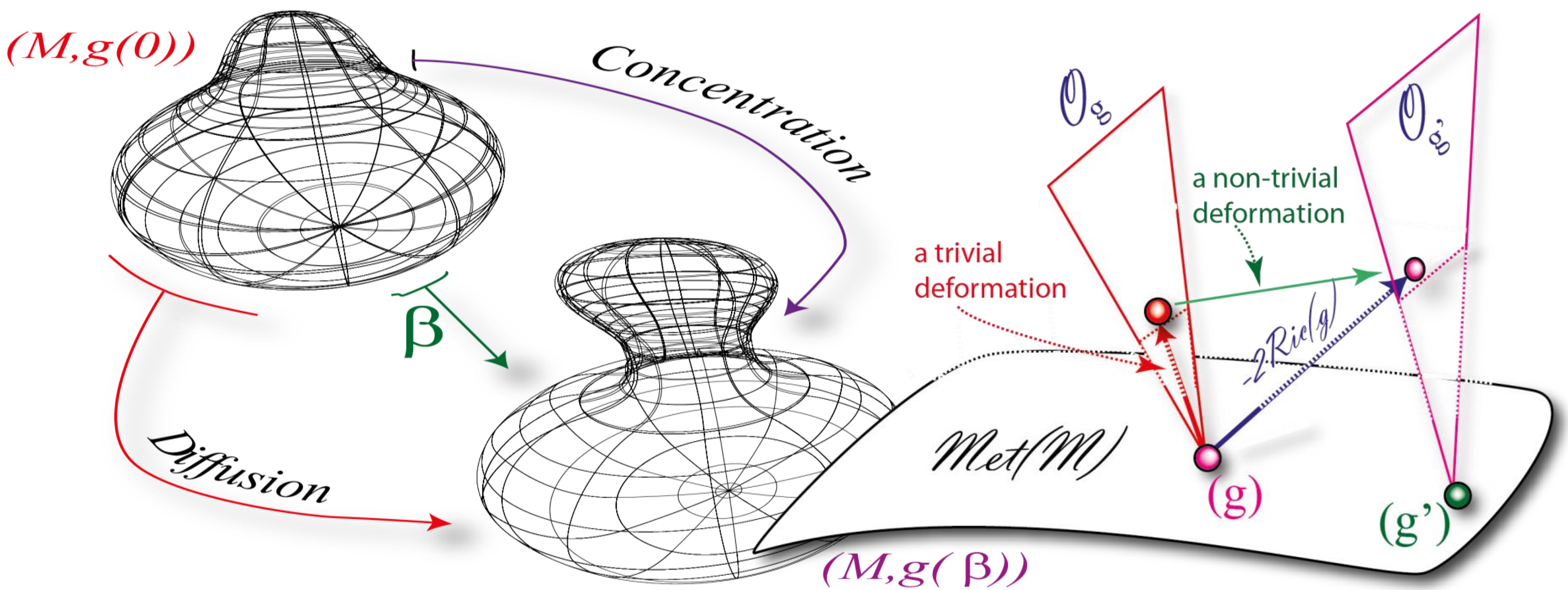
- By scale invariance and $\mathcal{D}iff(M)$–equivariance we compute

$$\frac{\partial}{\partial\beta}g(\beta) = -2\,\rho_{(g(\beta))}\,g(\beta) + \mathcal{L}_{V_{(g(\beta))}}\,g(\beta) = -2\,\mathrm{R}ic(g(\beta)) \;.$$

Hence, under the combined action of this family of diffeomorphisms and of the scaling, the quasi–Einstein metric g generates a self–similar solution $g(\beta) := \lambda(\beta)\,\phi^*_\beta\, g,\ 0 \leq \beta < \epsilon,$ of the *Ricci flow* , (R. Hamilton, 1982)

$$\begin{aligned} &\frac{\partial}{\partial\beta} g_{ab}(\beta) = -2\,\mathrm{R}_{ab}(\beta), \\ &g_{ab}(\beta = 0) = g_{ab} \quad , \quad 0 \leq \beta < \frac{1}{2\rho_{(g)}}\,. \end{aligned} \tag{1}$$

These solutions are known as *Ricci solitons*, (R. Hamilton, 1988).



Appendix: Ricci Flow as dynamical system on $\mathcal{M}et(M)$ ...

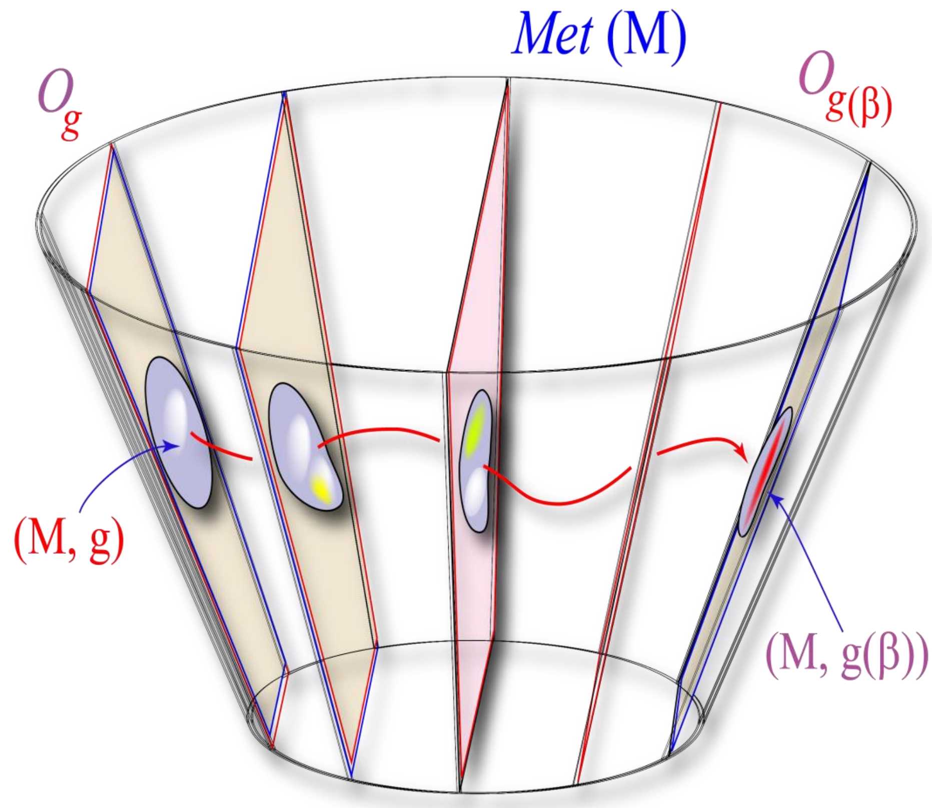
The Ricci flow can be thought of as a (weakly–parabolic) dynamical system on $\mathcal{M}et(M)$.

$$\begin{array}{ccc} \mathcal{M}et(M) & \longrightarrow & \mathcal{M}et(M) \\ (M,g) & \mapsto & (M,g(\beta))\ , \end{array}$$

defined by deforming the metric (M,g) in the direction of $-2\mathcal{R}ic(g)$ thought of as a (non–trivial) vector in $T_g\,\mathcal{M}et(M)$, *i.e.*,

$$\begin{cases} \dfrac{\partial}{\partial\beta} g_{ab}(\beta) = -2R_{ab}(\beta), \\ \\ g_{ab}(\beta = 0) = g_{ab}, \end{cases}$$

$$0 \leq \beta \leq \beta^* < T_0$$

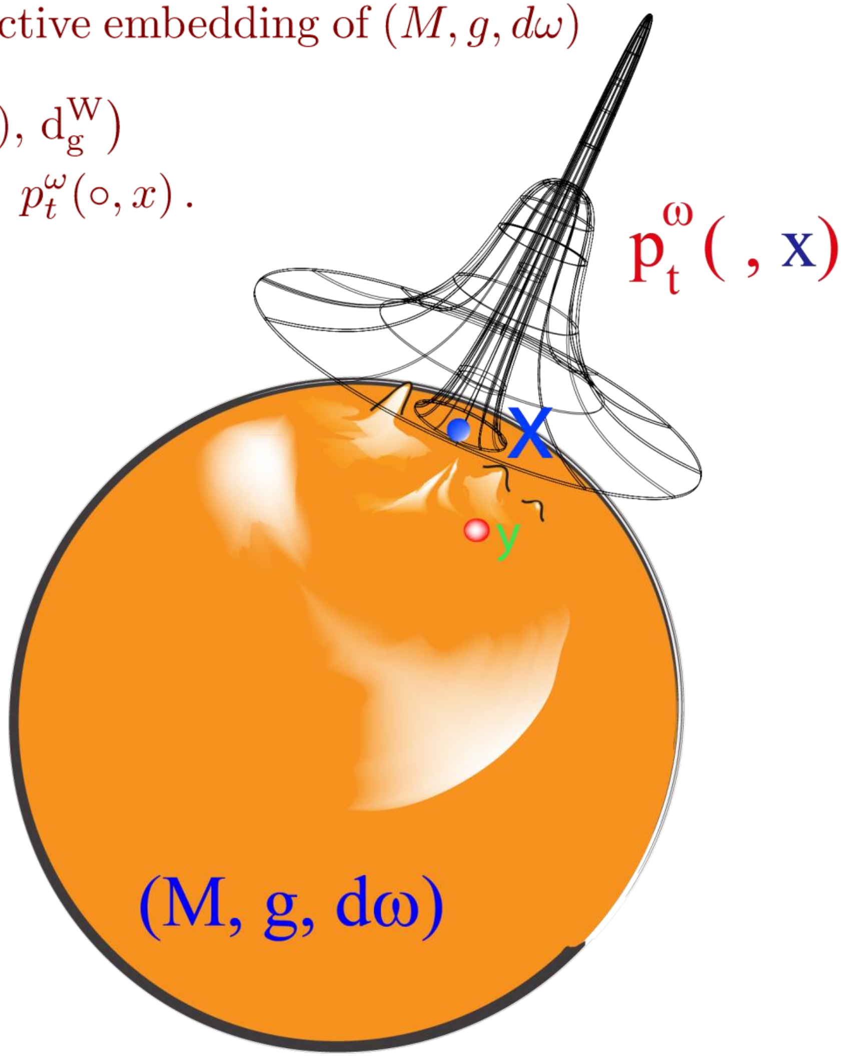


A deeper rationale ...

... underlying the connection among Ricci curvature, Ricci flow and weighted Riemannian manifolds, suggested by scaling and $Diff(M)$-equivariance, follows by observing that we can use the (weighted) heat kernel of $(M, g, d\omega)$, $(t, \delta_x) \longmapsto p_t^\omega(\circ, x)$, with source at $x \in M$, to generate an injective embedding of $(M, g, d\omega)$

$$\begin{aligned} \iota_{p_t^\omega} : (M, g) &\hookrightarrow \left(\mathrm{Prob}\,(\mathrm{M}),\, \mathrm{d}_{\mathrm{g}}^{\mathrm{W}}\right) \\ x &\longmapsto \iota_{p_t^\omega}(x) := p_t^\omega(\circ, x)\,. \end{aligned}$$

in the space $\left(\mathrm{Prob}(M),\ d_g^W\right)$ of all probability measures over M endowed with the quadratic Wasserstein distance d_g^W, (N.Gigli–C. Mantegazza, for the pure Riemannian case, M.C. for the general case of $(M, g, d\omega)$ and the relation with Renorm group for NLσM).

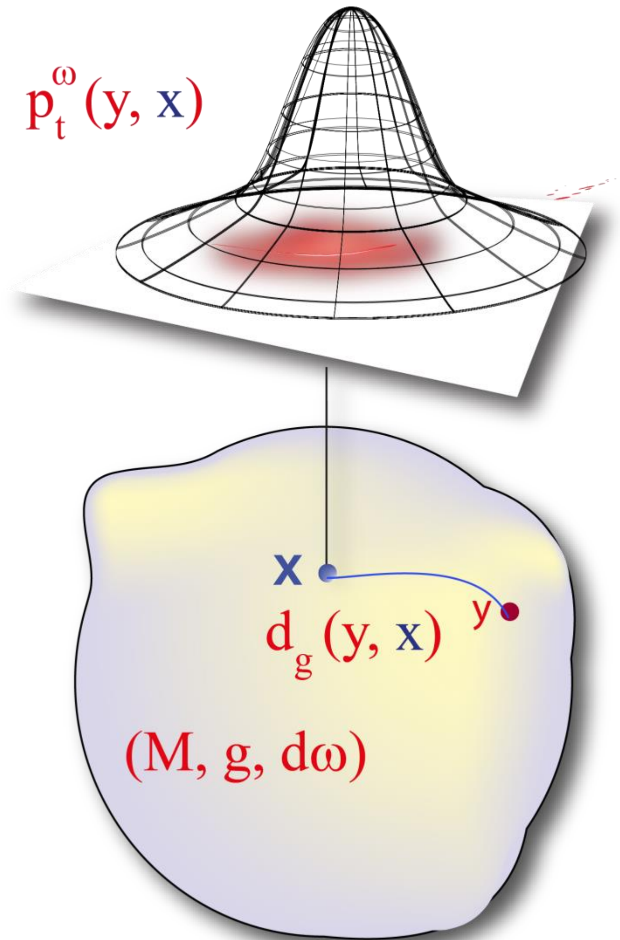


Let x vary in (M, g) and consider the (minimal positive) weighted heat kernel $p_t^\omega(y, x)$, centered at x, solution of

$$\left(\frac{\partial}{\partial t} - \triangle^\omega_{(g,y)}\right) p_t^\omega(y, x) = 0 \ ,$$

$$\lim_{t \searrow 0^+} \ p_t^\omega(y, x) \ = \ \delta_x(y) \ ,$$

where $\triangle^\omega_{(g,y)} := \triangle_{(g,y)} - \nabla_{(y)} f \cdot \nabla_{(y)}$ denotes the weighted Laplace–Beltrami operator on $(M, g, d\omega)$.



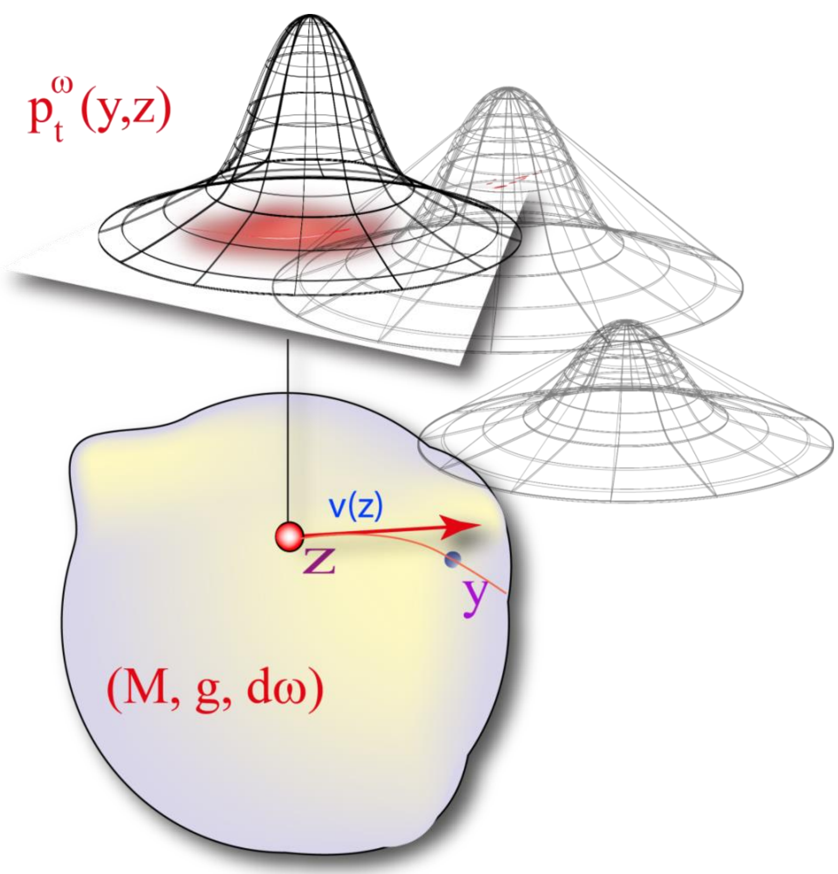
Varadhan's large deviation formula,

$$-\lim_{t \searrow 0^+} \ t \ \ln \left[p_t^{(\omega)}(y, z)\right] \ = \ \frac{d_g^2(y,z)}{4}$$

For any smooth vector field v over M, and $t > 0$, there exists a unique smooth solution $\widehat{\psi}_{(t,z,v)}$, smoothly depending on the data t, z, v, of the elliptic PDE

$$\nabla^i_{(y)} \left(p^\omega_t(y,z)\, \nabla^{(y)}_i \, \widehat{\psi}_{(t,z,v)}(y) \right) = -\, v^i(z)\, \nabla^{(z)}_i \, p^\omega_t(y,z) \; ,$$

and with $\nabla^{(y)}_i \, \widehat{\psi}_{(t,z,v)}(y) \not\equiv 0$ for all $v \neq 0$. (F.Otto's parametrization associated with the heat kernel (here based ate the generic point z))



$$\nabla^{(y)} \, \widehat{\psi}_{(t,z,v)}(y)$$

$$T_z(M) \rightarrow \mathcal{T}_{p^\omega_t} Prob(M)$$

$$v(z) \rightarrow \nabla^{(y)}_i \, \widehat{\psi}_{(t,z,v)}(y)$$

By exploiting this heat kernel parametrization of the vector fields of (M, g), and by *pulling–back* via the injective embedding

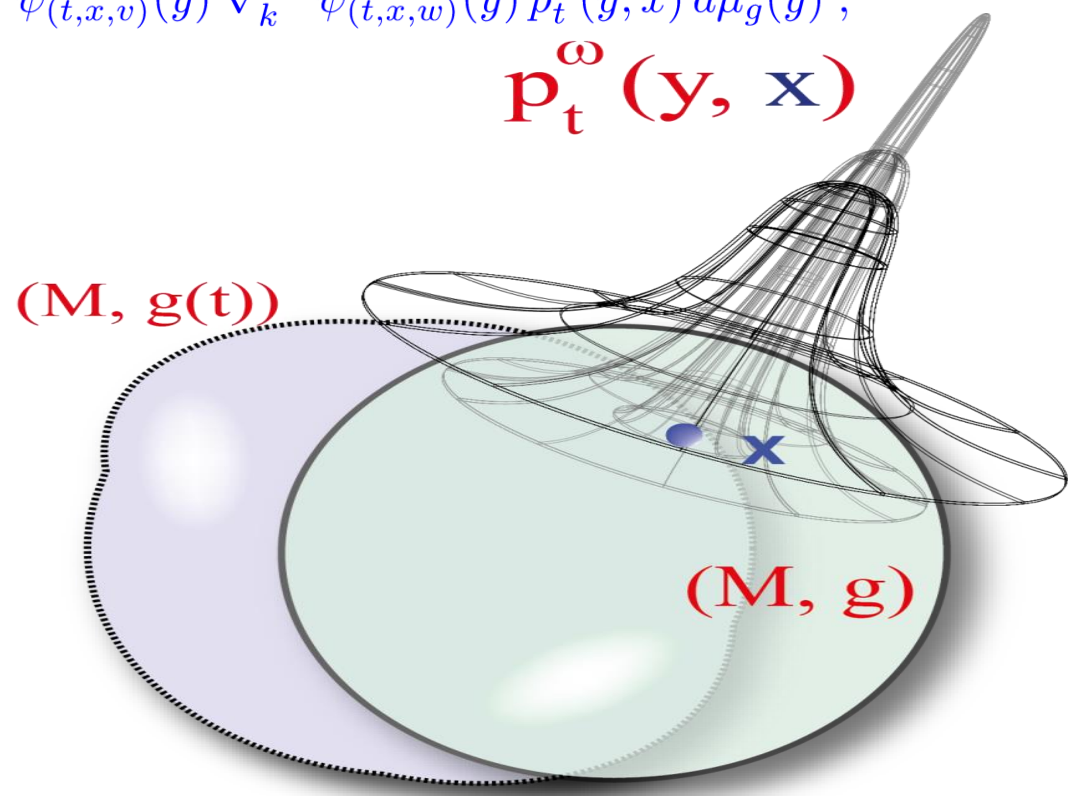
$$\begin{aligned} \iota_{p_t^\omega} \,:\, (M, g) \quad &\hookrightarrow \quad \left(\mathrm{Prob}\,(\mathrm{M}),\, \mathrm{d}_{\mathrm{g}}^{\mathrm{W}}\right) \\ x \quad &\longmapsto \quad \iota_{p_t^\omega}(x) \,:=\, p_t^\omega(\circ, x) \end{aligned}$$

the Wasserstein distance $d_g^W\,(p_t^\omega(\circ, x),\; p_t^\omega(\circ, y))$ to M one defines, for all $t \in (0, \infty)$, a t–dependent metric tensor g_t on M according to

$$g_t\,(v(x), w(x)) \,:=\, \int_{M_y} g^{ik}(y)\, \nabla_i^{(y)}\, \widehat{\psi}_{(t,x,v)}(y)\, \nabla_k^{(y)}\, \widehat{\psi}_{(t,x,w)}(y)\, p_t^\omega(y, x)\, d\mu_g(y)\;,$$

(Gigli-Mantegazza in the pure Riemannian case, m.c. for the general case). There is a similar time evolution for the measure $\omega \longmapsto \omega(t)$ induced by the constraint

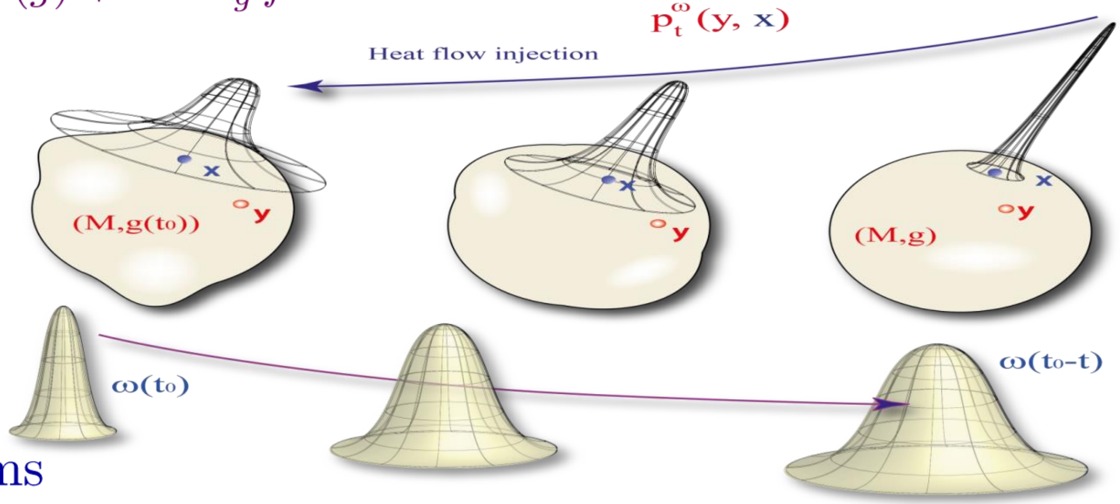
$\int_M e^{-f(t)} d\mu_{g(t)} = 1.$



This is the scale dependent metric induced on M by the heat kernel immersion. As $t \searrow 0$, the metric g_t reduces to g, *i.e.* $\lim_{t\searrow 0} g_t(v,v) = g(v,v)$, $v \in T_y M$, $y \in M$, and

$$g_t(v,v) \leq e^{-2K_g^{B-E}t} g(v,v) \, ,$$

where K_g^{B-E} denotes the lower bound of the Bakry–Emery Ricci curvature of $(M, g, d\omega))$, $Ric^{B-E}(g, d\omega) := Ric(g) + Hess_g f$.



A rather sophisticated use of the (weak) Riemannian geometry of the Wasserstein space $\left(Prob(M,g),\, d_g^W\right)$, (related to the Riemannian geometry of the diffeomorphisms group $\mathcal{D}iff(M)$ *a' la Arnold*, *e.g.* A. F. Solov'ev, *Curvature of a distribution*, Matematicheskie Zametki, **35** (1984) 111-124; N. K. Smolentsev, *Curvatures of the diffeomorphism group and the space of volume elements*, Sibirskii Matematicheskii Zhurnal, **33** (1992) 135-141; A. M. Lukatsky, *On the curvature of the diffeomorphisms group*, Annals of Global Analysis and Geom. **11** (1993) 135-140; B. Khesin, J. Lenells, G. Misiolek, and S. C. Preston, *Geometry of diffeomorphisms groups, complete integrability and geometric statistics*, arXiv: 1105.0643v1)...

allows us to get full–fledged flows for the metric $t \longmapsto g(t)$ and for the measure field $t \longmapsto f(t)$:

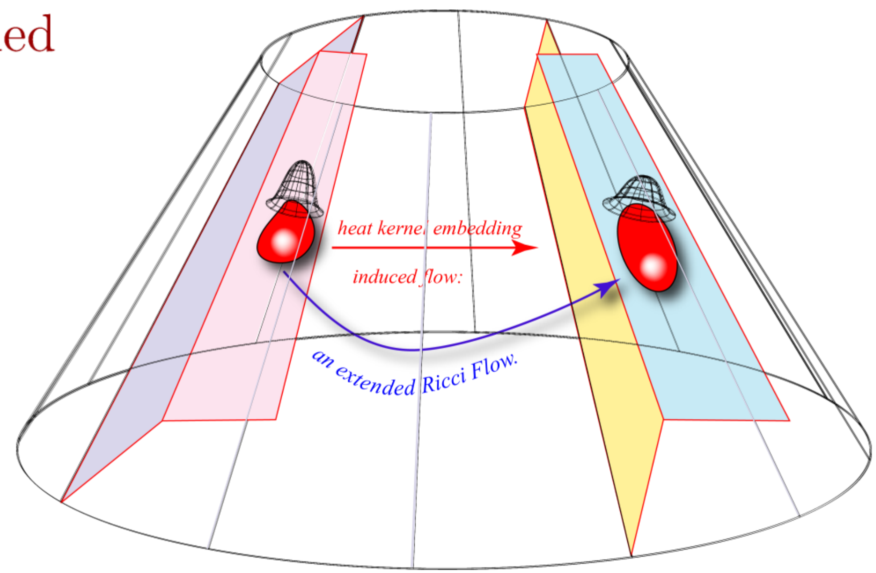
$$\frac{\partial}{\partial t} f = - \triangle_{\omega}^{(z)} f$$

$$\frac{\partial}{\partial t} g_t(u,w) = -2\,Ric^{(t)}(u,w) - 2\,Hess\,f\,(u,v)$$

$$-2 \int_M \left(Hess\,\widehat{\psi}_{(t,u)} \cdot Hess\,\widehat{\psi}_{(t,w)} \right) p_t^{(\omega)}(y,z)\,d\omega(y)\ ,$$

where $Ric^{(t)}$ denotes the Ricci curvature of the evolving metric (M, g_t), and where $\widehat{\psi}_{(t,u)}$, $\widehat{\psi}_{(t,W)}$ are the *tangent vectors* in Prob(M) representing the manifold tangent vectors u and w, respectively.

Hence we get an extended Ricci flow coupled with a (backward) parabolic evolution for the measure $d\omega = e^{-f}\,d\mu_g$.

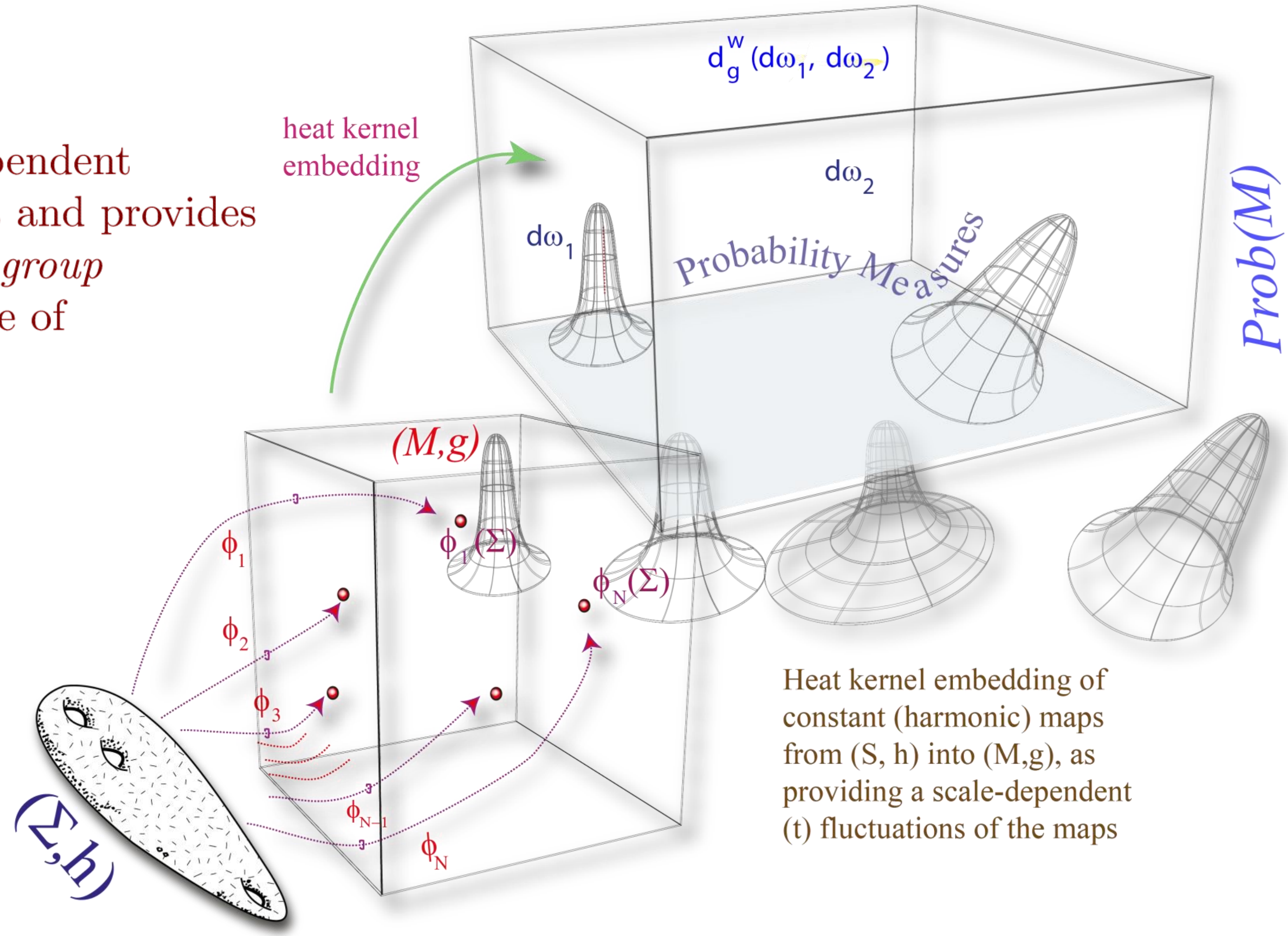


M.C. *The Wasserstein geometry of nonlinear σ models and the Hamilton - Perelman Ricci flow*, Reviews in Mathematical Physics Vol. 29, No. 1 (2017) 1750001

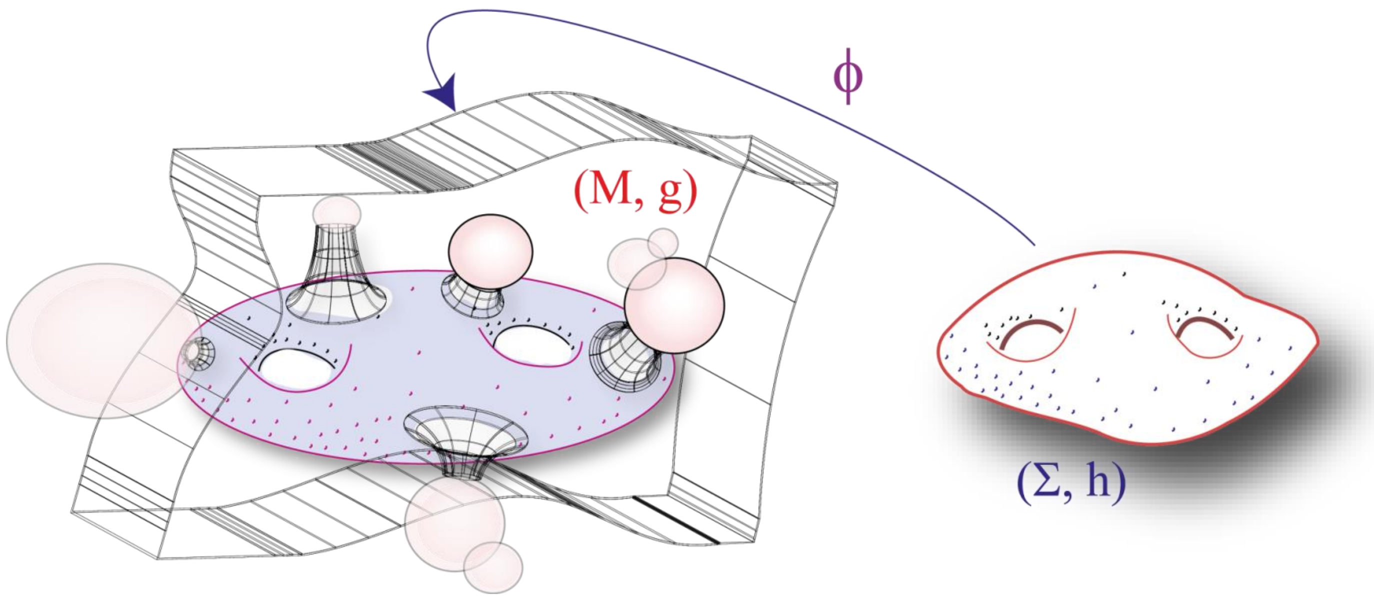
A QFT perspective (slightly expanded w.r.t. the talk)

These flows generated by the Wasserstein space construction can be reinterpreted in terms of the (perturbative) analysis of the Renormalization Group for (Dilatonic) Non-Linear σ Model (NLσM), the quantum field theory avatar of harmonic maps from a Riemann surface (Σ, h) into the weighted Riemannian manifold $(M, g, d\omega)$.

This describes scale–dependent fluctuations of the maps and provides a *QFT renormalization group* perspective of the nature of Ricci curvature.



The dilatonic NLσM action



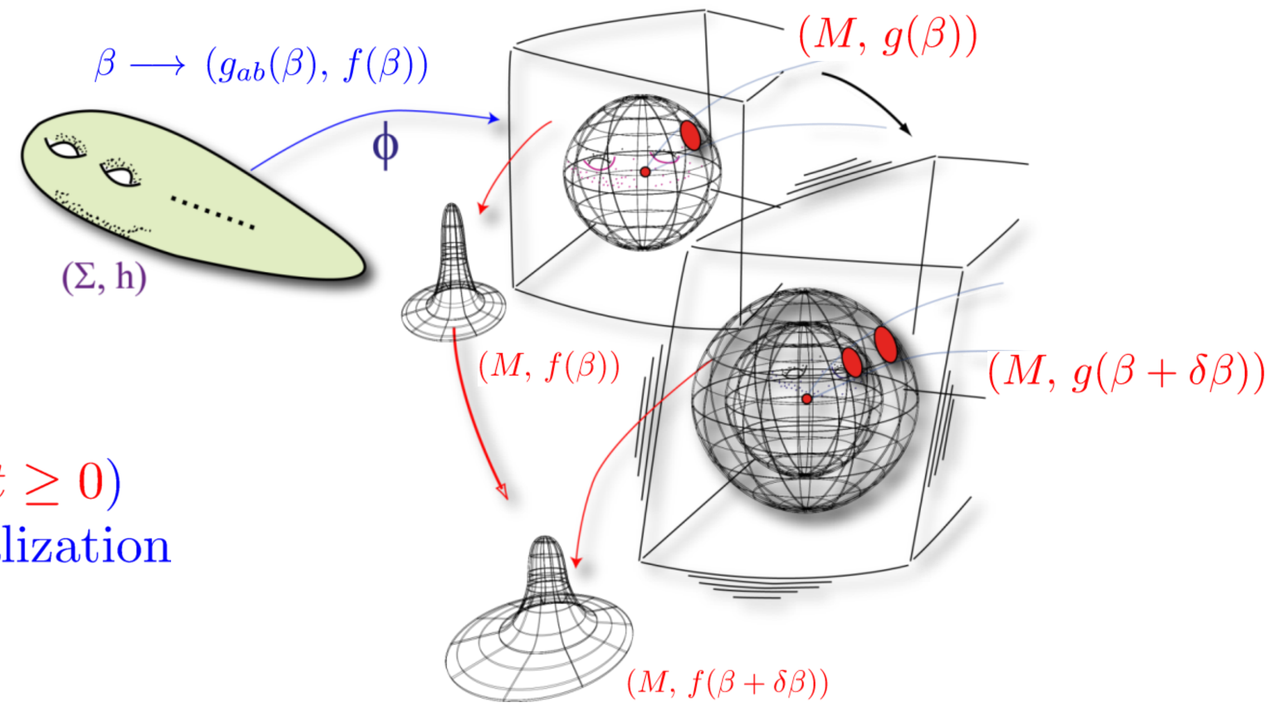
$$S[\phi;\ a,\ f,\ g] := \frac{1}{2a}\int_\Sigma d^2x\,\sqrt{h}\ \left[h^{\mu\nu}\partial_\mu\phi^i\partial_\nu\phi^k\ \ g_{ik} + 2a\,\mathcal{K}_h(x)\,f(\phi(x))\right]$$

$\mathcal{K}_h$: Gauss curvature of the surface (Σ, h)

$a^{-1}\,g$ ... the metric coupling

$f : M \longrightarrow \mathbb{R}$ ... the dilaton coupling

Quantum (actually, random) fluctuations of $\phi : \Sigma \longrightarrow M$ around a classical configuration ϕ_{cm}, (a center of mass of a large collection ($\rightarrow \infty$) of constant maps $\{\phi_{(i)}\}$, i.i.d. distributed with respect to a sampling functional measure), can modify the geometry of (M, g, ω)



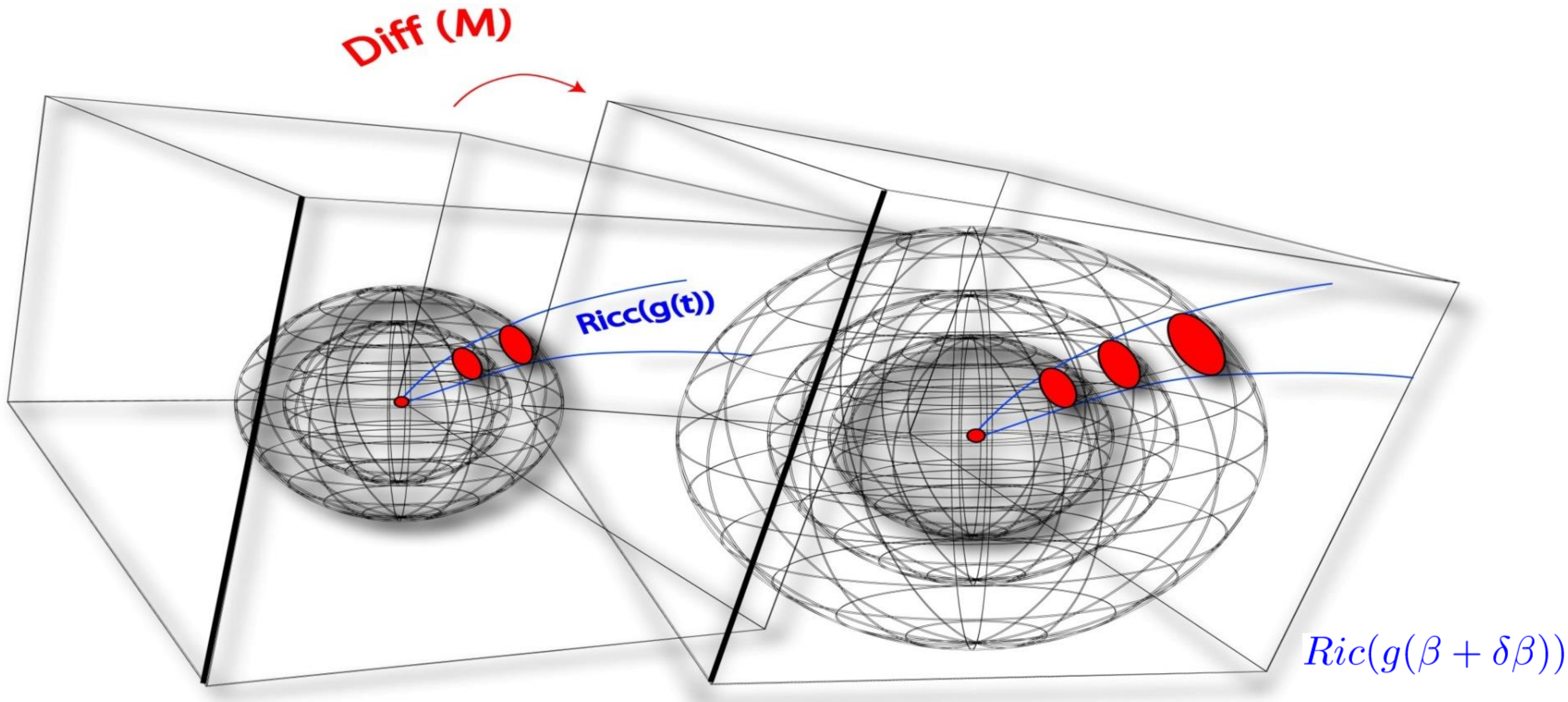
by exploiting a scale dependent ($\beta := a\, t, t \geq 0$) perturbative renormalization

we get a *RG-Flow* controlled by a large deviation mechanism w.r.t. the Gaussian fluctuations around the (classical) background ϕ_{cm}, (*i.e.* by the control of the exponential decline of large field fluctuations, around ϕ_{cm}, as the energy ($= length^2$) scale β varies).

- This procedure (re)constructs perturbatively the geometry in a ball around ϕ_{cm} as a function of the parameter β according to

$$\frac{\partial}{\partial\beta}\, g_{ik}(\beta) = -2\, R_{ik}(\beta) - 2\nabla_i\nabla_k f - \frac{a}{2}\left(R_{ilmn}R_k^{lmn}\right) + \mathcal{O}(a^2)$$

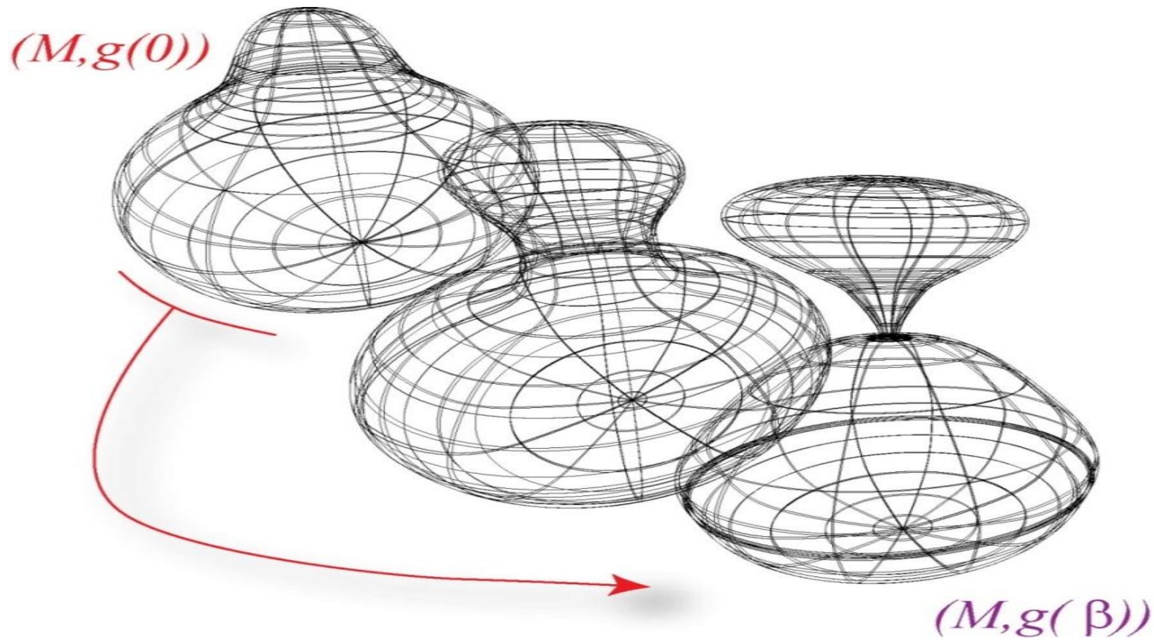
$$\frac{\partial}{\partial\beta}\, f(\beta) = \Delta f(\beta) - |\nabla f(\beta)|^2 + \mathcal{O}(a^2).$$



- As long as we are in the weak coupling regime, $a\,|\mathcal{R}m(g(\beta))|^{1/2} << 1$, we have the connection with Ricci flow in the DeTurck version:

$$\begin{cases} \frac{\partial}{\partial\beta} g_{ab}(\beta) = -2R_{ab}(\beta) - 2\nabla_a\nabla_b\, f, \\ \\ g_{ab}(\beta=0) = g_{ab}, \quad 0 \leq \beta \leq \beta^* < T_0 \end{cases} \qquad \begin{cases} \frac{\partial}{\partial\beta} f(\beta) = \Delta^{(\omega)}_{g(\beta)} f \\ \\ f(\beta=0) = f_0. \end{cases}$$

Rigorous AQFT approach possible (M.C., C. Dappiaggi, N. Drago, P. Rinaldi, CMP vol. 374, p. 241-276, (2020) arXiv:1809.07652)



However, a full-fledged analysis of these (weakly) parabolic PDEs requires a change of pespective in the role of the dilaton coupling f.

- We need to impose the *Perelman Coupling*: *viz.* we need to conjugate the dilaton f rescaling to the β–evolving Riemannian measure $d\mu_{g(\beta)}$, by replacing $f \longrightarrow \widetilde{f}$, with

$$\frac{\partial}{\partial\beta} e^{-\widetilde{f}(\beta)} d\mu_{g(\beta)} = 0, \implies \int_M e^{-\widetilde{f}(\beta)} d\mu_{g(\beta)} = const.$$

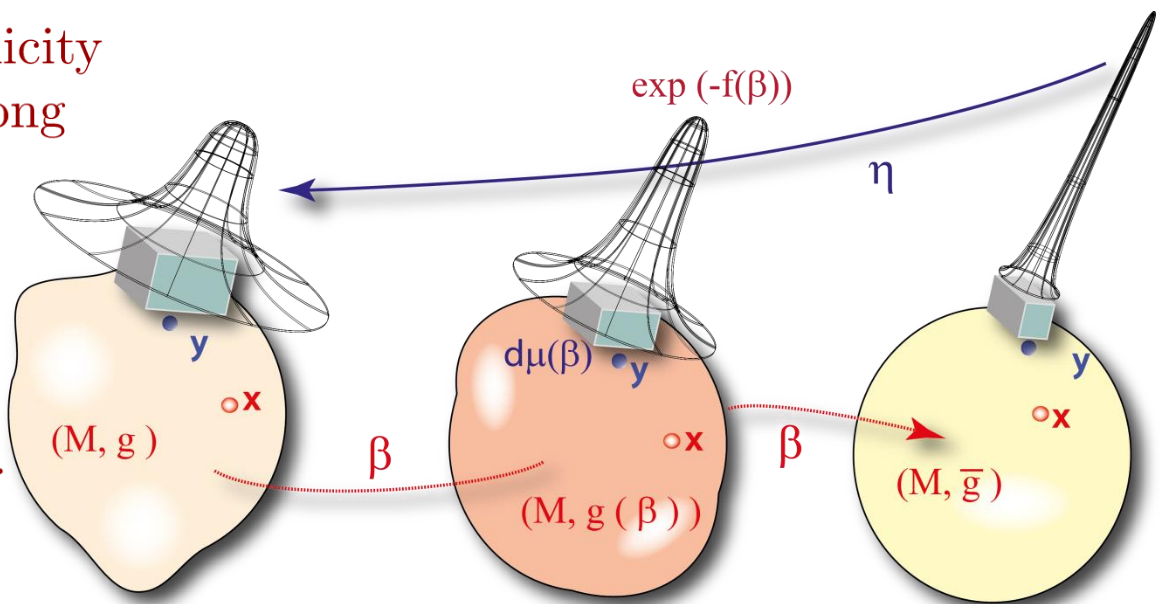
$$\begin{cases} \frac{\partial}{\partial\beta} g_{ab}(\beta) = -2R_{ab}(\beta) - 2\nabla_a\nabla_b \widetilde{f}, \\ \\ g_{ab}(\beta = 0) = g_{ab}, \end{cases} \qquad \begin{cases} \frac{\partial}{\partial\eta} \widetilde{f}(\eta) = \Delta_{g(\eta)} \widetilde{f}(\eta) - R(\eta)\, \widetilde{f}(\eta), \\ \\ \widetilde{f}(\eta = 0) = \widetilde{f}_0. \end{cases}$$

$$\eta \doteq \beta^* - \beta$$

In this case we have monotonicity of Perelman's $F[g]$–energy along the flow (*i.e.* we have an entropic functional at work

$$\mathrm{F}[g] \doteq \inf \int_M \mathrm{R}^{Per}(g)\, d\omega$$

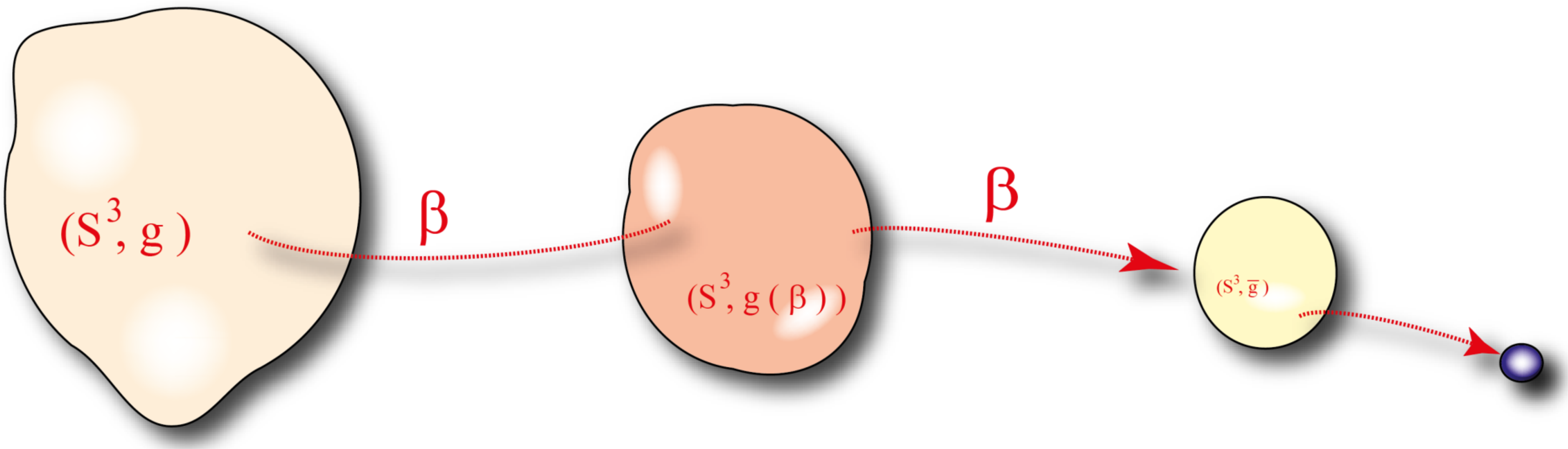
and the flow is gradient w.r.t. the $\mathcal{F}(g, f)$-energy



... This change of perspective is not confined to the math analysis of the one-loop contribution (Ricci flow) ...

- ... but also to the higher order terms such as $a\,(R_{ilmn}R_k^{lmn})$, *i.e.* if we consider the two-loop contribution to the RG flow

$$\frac{\partial}{\partial\beta}\,g_{ik}(\beta) = -2\,R_{ik}(\beta) - \frac{a}{2}\,(R_{ilmn}R_k^{lmn})$$



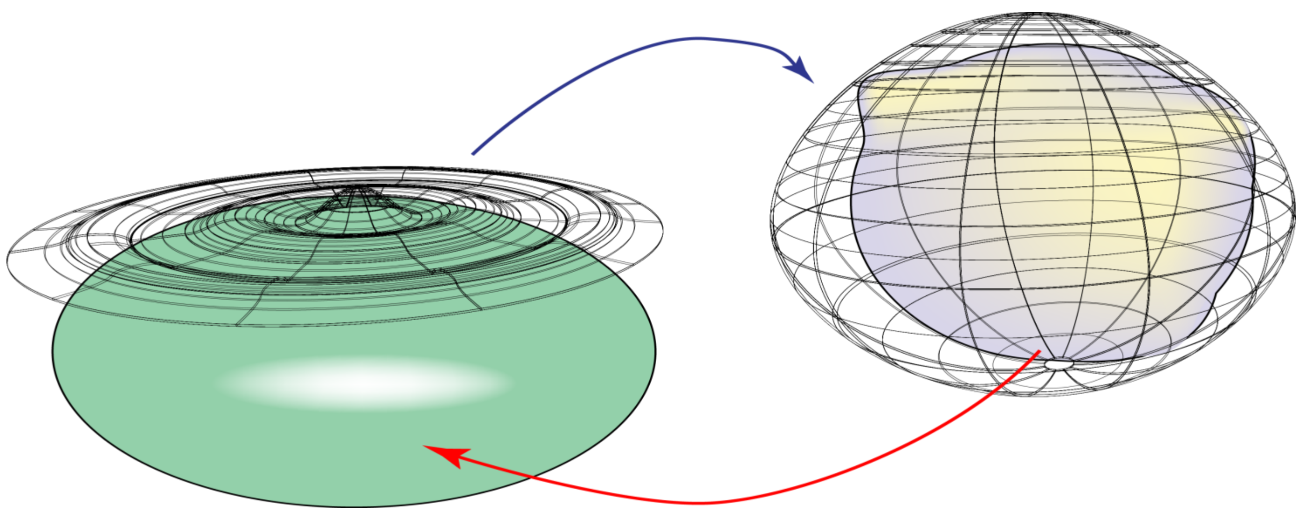
For instance, in a recent work with Christine Guenther (*Scaling and Entropy for the RG-2 Flow*, CMP vol. 378, p. 369-399, (2020) arXiv:1805.09773 math.DG) we prove that

- If we require that the two–loop RG flow gives rise to a scale invariant flow we need to relate the coupling a to Perelman's version of the dilaton field according to $(a)^{n/2} = \int_M e^{-\widetilde{f}(\beta)} d\mu_{g(\beta)}$.

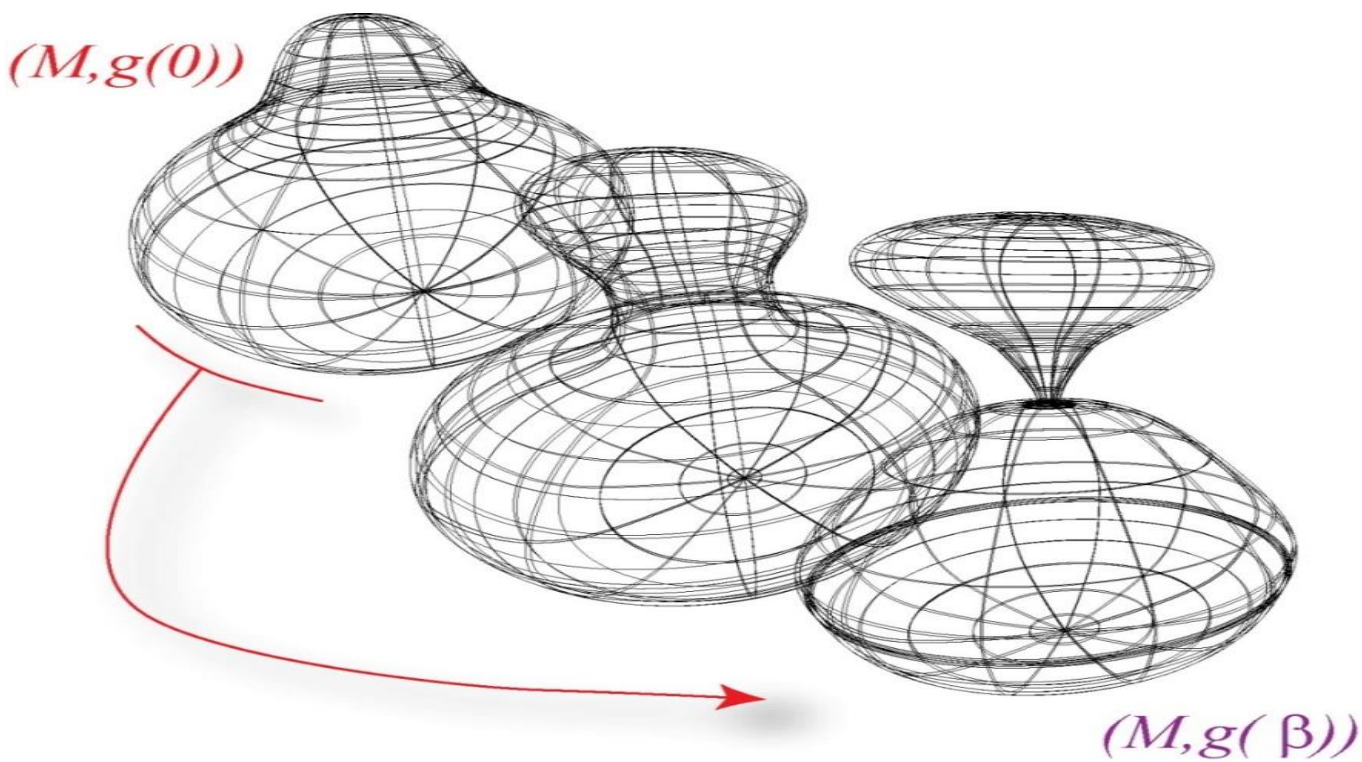
- Only in this case there is an entropy functional (extending Perelman's $\mathcal{F}[g]$–energy) which is monotonic along the flow

- The flow is not a gradient flow with respect to this entropy.

- The natural 2-loop gradient flow (w.r.t. a natural geometric entropy) is a fourth-order geometric flow extending Ricci flow in a form suggested by the supersymmetric version of NLSM



This QFT perspective on the Wasserstein geometry of Ricci curvature comes indeed to full circle ... since quasi-Einstein metrics originated from theoretical physics (D. Friedan, 1980), precisely in the analysis of the Renormalization Group for (Dilatonic) NLσM, showing that the Wasserstein geometry of Ricci provides a guiding principle in a mathematically rigorous analysis of the renormalization group flow in quantum geometry.



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