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Two periodic Aztec diamond and matrix valued orthogonality

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Integrable Systems in Geometry and Mathematical Physics, Conference in Memory of Boris Dubrovin, SISSA Trieste (online), 30 June 2021

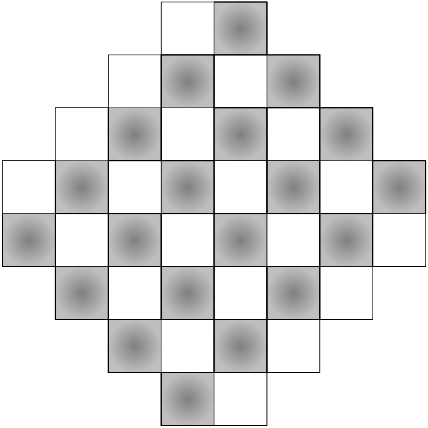
0 References

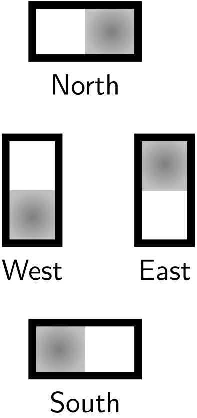
Based on

- M. Duits and A.B.J. Kuijlaars,
  The two periodic Aztec diamond and matrix valued orthogonal polynomials,
  Journal of the European Mathematical Society 23 (2021), 1075–1131, extended preprint arXiv:1712.05636.

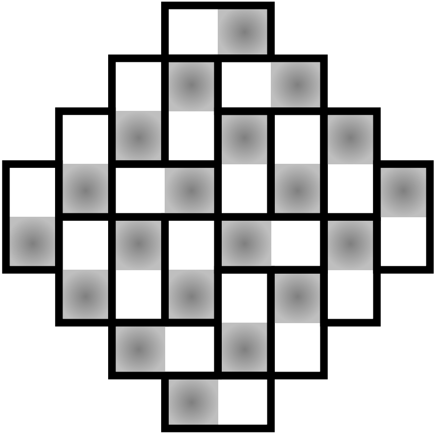
- C. Charlier, M. Duits, A.B.J. Kuijlaars, and J. Lenells,
  A periodic hexagon tiling model and non-Hermitian orthogonal polynomials,
  Communications in Mathematical Physics 378 (2020), 401–466, arXiv:1907.02460.

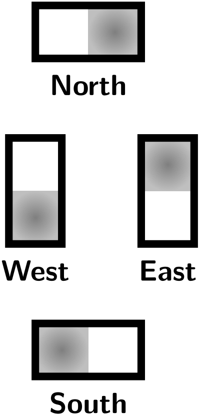
1 Aztec diamond





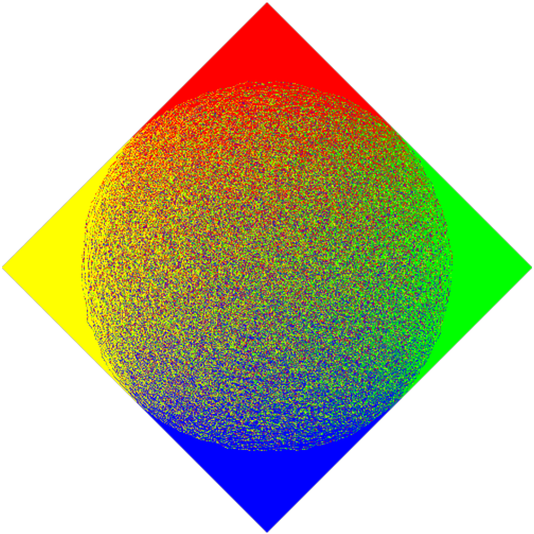
1 Tiling of an Aztec diamond





- **Tiling with 2×1 and 1×2 rectangles (dominos)**
- **Four types of dominos**

1 Aztec diamond: Large random tiling



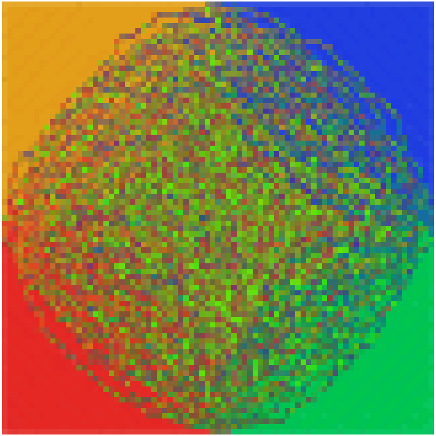
Deterministic pattern near corners
Solid region
or Frozen region

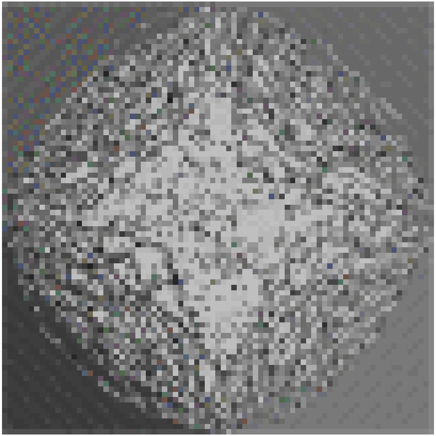
Disorder in the middle Liquid region
or Rough region

Boundary curve
Arctic circle

Jockush, Propp, Shor (1995)
Johansson (2002)

1 Aztec diamond: two periodic weighting





Chhita, Johansson (2016)
Beffara, Chhita, Johansson (2018)

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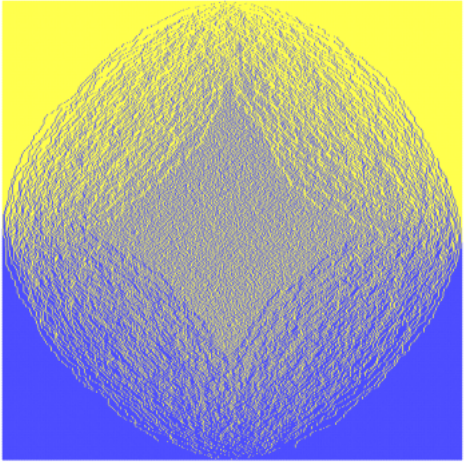
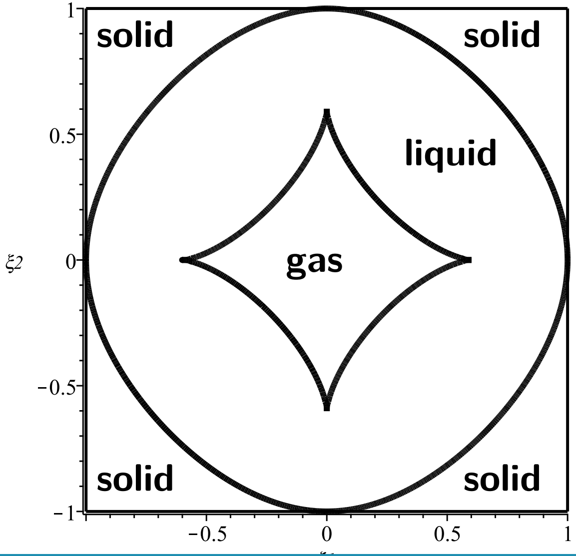
1 Aztec diamond; two-periodic weighting

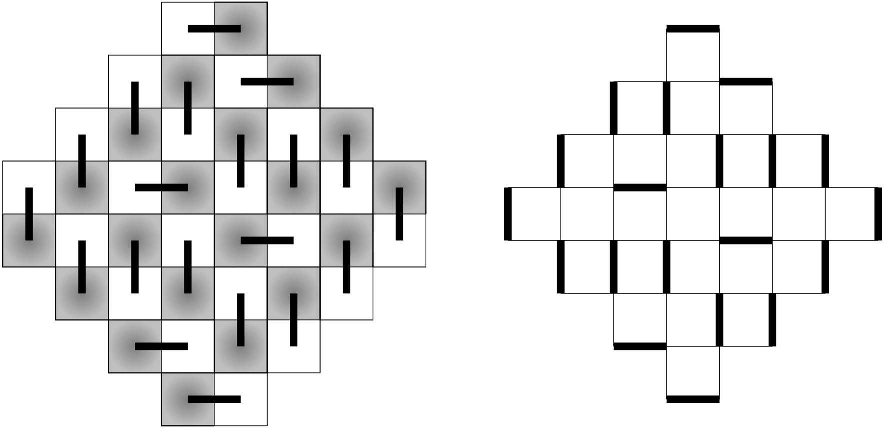


New phase within the liquid region: gas region (smooth region)

1 Two periodic Aztec diamond: Phase diagram



2 Aztec diamond as a dimer model



A domino tiling of the Aztec diamond is a dimer configuration on part of the square lattice (a.k.a. perfect matching)

survey on dimer model Kenyon (2006)

2 Aztec diamond with periodic weights

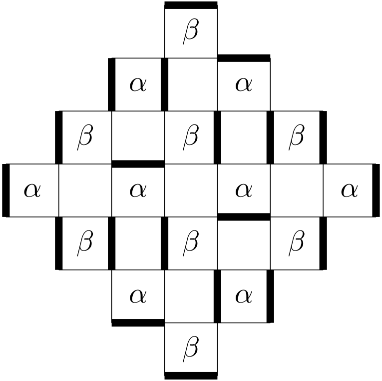
Put weights on the faces

Weight of a dimer (domino) is the product of the weights of adjacent faces

Weight of a domino tiling is the product of the weights of dominos

Probability of a domino tiling is proportional to its weight

$$\text{Prob}(\mathcal{T}) = \frac{w(\mathcal{T})}{\sum_{\text{tilings } \mathcal{T}'} w(\mathcal{T}')}$$



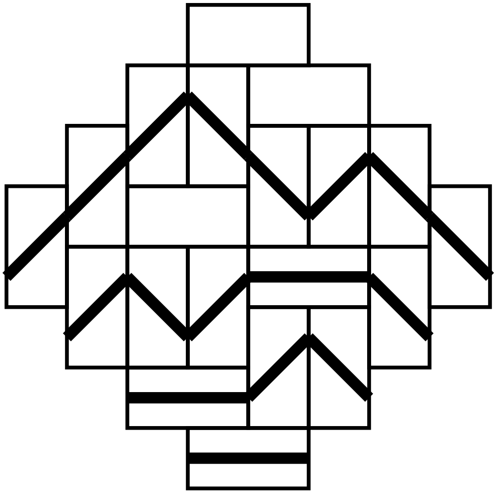
Empty faces have weight 1

We may assume

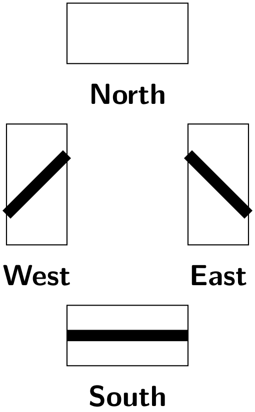
$$\boxed{\alpha\beta = 1, \quad \alpha > 1}$$

$\alpha = 1$ **is the uniform model**

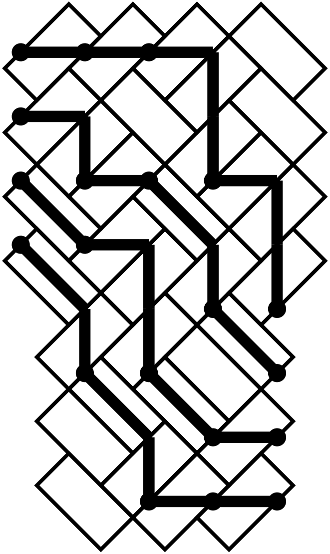
2 Paths in the Aztec diamond



Line segments on West, East and South dominos



2 Transformations and extension; particle system



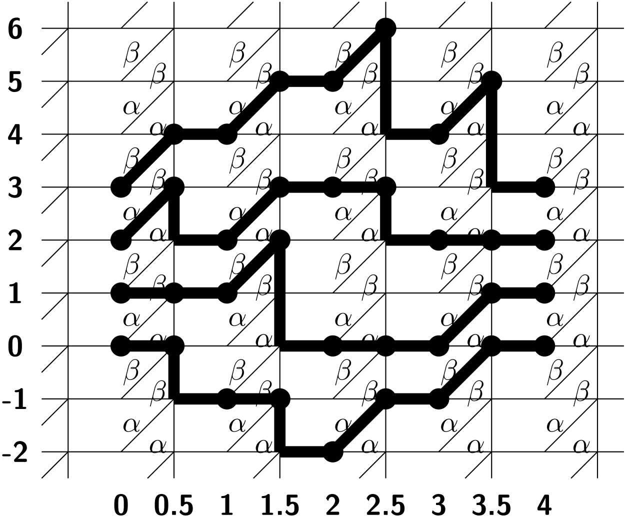
- Rotate the Aztec diamond
- Extend the tiling to a **double Aztec diamond**
- Put particles on the paths
- Particles are a **determinantal point process**

2 Non-intersecting paths on a weighted graph

- Apply affine transformation

Two types of steps

- Bernoulli step up: weight α/β
- Steps down followed by horizontal step: weight 1
- Weight of a path system: product of weights of edges



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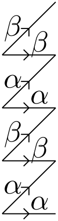
2 Transition matrices (two types)

- **Bernoulli step with transition matrix**

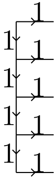
$$T_B(x,x) = T_B(x,x+1) = \begin{cases} \alpha & \text{if } x \text{ is even}, \\ \beta & \text{if } x \text{ is odd}, \end{cases}$$

and $T_B(x,y) = 0$ otherwise.



- **Geometric step down with transition matrix**

$$T_G(x,y) = \begin{cases} 1 & \text{if } y \leq x, \\ 0 & \text{if } y > x. \end{cases}$$



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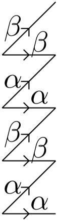

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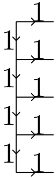

$$T_G(x,y) = \begin{cases} 1 & \text{if } y \le x, \\ 0 & \text{if } y > x. \end{cases}$$

Transition matrices are **block Toeplitz** with respective **symbols**

$$\Phi_B(z) = \begin{pmatrix} \alpha & \alpha \\ 0 & \beta \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \beta & 0 \end{pmatrix} z = \begin{pmatrix} \alpha & \alpha \\ \beta z & \beta \end{pmatrix}$$

$$\Phi_G(z) = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} z^{-1} + \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} z^{-2} + \cdots = \frac{1}{z-1} \begin{pmatrix} z & 1 \\ z & z \end{pmatrix}$$

3 Determinantal point process

At each level $m = 0, 1, \ldots, L = 2N$ there are N particles $x_j^{(m)}$

Proposition [Lindström Gessel-Viennot lemma]

$$\mathrm{Prob}\left[\left(x_j^{(m)}\right)_{j=0,m=1}^{N-1,L-1}\right] = \frac{1}{Z_n}\prod_{m=0}^{L-1}\det\left[T_m\left(x_j^{(m)}, x_k^{(m+1)}\right)\right]_{j,k=0}^{N-1}$$

with $T_m = T_B$ if m is even, $T_m = T_G$ if m is odd.

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with $T_m = T_B$ if m is even, $T_m = T_G$ if m is odd.

- **This is determinantal on $\mathcal{X} = \{0, 1, \ldots, L\} \times \mathbb{Z}$**
- **There is $K : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ such that for finite $\mathcal{A} \subset \mathcal{X}$,**

$$\operatorname{Prob}\left[\exists \text{ particle at each } (m,x) \in \mathcal{A}\right] = \det\left[K((m,x),(m',x'))\right]_{(m,x),(m',x') \in \mathcal{A}}$$

Eynard-Mehta (1998)

3 Double contour integral formula

Theorem (Duits, K (2021))

- **Suppose transition matrices are block Toeplitz with block symbols Φ_m of size $p \times p$.**
- **Suppose pN particles at each level with starting positions $x_j^{(0)} = j$ and ending positions $x_j^{(L)} = pM + j$.**

3 Double contour integral formula

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Then $K(m, px + j; m', py + i)$ for $i, j = 0, \ldots, p-1$ are entries of

$$-\frac{\chi_{m>m'}}{2\pi i} \oint_\gamma \prod_{j=m'}^{m-1} \Phi_j(z) z^{y-x} \frac{dz}{z} +$$

$$\frac{1}{(2\pi i)^2} \oint_\gamma \oint_\gamma \left(\prod_{j=m'}^{L-1} \Phi_j(w) \right) R_N(w,z) \left(\prod_{j=0}^{m-1} \Phi_j(z) \right) \frac{w^y}{z^{x+1} w^{M+N}} dz dw$$

3 What is R_N ?

$$\frac{1}{(2\pi i)^2} \oint_\gamma \oint_\gamma \left(\prod_{j=m'}^{L-1} \Phi_j(w) \right) R_N(w,z) \left(\prod_{j=0}^{m-1} \Phi_j(z) \right) \frac{w^y}{z^{x+1} w^{M+N}} dz dw$$

$R_N(w,z)$ **is the degree** N **reproducing kernel for matrix valued orthogonal polynomials (MVOP) with matrix weight**

$$W(z) = \frac{\prod_{j=0}^{L-1} \Phi_j(z)}{z^{M+N}} \qquad \text{on } \gamma$$

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- $R_N(w,z)=\sum_{k=0}^{N-1}P_k(w)^T H_k^{-1}P_k(z)$ **with** $P_k(z)=I_p z^k+$ **l.o.t.**

$$\frac{1}{2\pi i}\oint_\gamma P_k(x)W(x)P_j^T(x)dx=\delta_{j,k}H_k \qquad \det H_k\neq 0$$

4 MVOP in our setting

Weight matrix for two periodic Aztec diamond of size $2N$ takes the form

$$W^N(z) = \frac{(\Phi_B(z)\Phi_G(z))^{2N}}{z^N}$$

with

$$W(z) = \frac{1}{z}\left(\begin{pmatrix} \alpha & \alpha \\ \beta z & \beta \end{pmatrix} \frac{1}{z-1} \begin{pmatrix} z & 1 \\ z & z \end{pmatrix}\right)^2$$
$$= \frac{1}{(z-1)^2}\begin{pmatrix} (z+1)^2 + 4\alpha^2 z & 2\alpha(\alpha+\beta)(z+1) \\ 2\beta(\alpha+\beta)z(z+1) & (z+1)^2 + 4\beta^2 z \end{pmatrix}$$

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Asymptotic analysis as $N \to \infty$ has two parts

- **Riemann-Hilbert steepest descent analysis of MVOP and reproducing kernel**
- **Saddle point analysis of double integral.**

4 Riemann Hilbert problem for MVOP

RH problem of size 4×4 **(in general $2p \times 2p$)**

$$Y_+ = Y_- \begin{pmatrix} I_2 & W^N \\ 0_2 & I_2 \end{pmatrix} \quad \textbf{on } \gamma$$

$$Y(z) = \left(I_4 + O(z^{-1})\right) \begin{pmatrix} z^N I_2 & 0_2 \\ 0_2 & z^{-N} I_2 \end{pmatrix} \quad \textbf{as } z \to \infty.$$

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- P_N **is 2×2 left upper block of Y**
- **Reproducing kernel is** **Delvaux (2010)**

$$R_N(w,z) = \frac{1}{z-w} \begin{pmatrix} 0_2 & I_2 \end{pmatrix} Y^{-1}(w) Y(z) \begin{pmatrix} I_2 \\ 0_2 \end{pmatrix}$$

- **Extension of Fokas-Its-Kitaev RH problem for OP.**

Grünbaum, de la Iglesia, Martínez-Finkelshtein (2011)

Cassatella Contra, Mañas (2012)

4 Riemann Hilbert problem for MVOP

$$Y_+ = Y_- \begin{pmatrix} I_2 & W^N \\ 0_2 & I_2 \end{pmatrix} \quad \textbf{on } \gamma$$

- **First step in RH analysis uses eigenvalues of W**

$$\lambda_{1,2}(z) = \frac{\left((\alpha+\beta)z \pm \sqrt{z(z+\alpha)(z+\beta)}\right)^2}{z(z-1)^2}$$

- **Eigenvalues live on spectral curve $w^2 = z(z+\alpha)(z+\beta)$ that has genus 1**
- **Matrix orthogonality is (in our case, but maybe more general...)**

scalar orthogonality on the Riemann surface

4 MVOP for two periodic Aztec diamond

$$W(z) = \frac{1}{(z-1)^2} \begin{pmatrix} (z+1)^2 + 4\alpha^2 z & 2\alpha(\alpha+\beta)(z+1) \\ 2\beta(\alpha+\beta)z(z+1) & (z+1)^2 + 4\beta^2 z \end{pmatrix}$$

- **MVOP of degree N with respect to W^N has explicit formula (if N is even)**

$$\boxed{P_N(z) = (z-1)^N W(\infty)^{N/2} W^{-N/2}(z)}$$

- **The reproducing kernel is not that simple, but still the double contour integral for the correlation kernel simplifies considerably**
- **Different approach is due to Berggren-Duits (2019)**

5 Saddle point analysis

The double contour integral that remains is deformed to (essentially)

$$\frac{1}{(2\pi i)^2}\oint_{\gamma_z}\frac{dz}{z}\oint_{\gamma_w}\frac{dw}{z-w}F(w)F(z)e^{N(S(w)-S(z))}$$

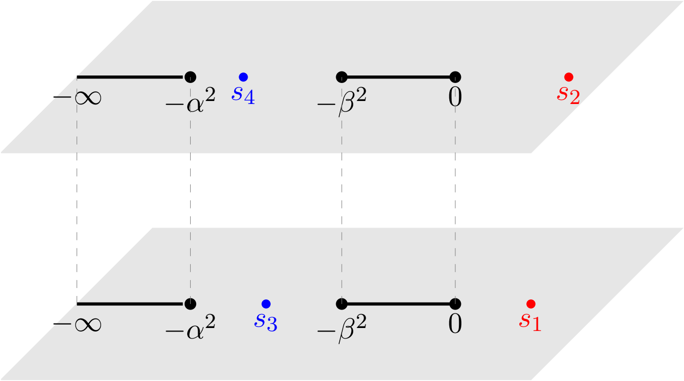
with

$$S(z)=S(z;\xi_1,\xi_2)=\log(z-1)-\frac{1+\xi_2}{2}\log z+\frac{\xi_1}{2}\log\lambda(z)$$

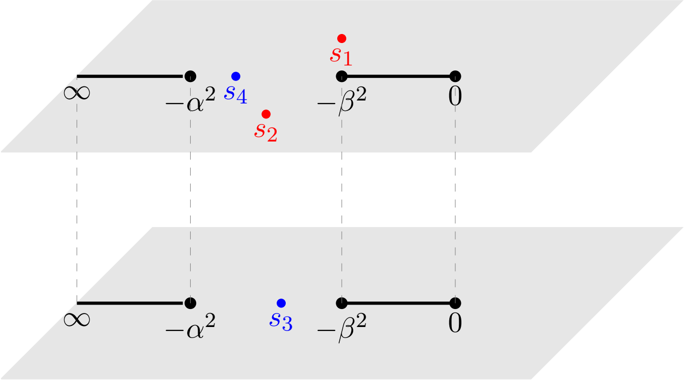
that depends on the asymptotic coordinates (ξ_1,ξ_2) of a point in the Aztec diamond.

- **The four zeros of dS on the spectral curve are the saddles**
- **Two saddles are in the gap $[-\alpha^2,-\beta^2]$.**
- **Location of other two saddles determines the phase.**

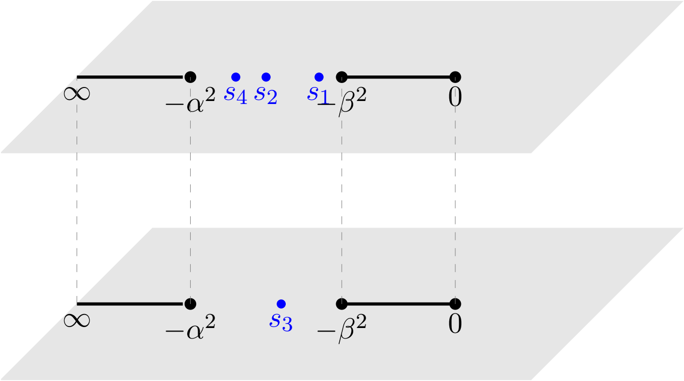
5 Solid phase: saddles s_1 and s_2 are in $[0, \infty)$



5 Liquid phase: saddles s_1 and s_2 are not on the real part



5 Gas phase: all saddles are in $[-\alpha^2, -\beta^2]$



5 Thank you for your attention !