Numerical Study of Davey-Stewartson systems

C. Klein (Dijon)
with
J.-C. Saut (Orsay), P. Miller (Ann Arbor), K. McLaughlin (Fort Collins), O. Assainova, K. Roidot, J. Sjöstrand, N. Stoilov,(Dijon)

Outline

- Introduction
- Integrable versus non-integrable cases
- Semiclassical limit
- Blow-up in focusing DS II
- Inverse scattering

Davey-Stewartson equations

- modulation of waves (rigorous justification in the context of water waves, Lannes (2013))

$$
\begin{aligned}
i\partial_t\psi + a\partial_x^2\psi + b\partial_y^2\psi &= (\nu_1|\psi|^2 + \nu_2\partial_x\phi)\psi,\\
\partial_x^2\phi + c\partial_y^2\phi &= -\delta\partial_x|\psi|^2,
\end{aligned}
$$

where $a > 0$ and $\delta > 0$.

- Ghidaglia, Saut (1990):
  - elliptic-elliptic if $(\mathrm{sgn}\, b, \mathrm{sgn}\, c) = (+1, +1)$,
  - hyperbolic-elliptic if $(\mathrm{sgn}\, b, \mathrm{sgn}\, c) = (-1, +1)$,
  - elliptic-hyperbolic if $(\mathrm{sgn}\, b, \mathrm{sgn}\, c) = (+1, -1)$,
  - hyperbolic-hyperbolic if $(\mathrm{sgn}\, b, \mathrm{sgn}\, c) = (-1, -1)$.

- integrable cases:
  DS I: elliptic-hyperbolic
  DS II: hyperbolic-elliptic, $a = 1, b = -1, \delta = 1, \nu_1 = \frac{\nu_2}{2}$, focusing when $\nu_2 > 0$ and defocusing when $\nu_2 < 0$.

Nonlocal NLS equation

- special case (infinite depth): hyperbolic NLS equation:

$$i\partial_t\psi + \partial_{xx}\psi - \partial_{yy}\psi + |\psi|^2\psi = 0$$

global well posedness (Totz 2016)

- DS II: nonlocal hyperbolic NLS equation:

$$i\partial_t\psi + \partial_{xx}\psi - \partial_{yy}\psi + 2\rho\Delta^{-1}[(\partial_{yy} + (1-2\beta)\partial_{xx})\,|\psi|^2]\psi = 0,$$

- integrable for $\beta = 1$:

$$i\partial_t\psi + \Box\psi - 2\rho[(\Delta^{-1}\Box)|\psi|^2]\psi = 0,$$

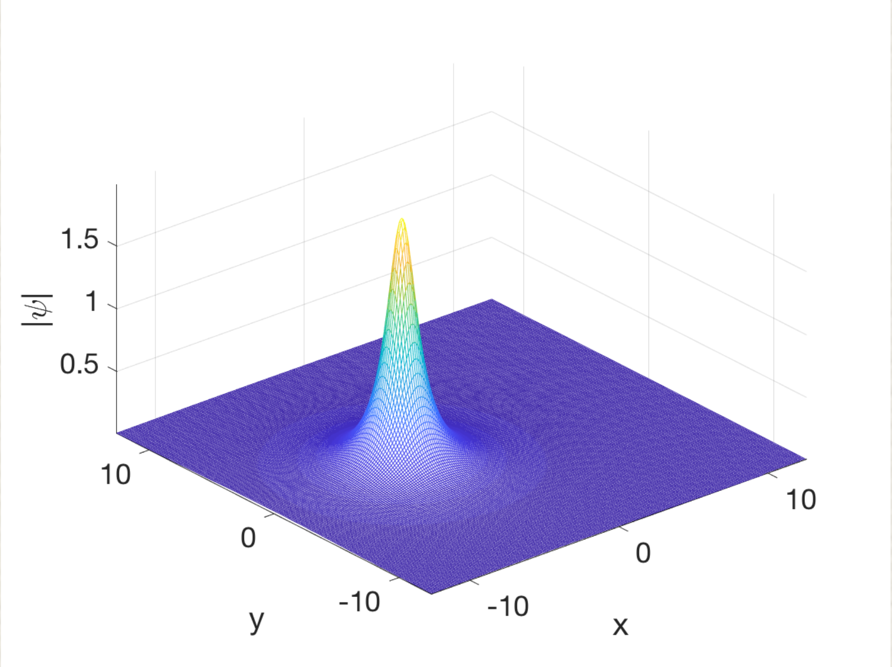
where $\Box = \partial_{xx} - \partial_{yy}$.

Lump

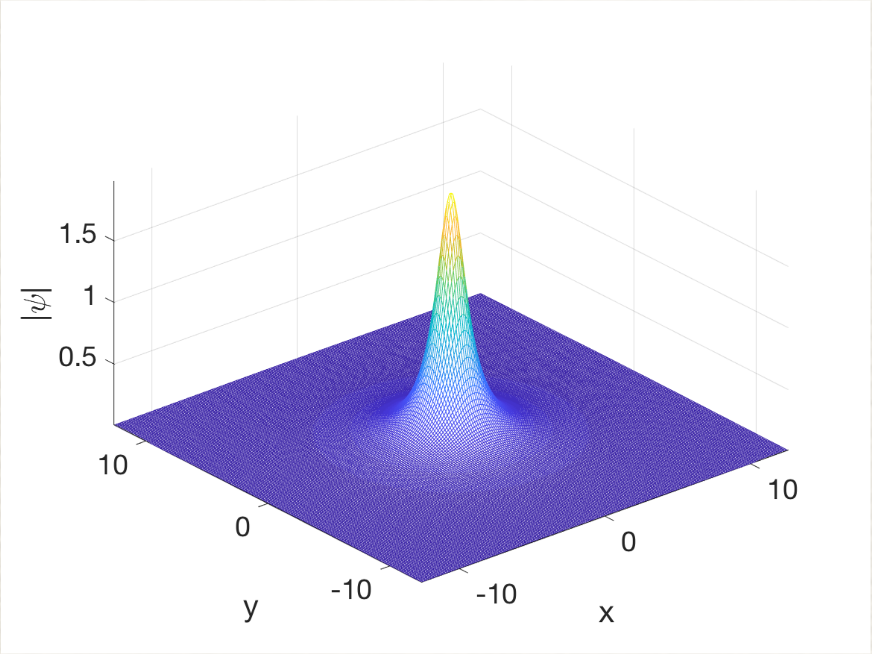
- $$\psi(x,y,t) = 2c\frac{\exp\left(-2i(\xi x - \eta y + 2(\xi^2 - \eta^2)t)\right)}{|x + 4\xi t + i(y + 4\eta t) + z_0|^2 + |c|^2}$$

where $(c, z_0) \in \mathbb{C}^2$ and $(\xi, \eta) \in \mathbb{R}^2$, constant velocity $(-4\xi, -4\eta)$

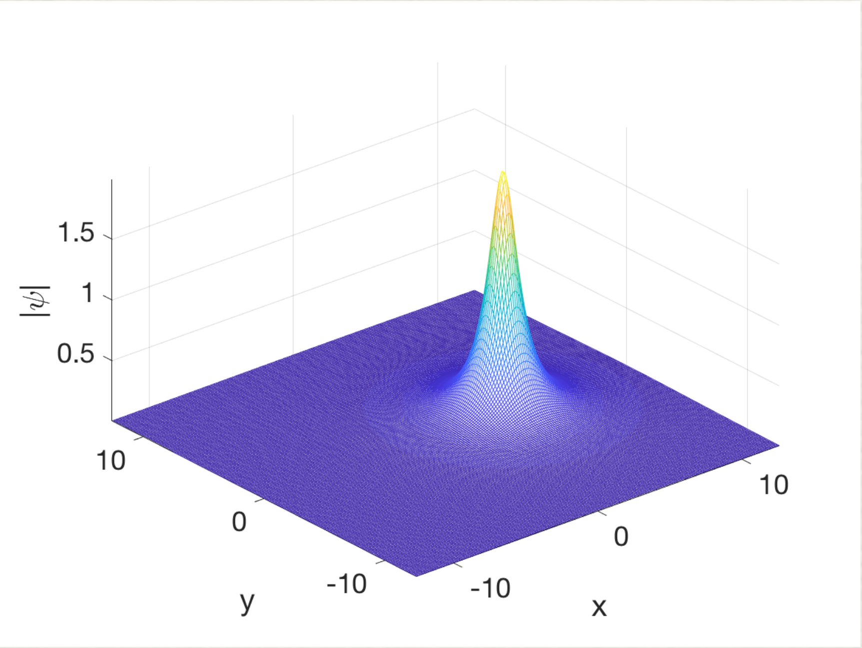
$t = -1$



$t = 0$



$t = 1$



$c = 1,\ \xi = -1,\ \eta = 0,\ z_0 = 0$

Dynamical rescaling

- scaling invariance: $\psi(x,y,t)$ a solution to DS II, so is $\lambda\psi(x/\lambda, y/\lambda, t/\lambda^2)$ with constant $\lambda \in \mathbb{R}/\{0\}$
- dynamical rescaling

$$X = \frac{x}{L(t)}, \quad Y = \frac{y}{L(t)}, \quad \tau = \int_0^t \frac{dt'}{L^2(t')}, \quad \Psi(\xi,\eta,\tau) = L(t)\psi(x,y,t).$$

Merle, Raphaël (2004)

$$L(t) \propto \sqrt{\frac{t\,^* - t}{\ln|\ln(t^* - t)|}},$$

t^*: blow-up time.

- pseudoconformal invariance: $\psi(x,y,t)$ a solution to DS II for $t > 0$, so is

$$\tilde{\psi}(x,y,t) = \exp\left(\frac{i(x^2 - y^2)}{4t}\right)\psi\left(\frac{x}{t}, \frac{y}{t}, \frac{1}{t}\right).$$

Ozawa

Theorem:
Let $a, b \in \mathbb{R}$ such that $ab < 0$ and $t^* = -a/b$. Let

$$\psi(x,y,t) = \exp\left(i\frac{b}{4(a+bt)}(x^2-y^2)\right)\frac{v(X,Y)}{a+bt}$$

where

$$v(X,Y) = \frac{2}{1+X^2+Y^2},\ X = \frac{x}{a+bt},\ Y = \frac{y}{a+bt}$$

Then, ψ is a DS II solution with

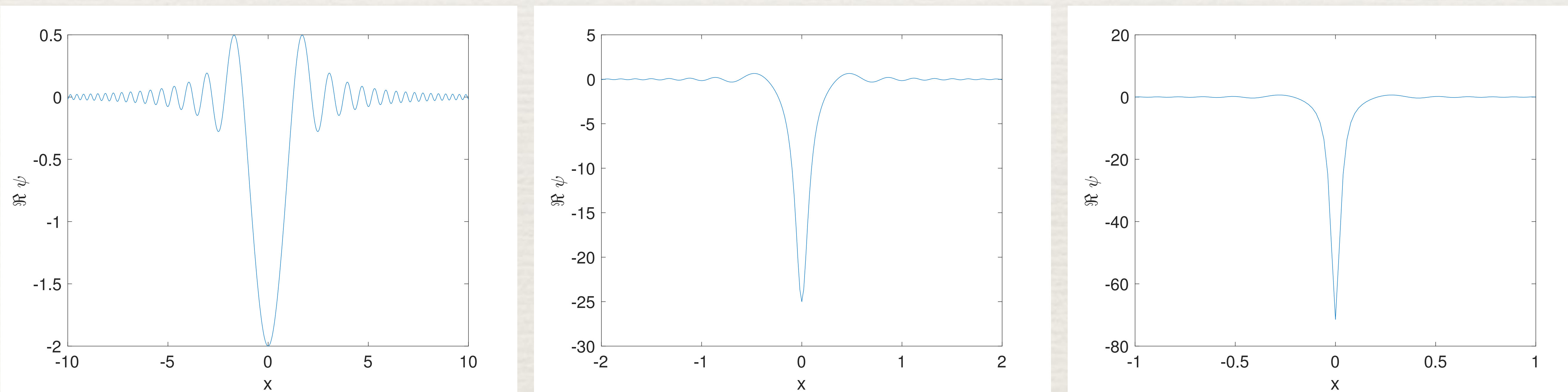
$$|\psi(t)|_2 = |v|_2 = 2\sqrt{\pi}$$

and

$$|\psi(t)|^2 \to 4\pi\delta \text{ in } \quad \mathcal{S}' \quad \text{when } t \to t^*.$$

where δ is the Dirac measure.

Solution on x-axis



soluoz

FIGURE 2. Real part of the Ozawa solution (9 for $a = 1$, $b = -4$ for various values of t: On the left for $t = 0$, on the middle for $t = 0.23$ and on the right for $t = 0.243$ close to the blow-up time $t^* = 0.25$.

Cauchy problem

- **Theorem**: (Sung)
  Let $\psi_0 \in \mathcal{S}(\mathbb{R}^2)$. Then the focusing DS II possesses a unique global solution ψ such that the mapping $t \mapsto \psi(\cdot, t)$ belongs to $C^\infty(\mathbb{R}, \mathcal{S}(\mathbb{R}^2))$ if

  $$|\widehat{\psi_0}|_1 |\widehat{\psi_0}|_\infty < C,$$

  where C is an explicit constant. There is no condition for the defocusing DS II. Global well-posedness for $\hat{\psi}_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and $\psi_0 \in L^p(\mathbb{R}^2)$ for some $p \in [1, 2)$,.

- Perry: generalization to $\psi_0 \in H^{1,1}(\mathbb{R}^2)$.

- Nachman, Regev and Tataru: generalization to $\psi_0 \in L^2(\mathbb{R}^2)$.

Semiclassical limit

- localized initial data varying on length scales of order $1/\epsilon$ for times of order $1/\epsilon$ with $\epsilon \ll 1$: $x \mapsto \epsilon x$, $y \mapsto \epsilon y$, and $t \mapsto \epsilon t$

$$i\epsilon\partial_t\psi + \epsilon^2\partial_{xx}\psi - \epsilon^2\partial_{yy}\psi + 2\rho\Delta^{-1}[(\partial_{yy} + (1-2\beta)\partial_{xx})\,|\psi|^2]\psi = 0,$$

- first numerical studies: White, Weideman (1994), Besse, Mauser, Stimming (2004), McConnel, Fokas, Pelloni (2005)

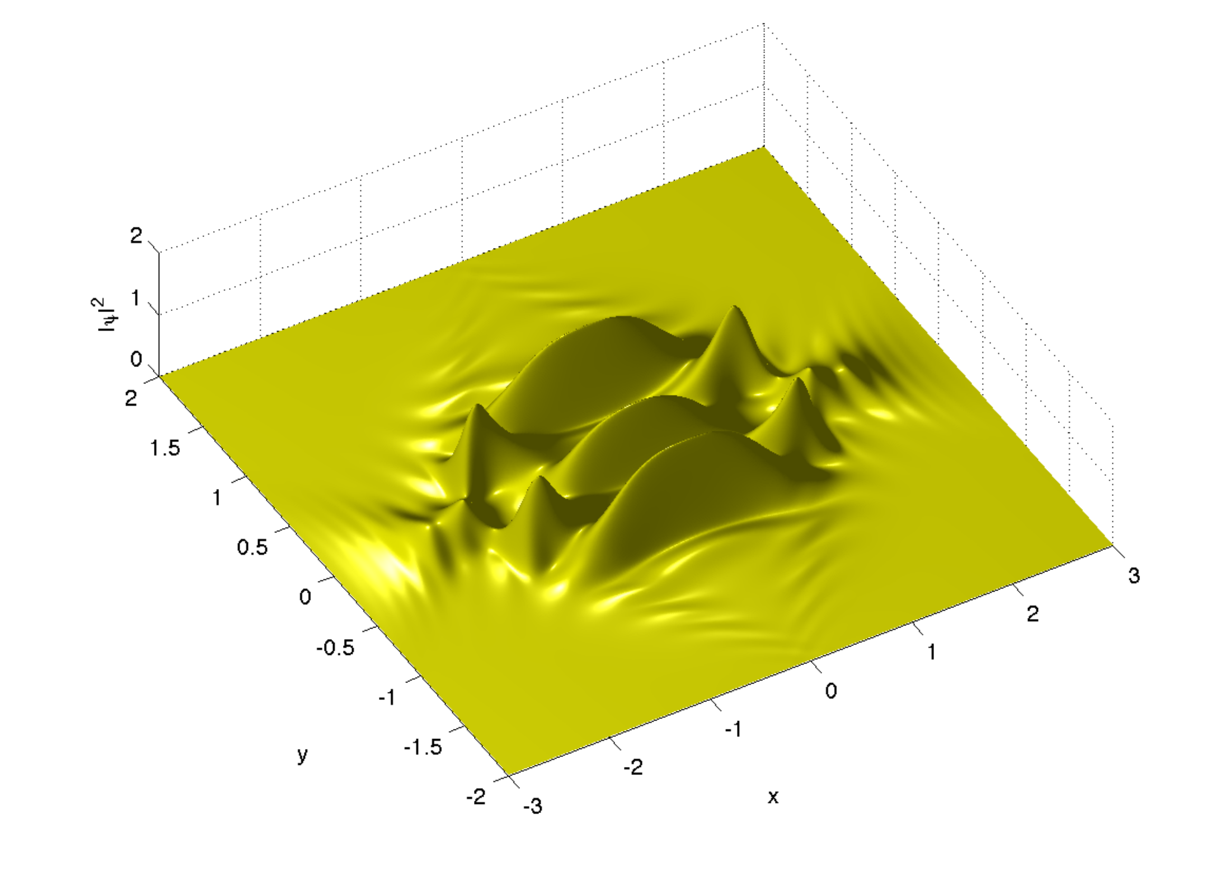
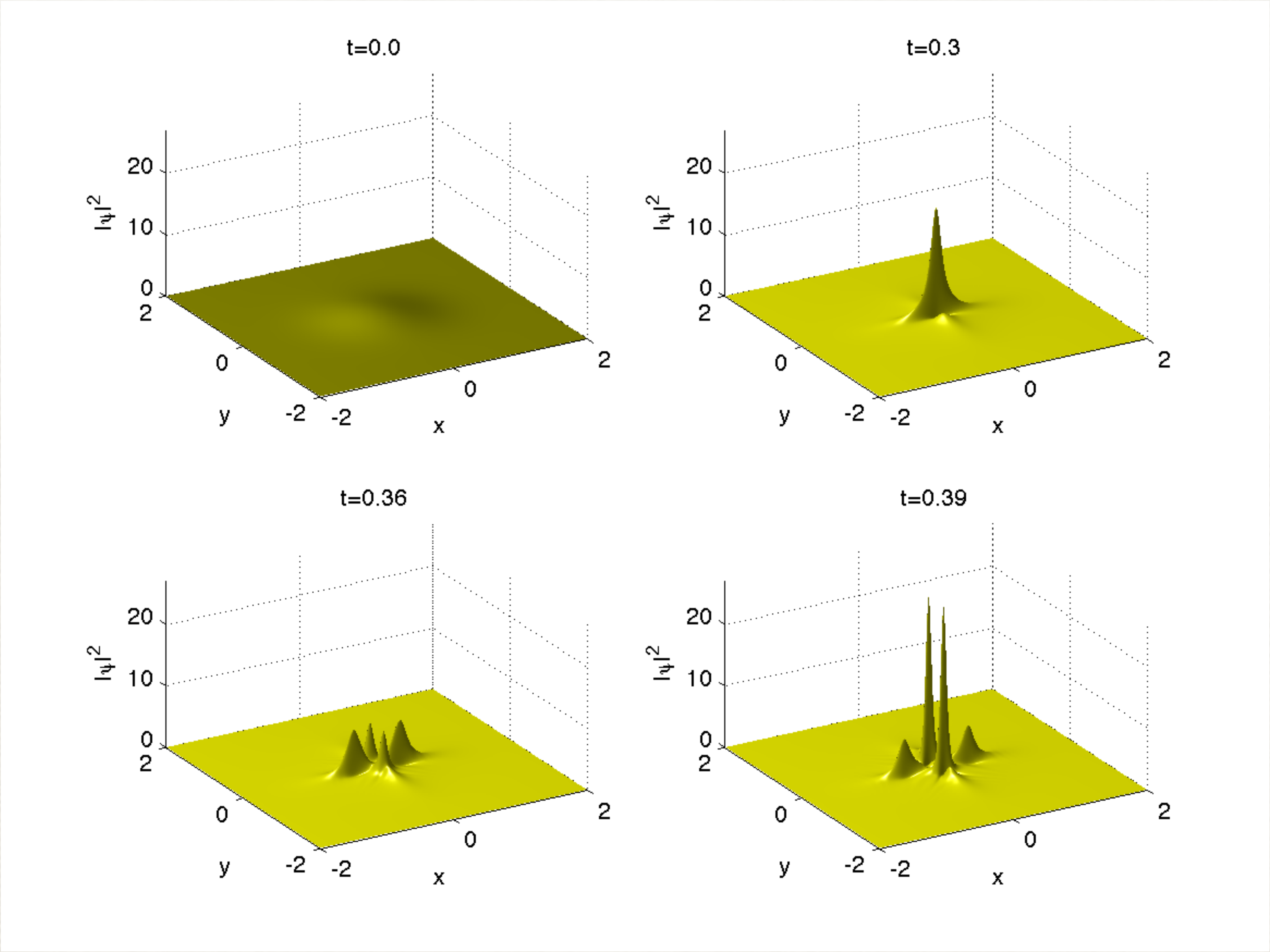
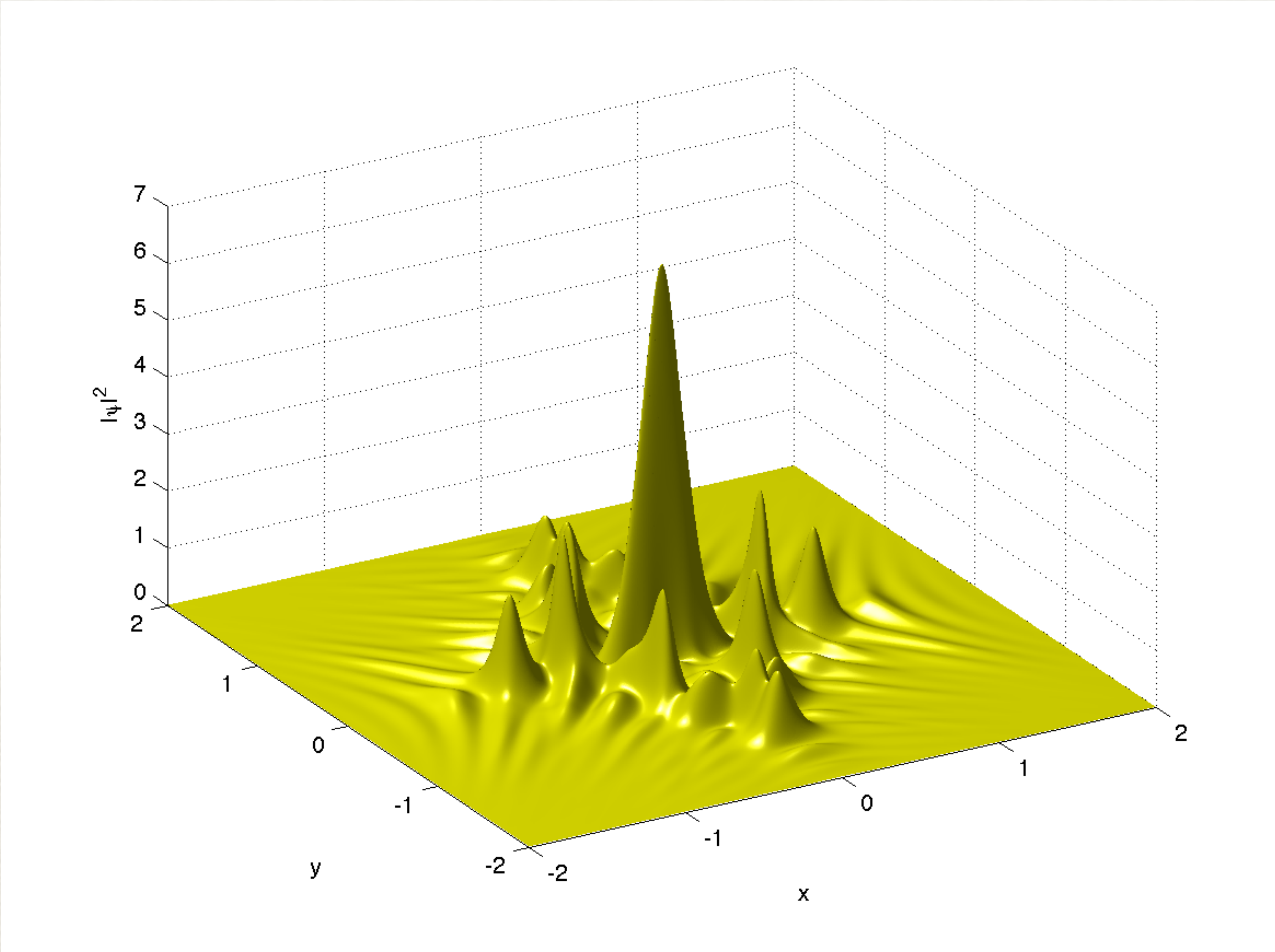


FIGURE 2. Solution to the hyperbolic NLS equation, equation (1) for $\beta = 0$, for the initial data $\psi_0 = \exp(-x^2 - y^2)$ for $\epsilon = 0.1$ at $t = 0.6$.

- Gaussian initial data, $\beta = 0.9$, $\epsilon = 0.1$



- Gaussian initial data, $\beta = 1.1$, $\epsilon = 0.1$, $t = 0.6$

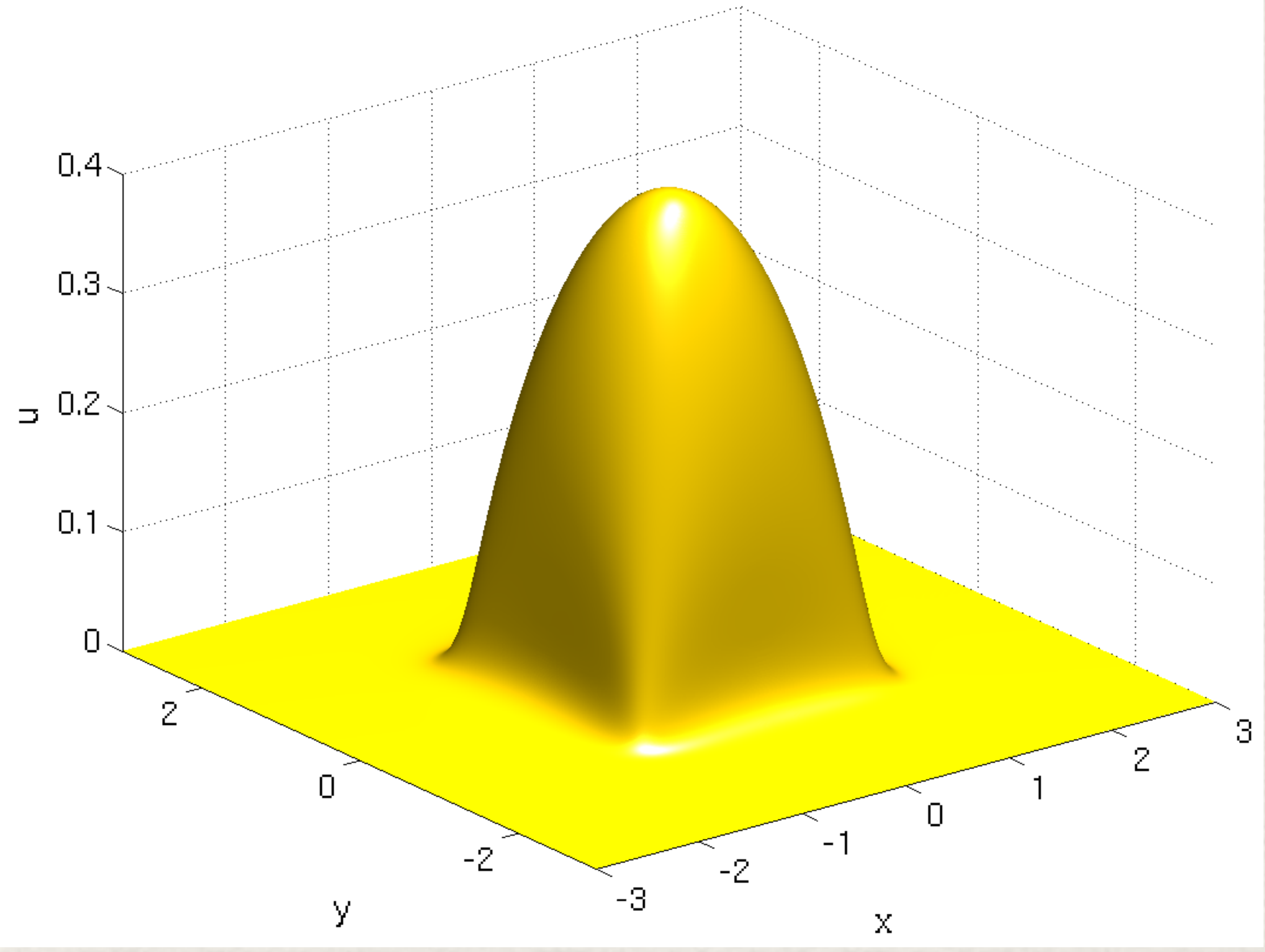


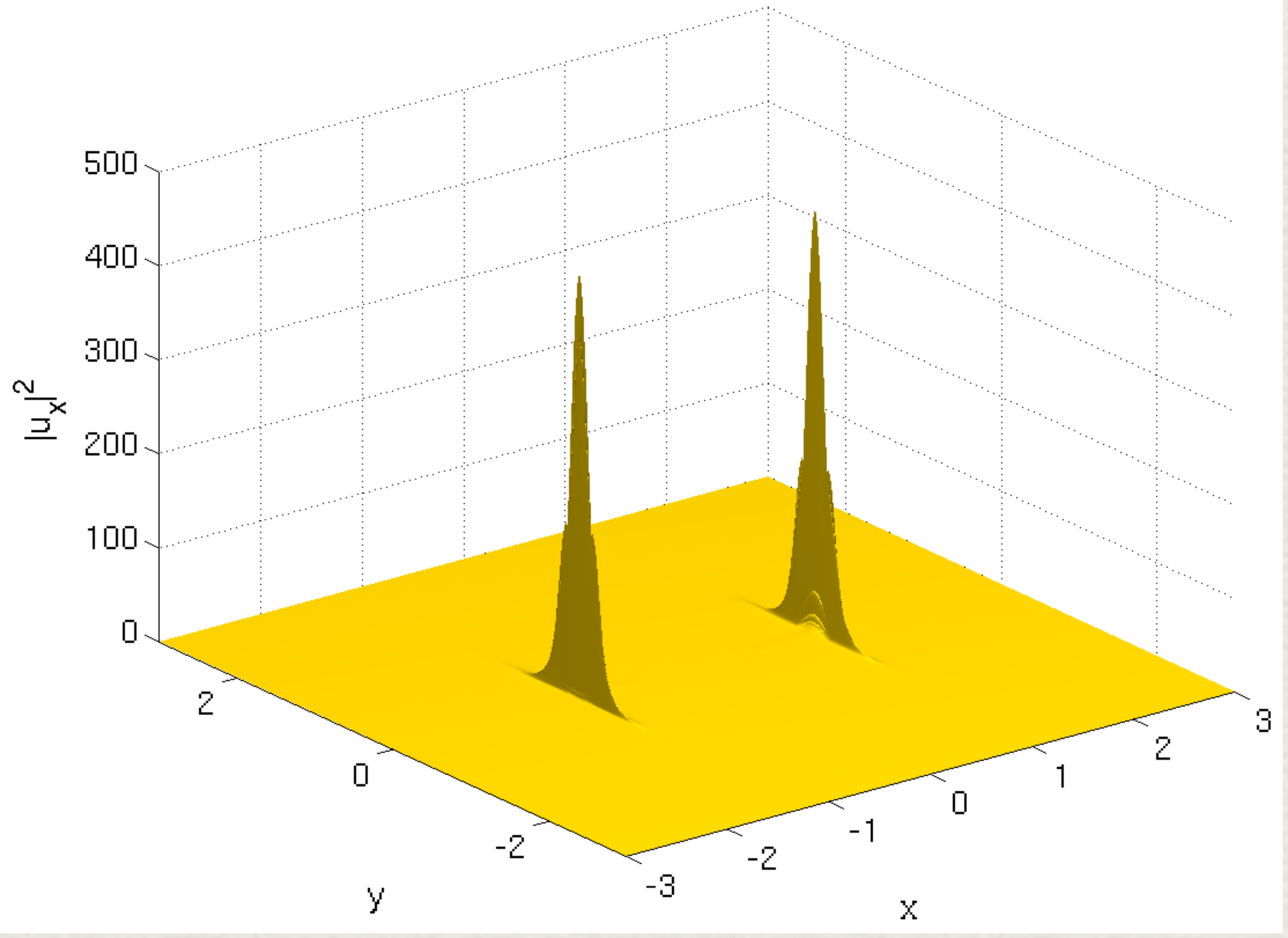
Semiclassical limit, integrable case

- semiclassical limit ($\Psi = \sqrt{u}e^{iS/\epsilon}$, $\epsilon \to 0$, $\mathcal{D}_\pm = \partial_x^2 \pm \partial_y^2$)

$$\begin{cases} S_t + S_x^2 - S_y^2 + 2\rho\mathcal{D}_+^{-1}\mathcal{D}_-(u) & = & \frac{\epsilon^2}{2}\left(\frac{u_x x}{u} - \frac{u_x^2}{u^2} - \frac{u_y y}{u} + \frac{u_y^2}{u}\right) \\ u_t + 2\left(S_x u\right)_x - 2\left(S_y u\right)_y & = & 0 \end{cases},$$

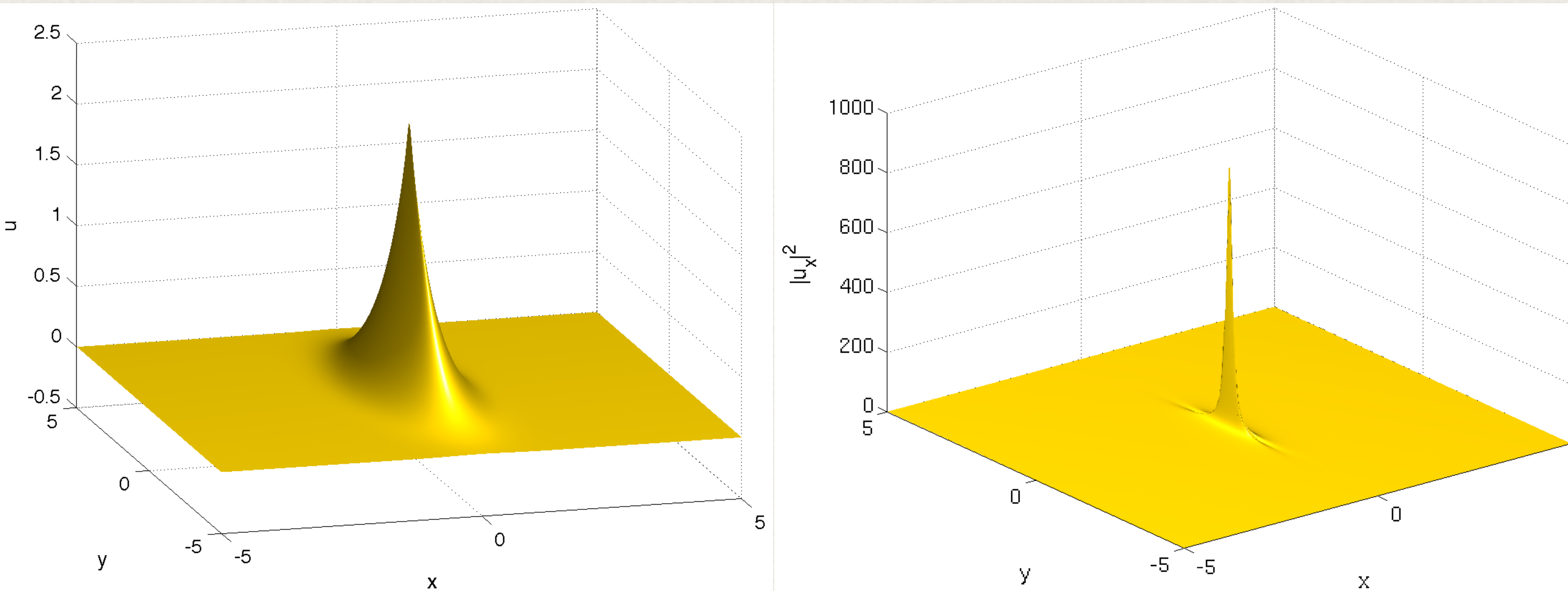
- defocusing case, $u_0 = \exp(-2(x^2+y^2))$, $S_0 = 0$



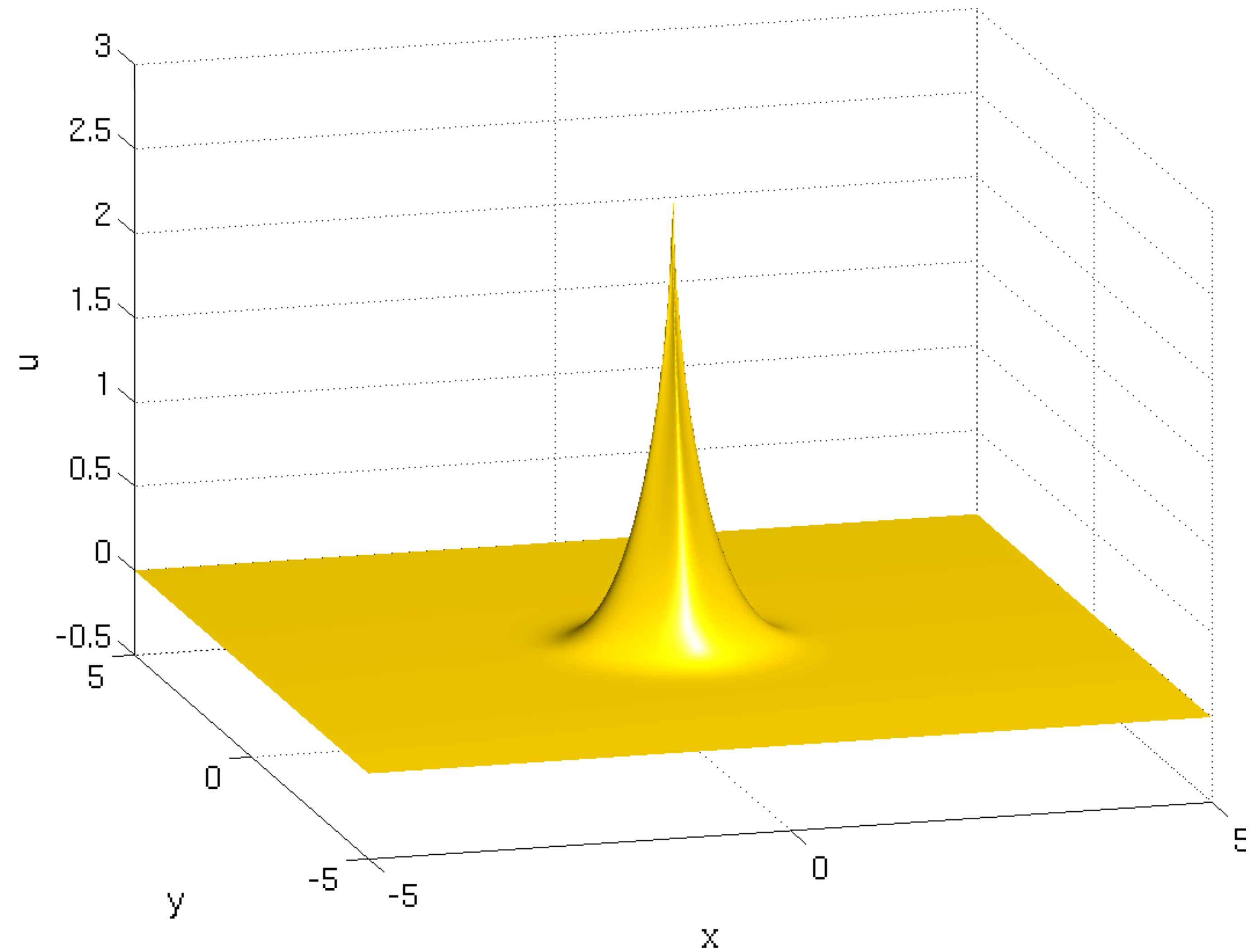


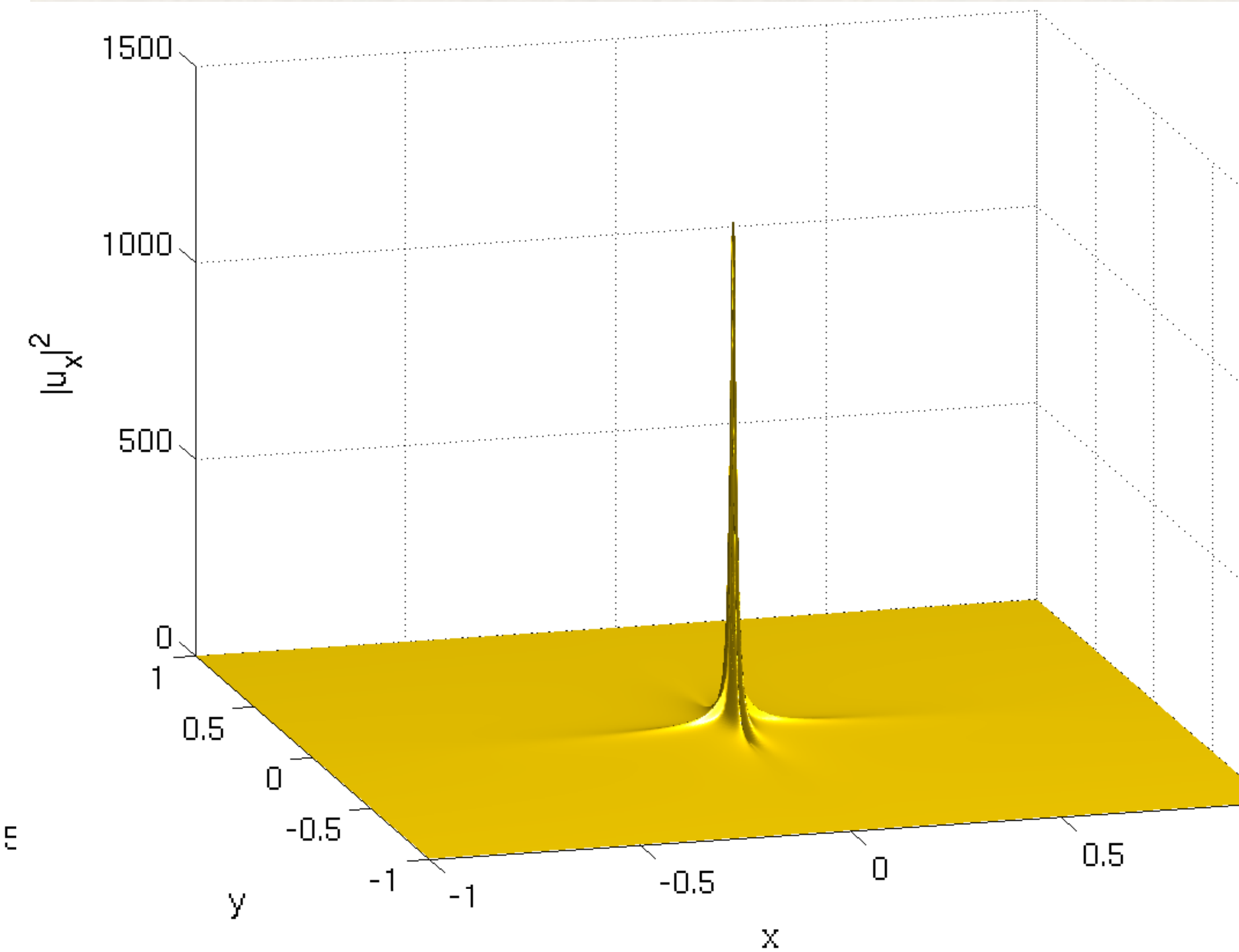
Focusing semiclassical DS II system

- $u_0 = \exp(-2(x^2 + 0.1y^2)),\ S_0 = 0$



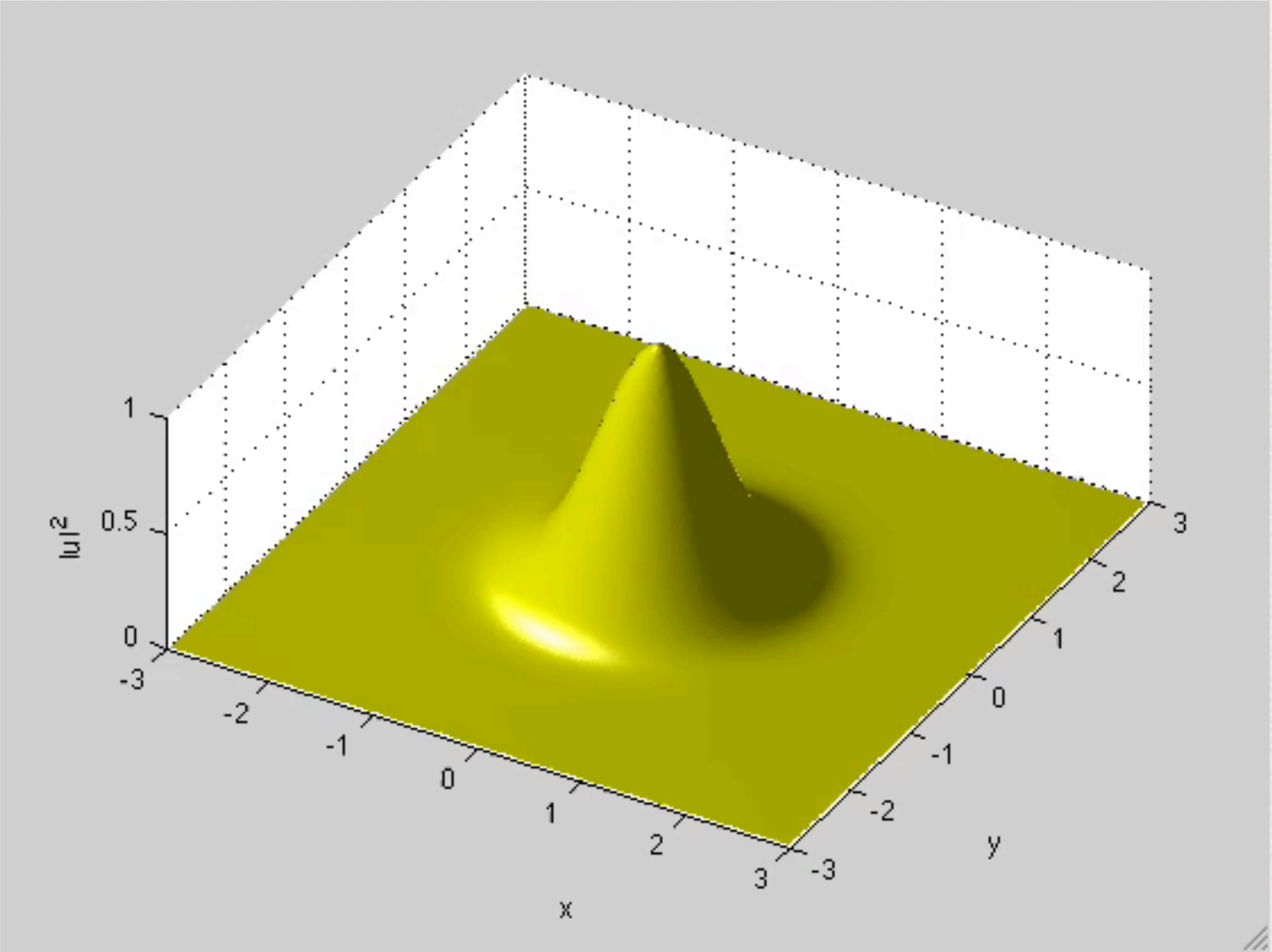
Symmetric initial data





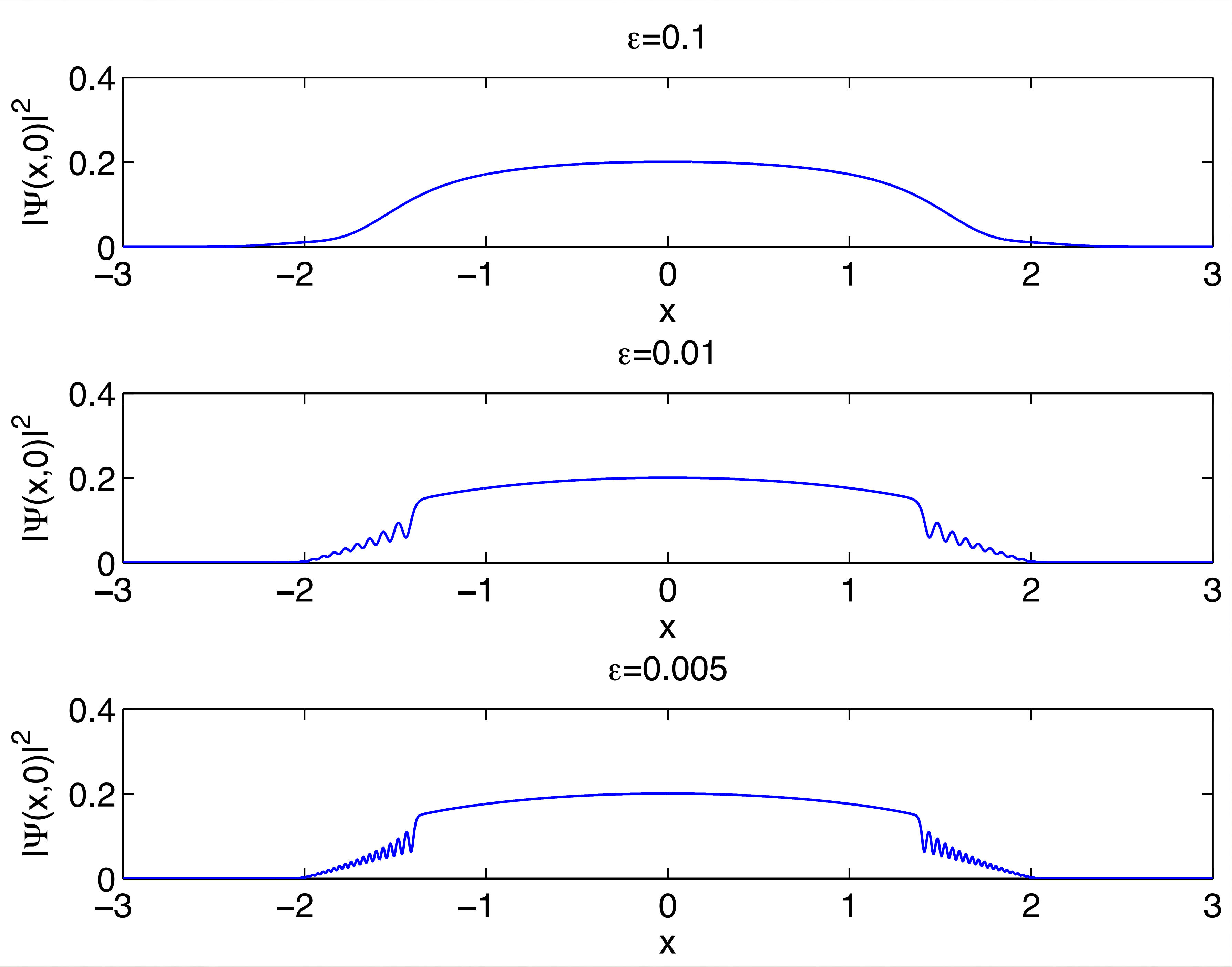
Defocusing DS II

$u_0 = \exp(-x^2 - y^2)$ $\beta = 1$



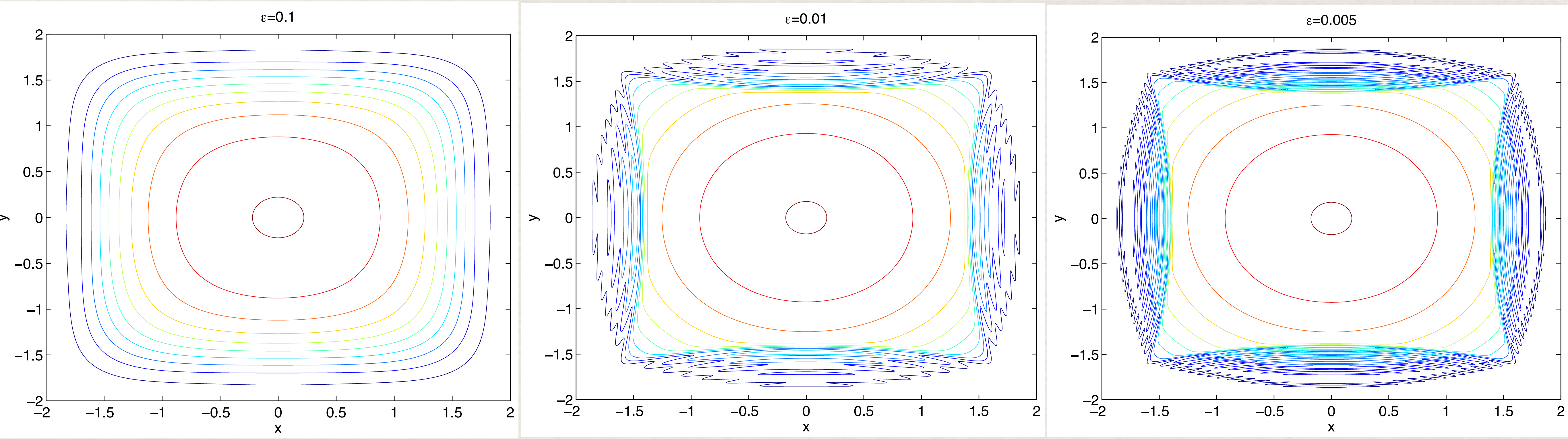
$\epsilon = 0.1$

Defocusing DS II



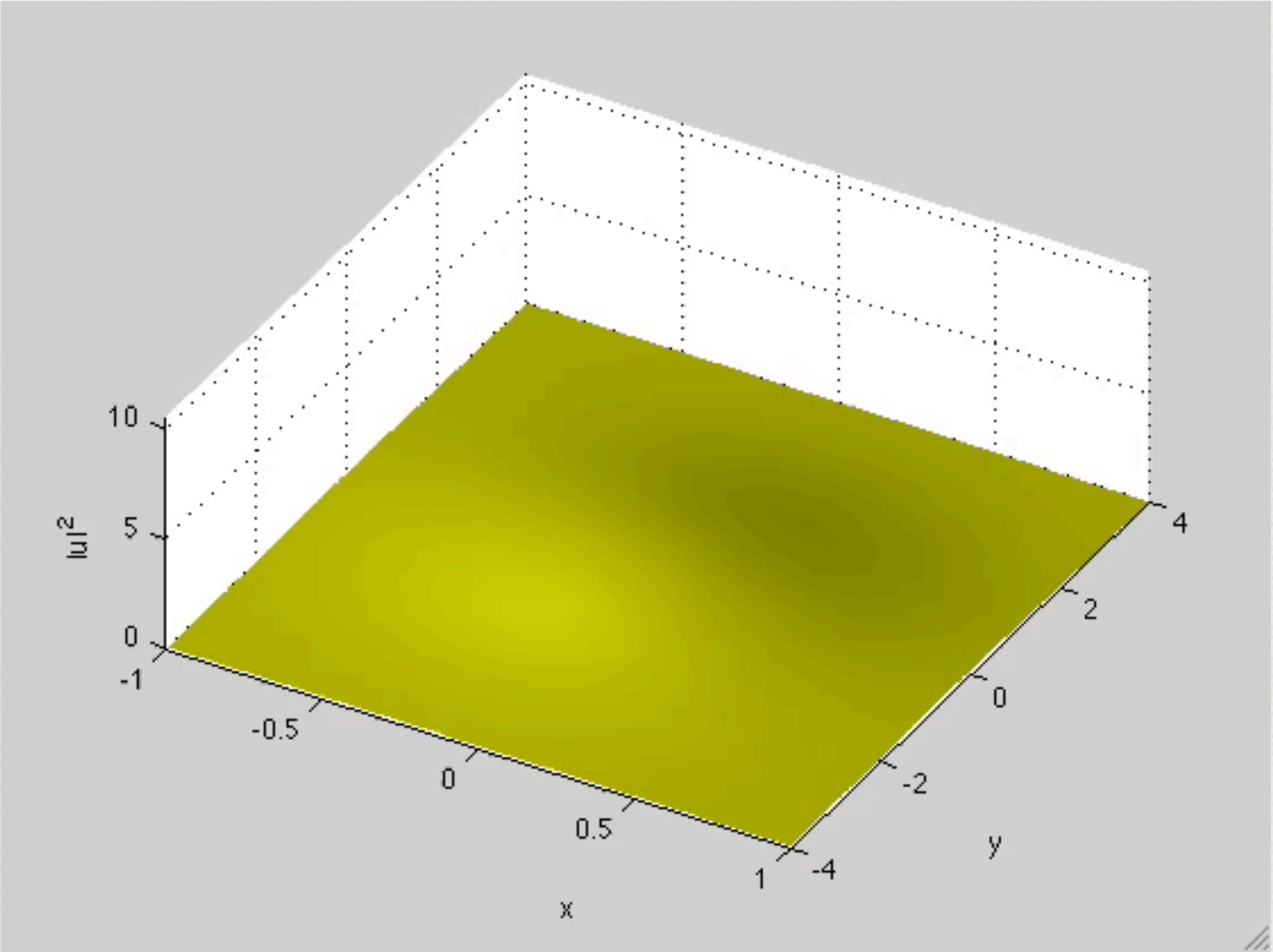
Defocusing DS II

- $t = t_c$: scaling of the difference between semiclassical and DS II solution proportional to $\epsilon^{2/7}$
- $t \gg t_c$: dispersive shock



Focusing DS

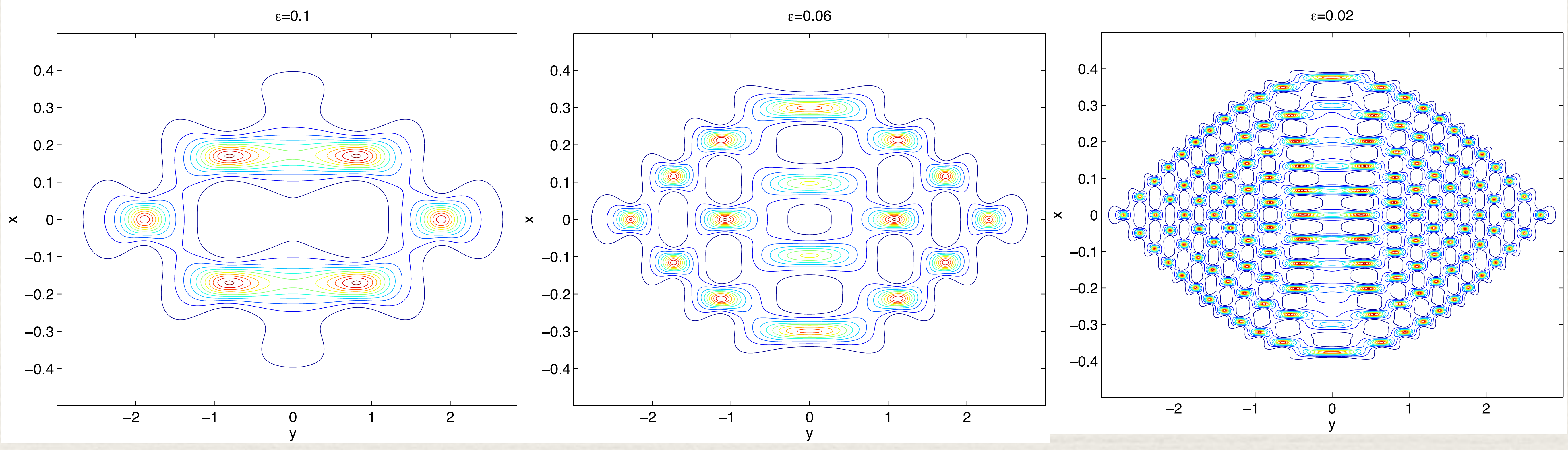
$\psi_0 = \exp(-x^2 - 0.1y^2)$



$\epsilon = 0.1$

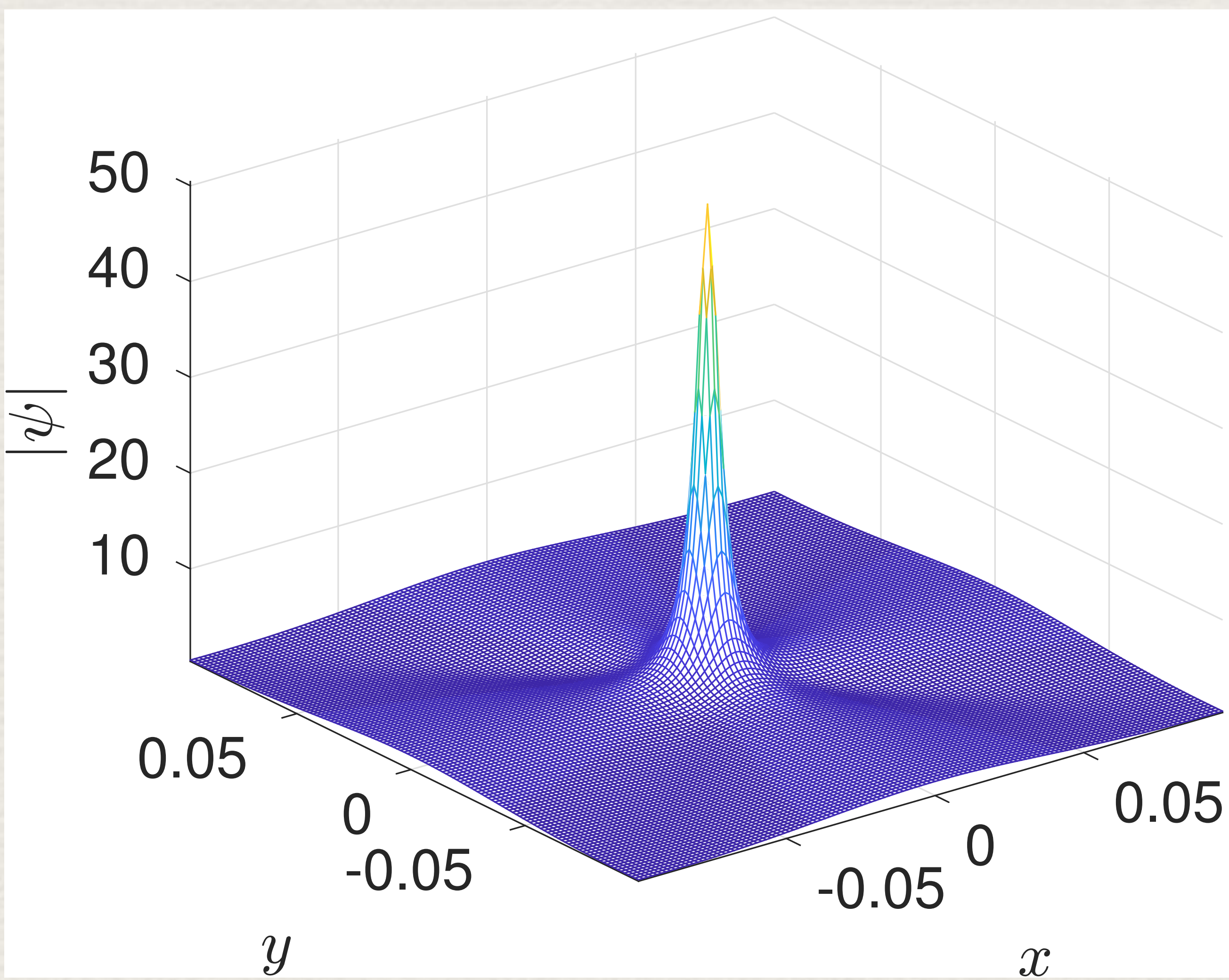
Focusing DS II

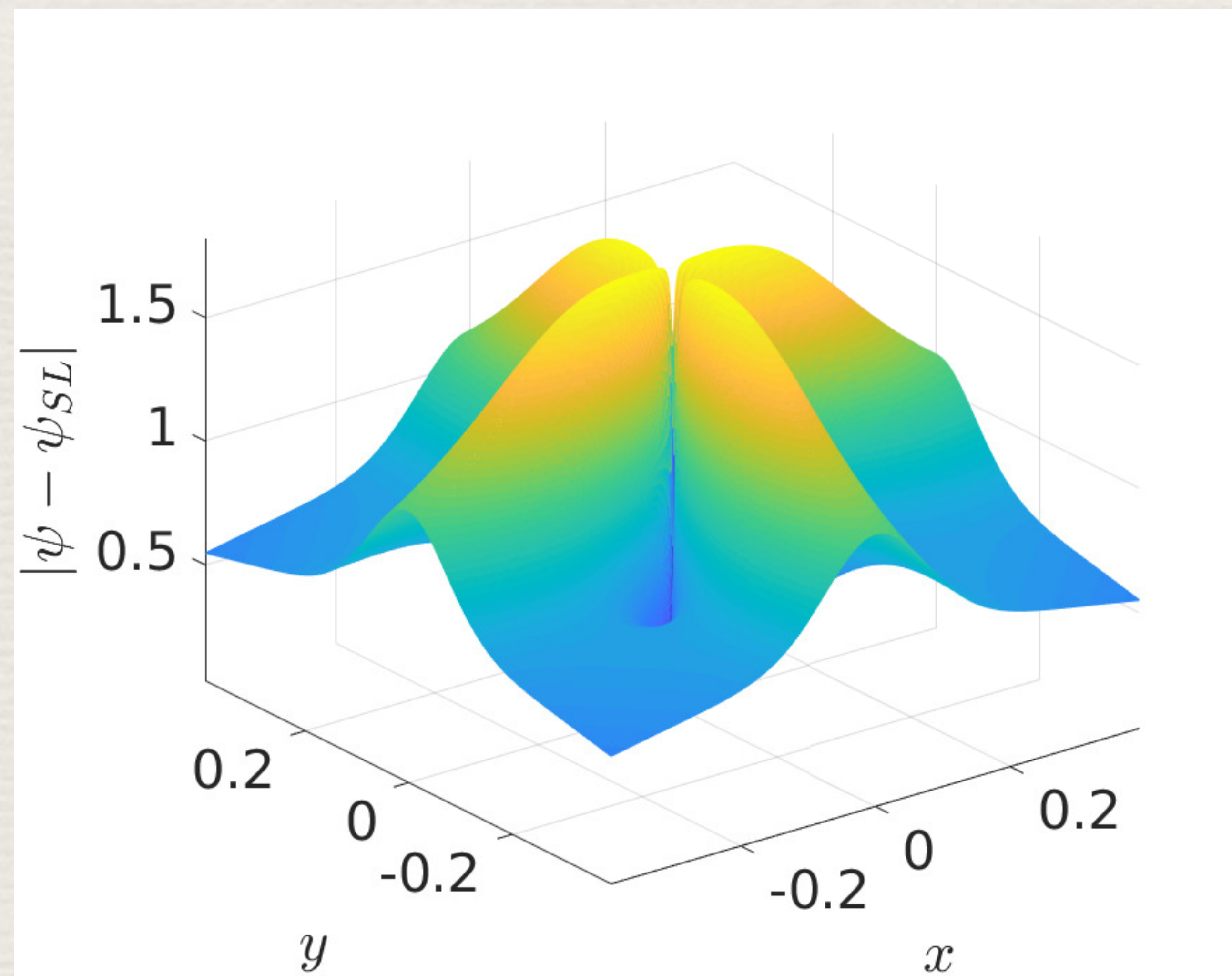
- $t = t_c$: scaling of the difference between semiclassical and DS II solution proportional to $\epsilon^{2/5}$
- $t \gg t_c$: dispersive shock for non-symmetric initial data



Blow-up

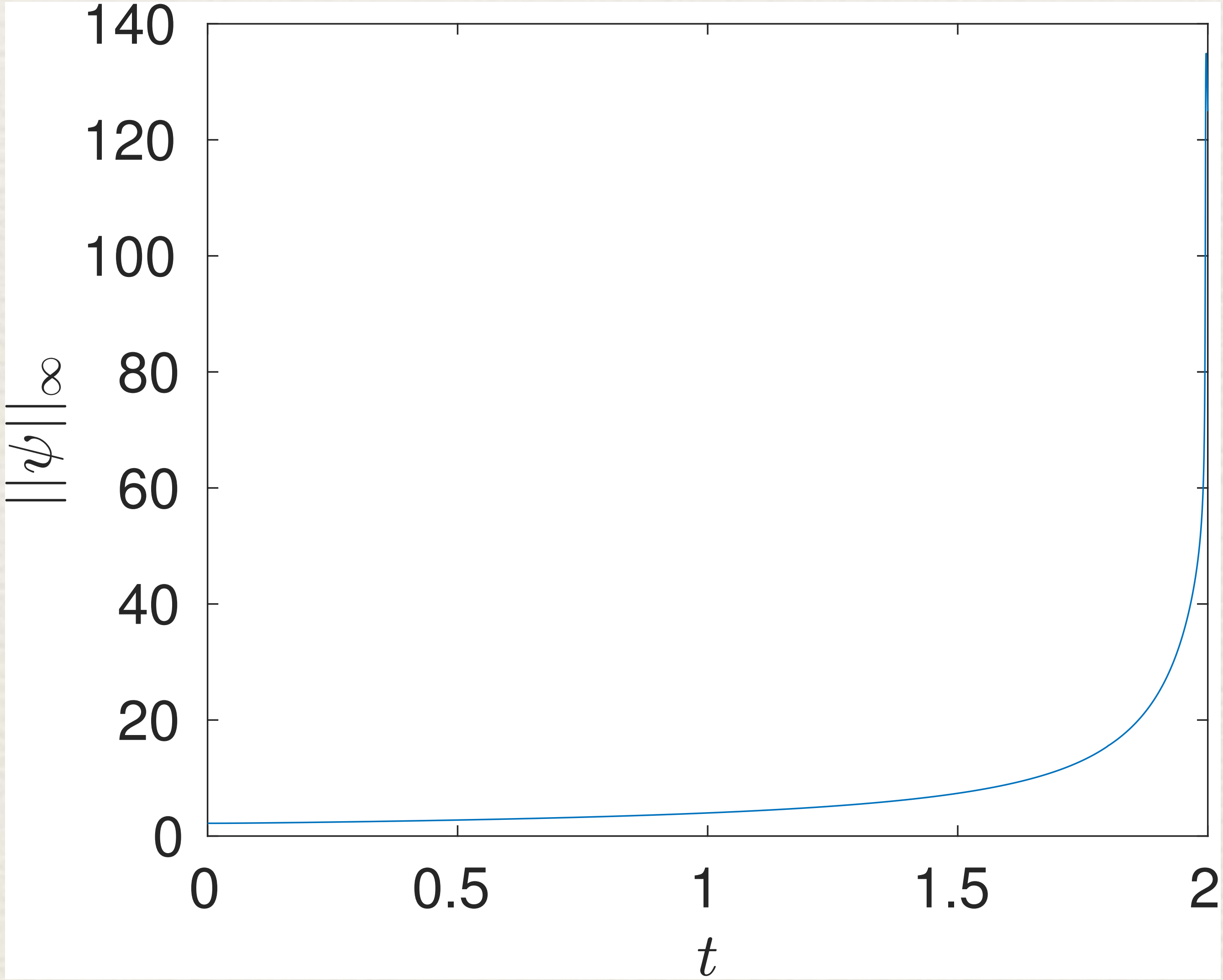
- Gaussian initial data, integrable case $\beta = 1$

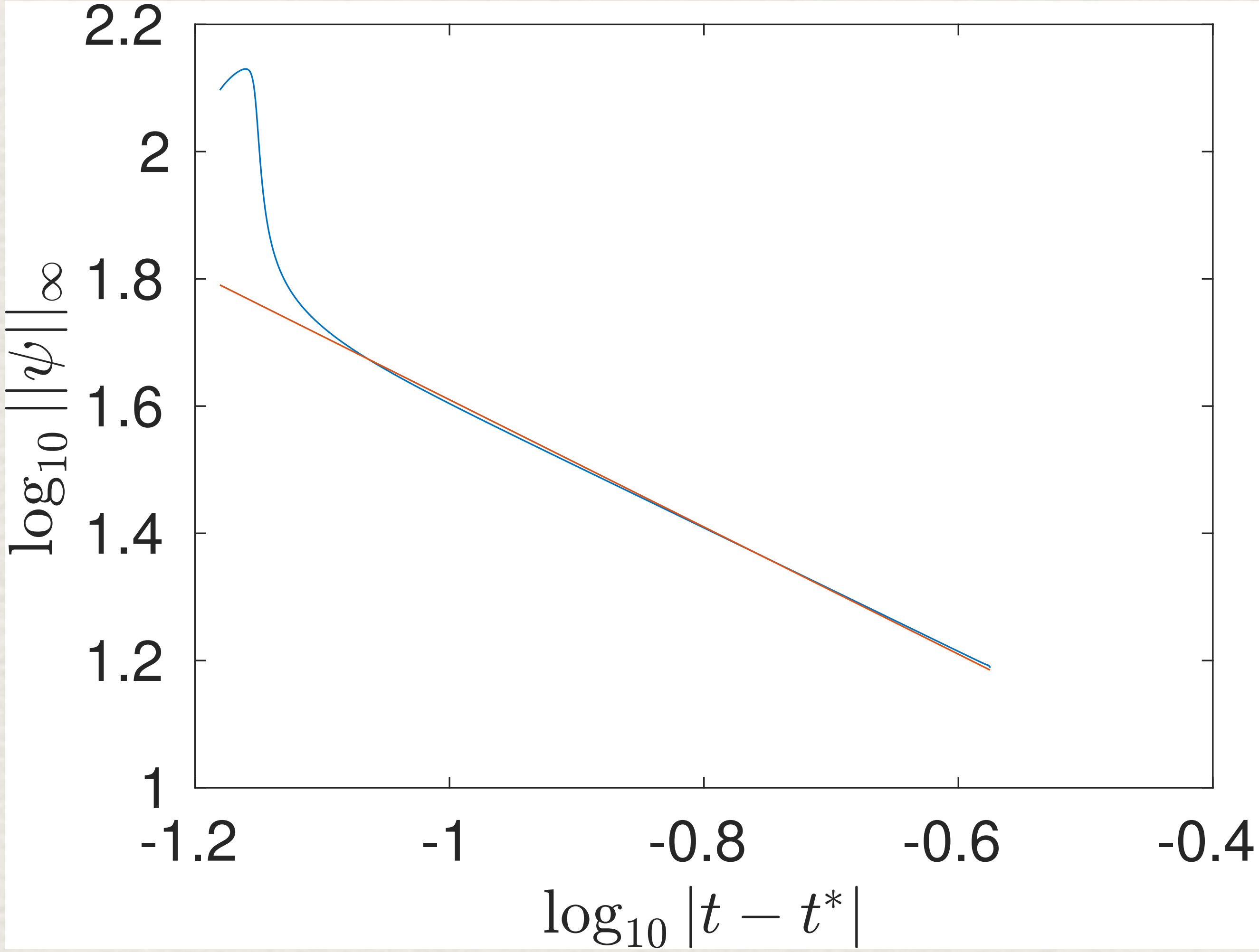




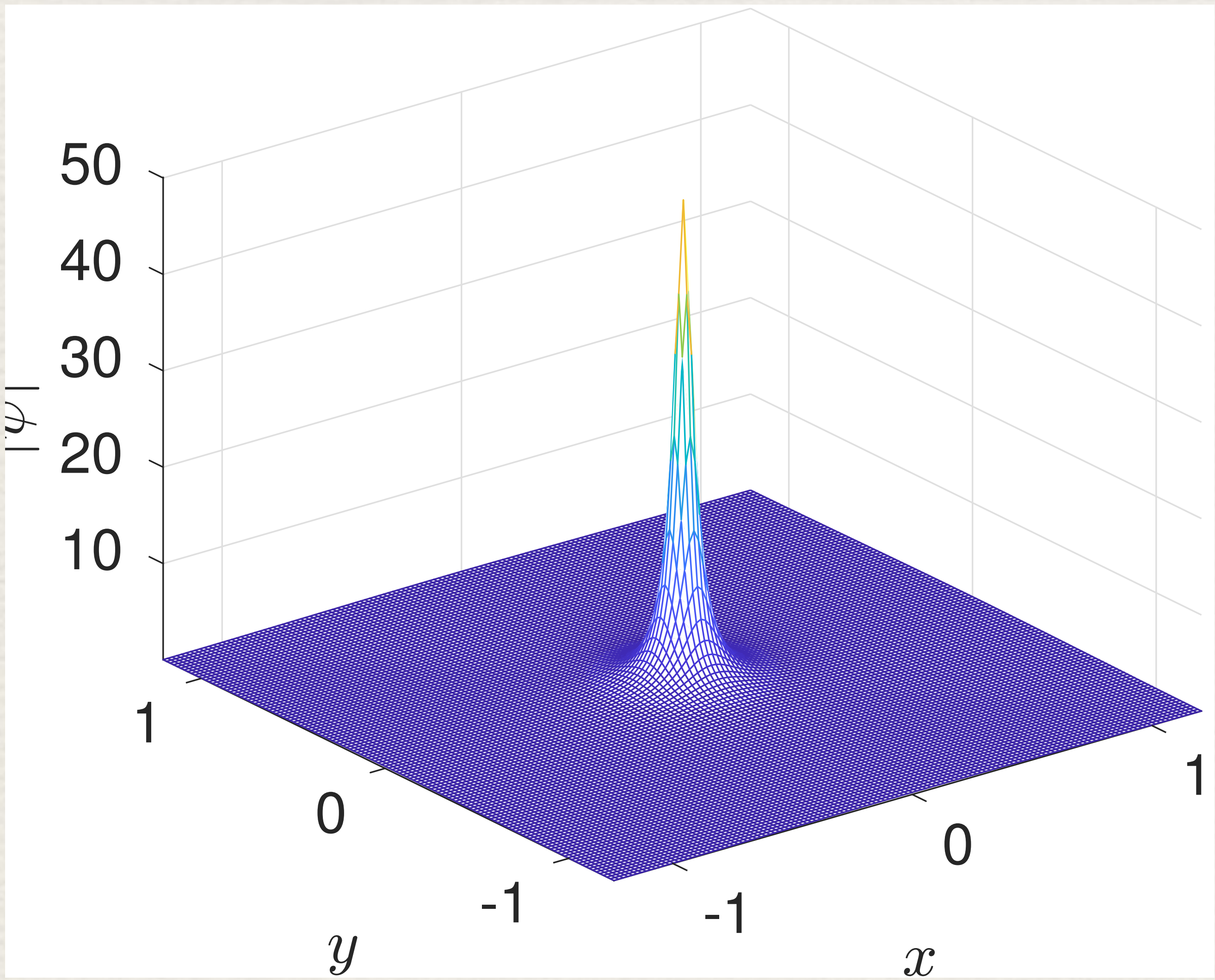
Perturbed lump

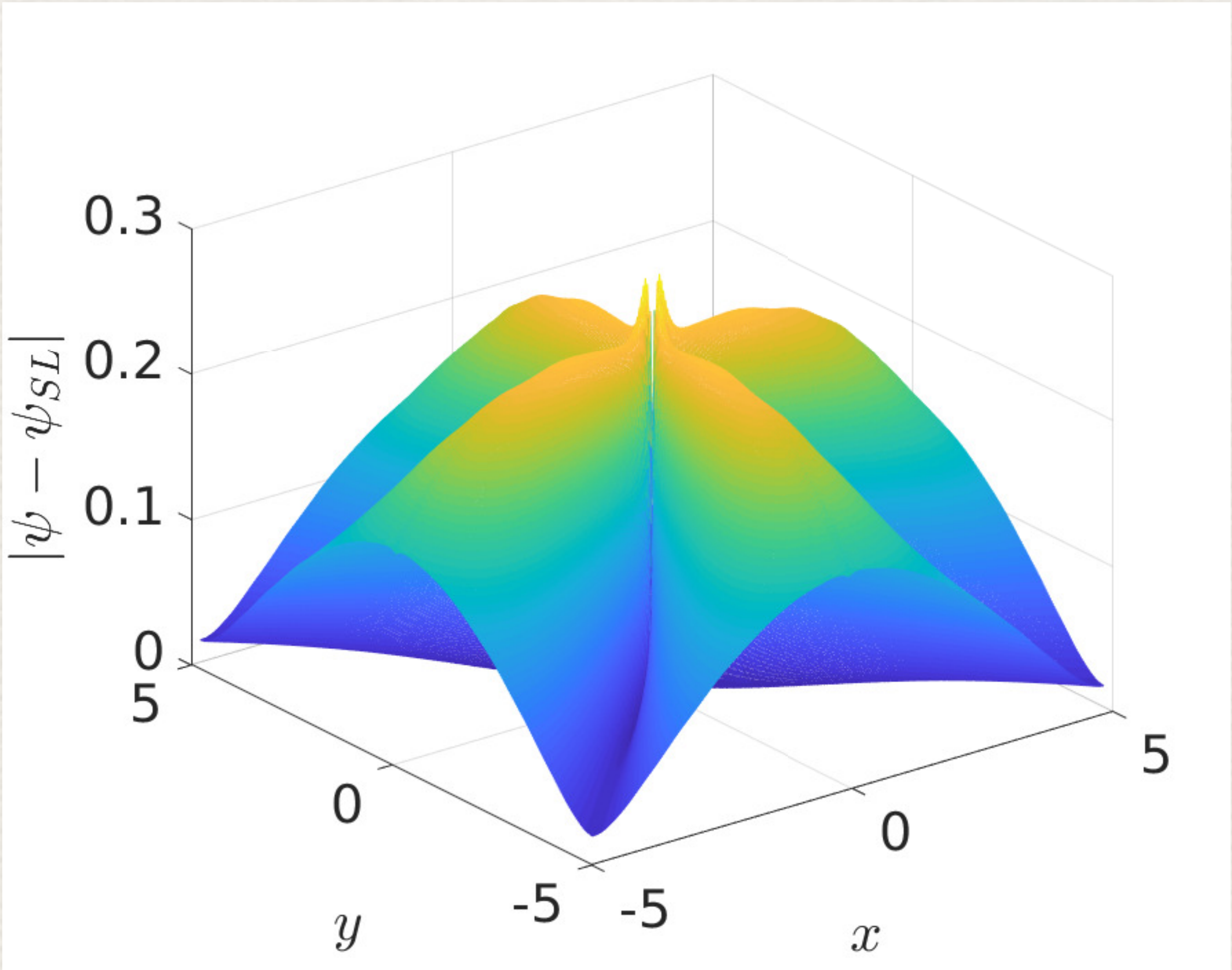
(multiplication with 1.1)





Blow-up profile





Conjecture

Conjecture 1.1. Consider initial data $\psi_0 \in C^\infty(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$ for the focusing DS II equation (1) with a single global maximum of $|\psi_0|$ such that the solution to DS II has a blow-up in finite time. Then the blow-up is self-similar according to (3) with a scaling factor $L(t)$ of the form (6) and the blow-up profile given by the lump, i.e.,

$$(7)\quad \psi(x,y,t) = \frac{P(X,Y)}{L(t)} + \tilde{\psi}, \quad P(X,Y) = \frac{2}{1+X^2+Y^2}, \quad L(t) \sim t^* - t,$$

where $\tilde{\psi}$ is bounded for all t.

Inverse scattering for DS II

- Dirac system of linear equations ($z = x + \mathrm{i}y$):

$$\epsilon\bar{\partial}\psi_1 = \frac{1}{2}q\psi_2$$
$$\epsilon\partial\psi_2 = \frac{1}{2}\overline{q}\psi_1$$

- complex geometrical optics solution, $k \in \mathbb{C}$:

$$\lim_{|z|\to\infty} \psi_1^\epsilon(z;k,t)\mathrm{e}^{-kz/\epsilon} = 1$$
$$\lim_{|z|\to\infty} \psi_2^\epsilon(z;k,t)\mathrm{e}^{-\overline{k}\overline{z}/\epsilon} = 0,$$

- reflection coefficient $R = R^\epsilon(k;t)$:

$$\mathrm{e}^{-kz/\epsilon}\overline{\psi_2^\epsilon(z;k,t)} = \tfrac{1}{2}R^\epsilon(k;t)z^{-1} + O(|z|^{-2}), \quad |z| \to \infty.$$

- time dependence

$$R^\epsilon(k;t) = R_0^\epsilon(k)\mathrm{e}^{4\mathrm{i}t\Re(k^2)/\epsilon}, \quad R_0^\epsilon(k) := R^\epsilon(k;0).$$

- Inverse transform

$$\nu_1 = \nu_1^\epsilon(k;z,t) := \mathrm{e}^{-kz/\epsilon}\psi_1 \quad \text{and} \quad \nu_2 = \nu_2^\epsilon(k;z,t) := \mathrm{e}^{-kz/\epsilon}\psi_2$$

satisfy

$$\begin{aligned} \epsilon\bar{\partial}_k\nu_1 &= \tfrac{1}{2}\overline{R^\epsilon(k;z,t)}\overline{\nu}_2 \\ \epsilon\bar{\partial}_k\nu_2 &= \tfrac{1}{2}\overline{R^\epsilon(k;z,t)}\overline{\nu}_1 \end{aligned}$$

where, writing $k = \kappa + \mathrm{i}\sigma$ for $(\kappa,\sigma) \in \mathbb{R}^2$,

$$\bar{\partial}_k := \frac{1}{2}\left(\frac{\partial}{\partial\kappa} + \mathrm{i}\frac{\partial}{\partial\sigma}\right),$$

asymptotic conditions

$$\lim_{|k|\to\infty} \nu_1^\epsilon(k;z,t) = 1 \quad \text{and} \quad \lim_{|k|\to\infty} \nu_2^\epsilon(k;z,t) = 0.$$

- inverse scattering problem

$$q^\epsilon(x,y,t) = 2\epsilon\overline{\left[\frac{\partial\psi_2}{\psi_1}\right]} = 2\epsilon\frac{\bar{\partial}\,\overline{\psi}_2}{\overline{\psi}_1} = 2\frac{\overline{k}\,\overline{\nu}_2 + \epsilon\bar{\partial}\,\overline{\nu}_2}{\overline{\nu}_1}.$$

Eikonal equation

- write $q = A\mathrm{e}^{S/\epsilon}$

$$\left[2\bar{\partial} f + \mathrm{i}\bar{\partial} S\right]\left[2\partial f - \mathrm{i}\partial S\right] = A^2,$$

with

$$\lim_{|z|\to\infty}\left(f + \frac{\mathrm{i}}{2}S - kz\right) = 0, \quad z = x + \mathrm{i}y.$$

- conjecture

$$\mathrm{e}^{-f(x,y;k)/\epsilon}\mathrm{e}^{-\mathrm{i}S(x,y)\sigma_3/(2\epsilon)}\boldsymbol{\psi}^\epsilon(x+\mathrm{i}y;k)$$
$$= \frac{\alpha_0(x,y;k)}{2k}\begin{bmatrix} 2\partial f(x,y;k) - \mathrm{i}\partial S(x,y) \\ A(x,y)\end{bmatrix} + o(1), \quad \epsilon \downarrow 0$$

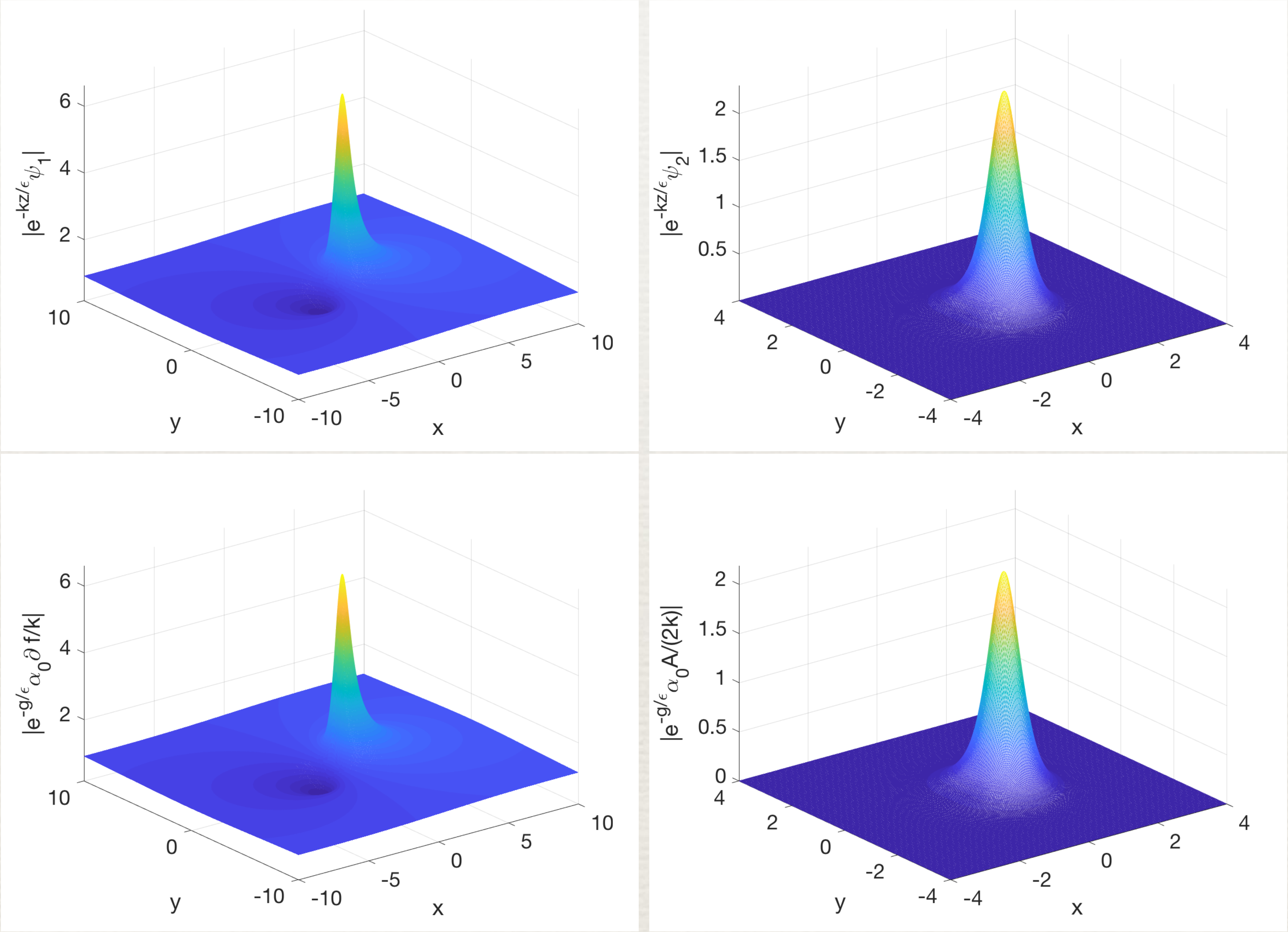


FIGURE 2. Comparison between the solution to the Dirac system (12)–(13) with Gaussian potential $e^{-(x^2+y^2)}$ for $k = 1$ and $\epsilon = 1/16$ with the WKB approximation. First row: the modulus of $e^{-kz/\epsilon}\psi_1$ (left) and of $e^{-kz/\epsilon}\psi_2$ (right). Second row: the corresponding WKB approximations of Conjecture 1.

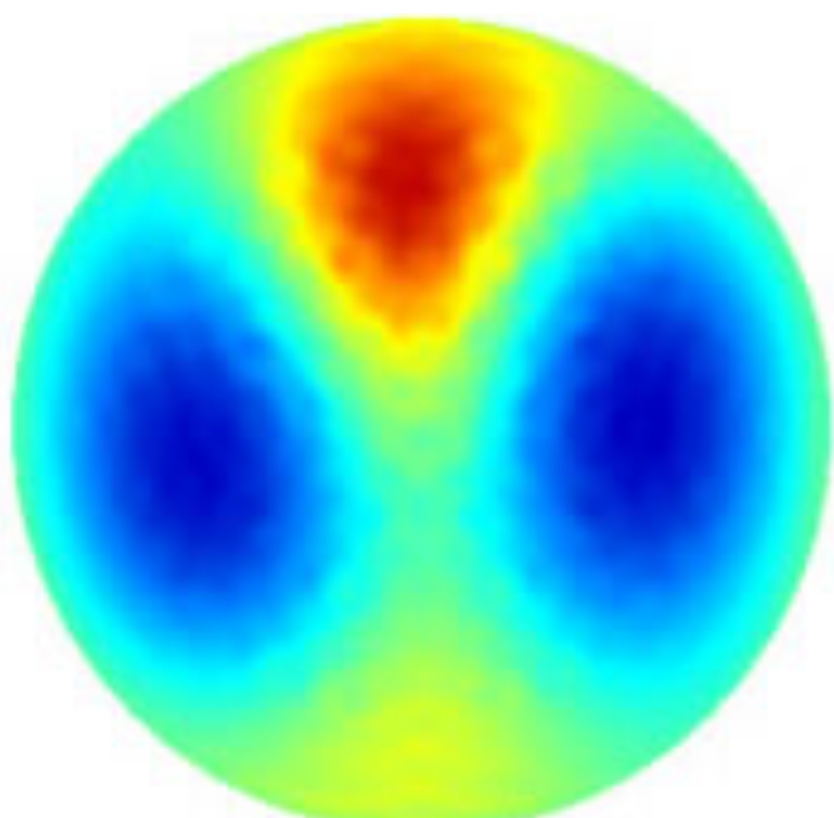
Applications

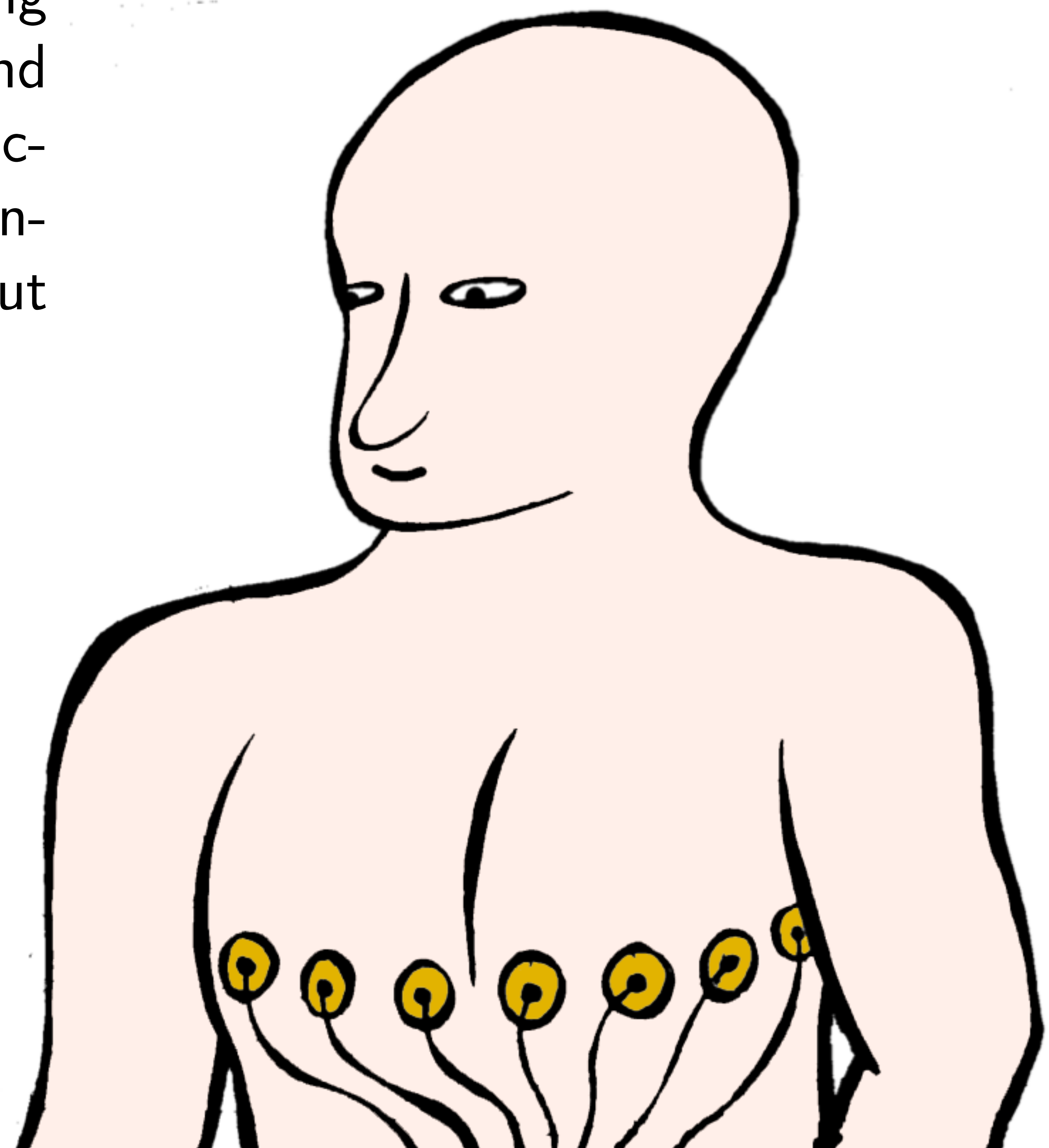
- Integrable systems
- Orthogonal polynomials
- Normal Matrix Models in Random Matrix Theory
- Electrical Impedance Tomography (EIT), Calderon's problem

- G. Uhlmann. Electrical impedance tomography and Calderóns problem. Inverse Problems, 25(12):123011, 2009.
- J.L. Mueller and S. Siltanen. *Linear and Nonlinear Inverse Problems with Practical Applications*, SIAM, 2012.
- C. Kenig, J. Sjöstrand, G. Uhlmann. The Calderón problem with partial data. Annals of Mathematics 165 (2007), 567-591.

The most successful application of EIT is chest imaging

Medical applications: monitoring cardiac activity, lung function, and pulmonary perfusion. Also, electrocardiography (ECG) can be enhanced using knowledge about conductivity distribution.





Reformulation of the d-bar problem

- functions with simple asymptotics: $\Phi_1 = e^{-kz}\psi_1$, $\Phi_2 = e^{-\bar{k}\bar{z}}\psi_2$

$$\bar{\partial}\Phi_1 = \frac{1}{2} q e^{\bar{k}\bar{z} - kz} \Phi_2,$$

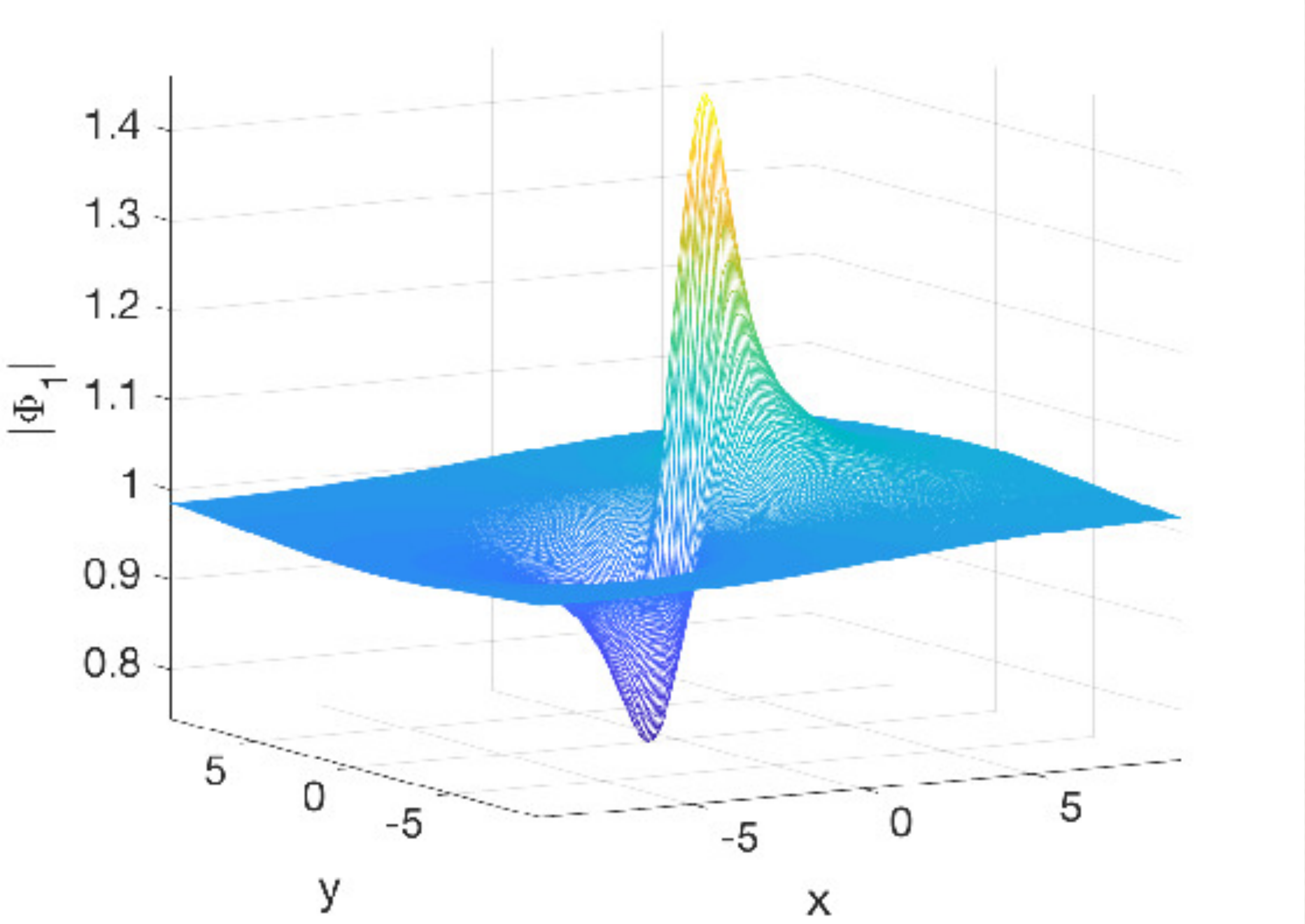
$$\partial\Phi_2 = \frac{1}{2} \bar{q} e^{kz - \bar{k}\bar{z}} \Phi_1.$$

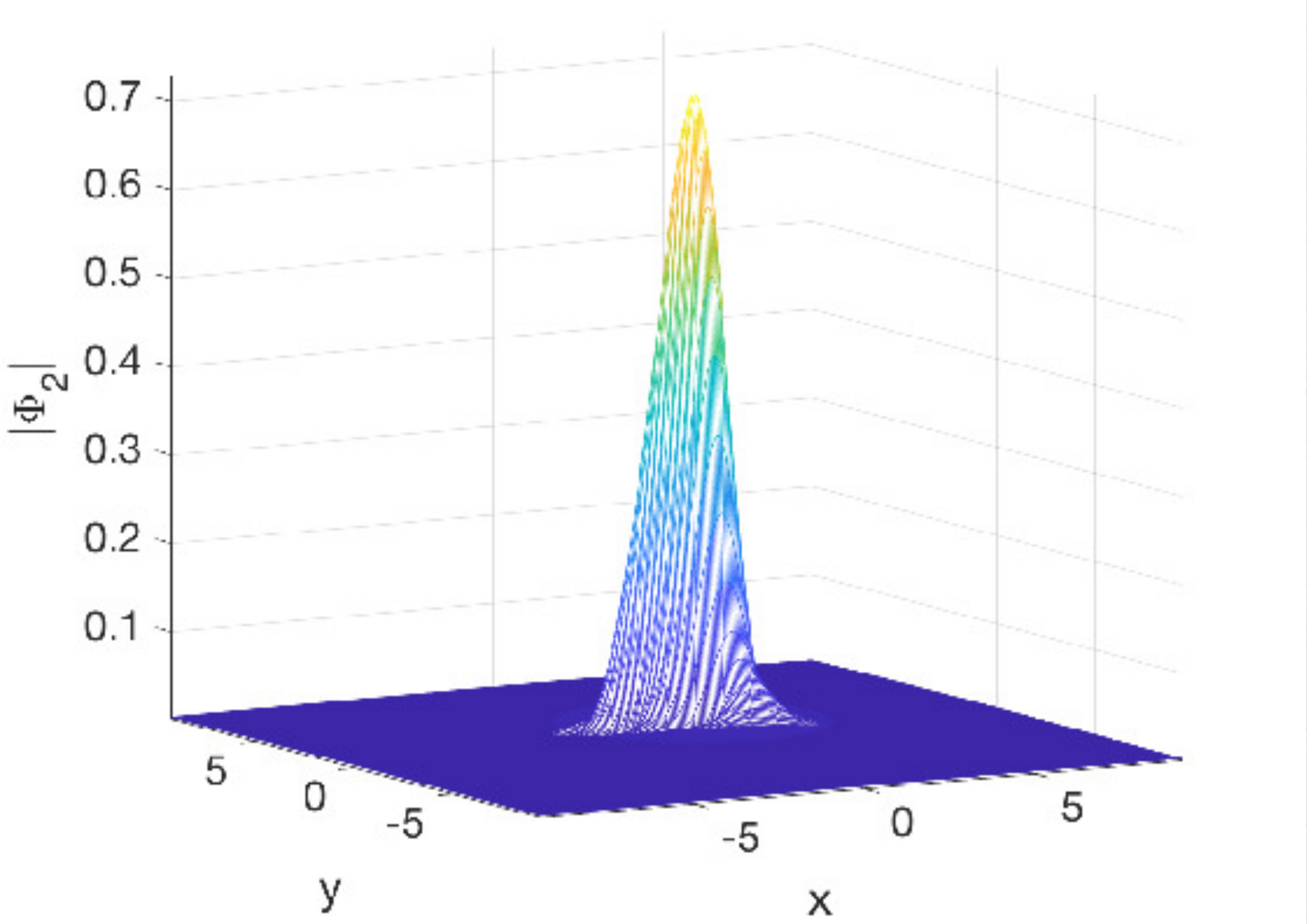
- diagonal form, vanishing functions at infinity: $m^{\pm} = \Phi_1 \pm \bar{\Phi}_2 - 1$

$$\bar{\partial} m^{\pm} = \frac{1}{2} q e^{\bar{k}\bar{z} - kz} (\overline{m}^{\pm} + 1)$$

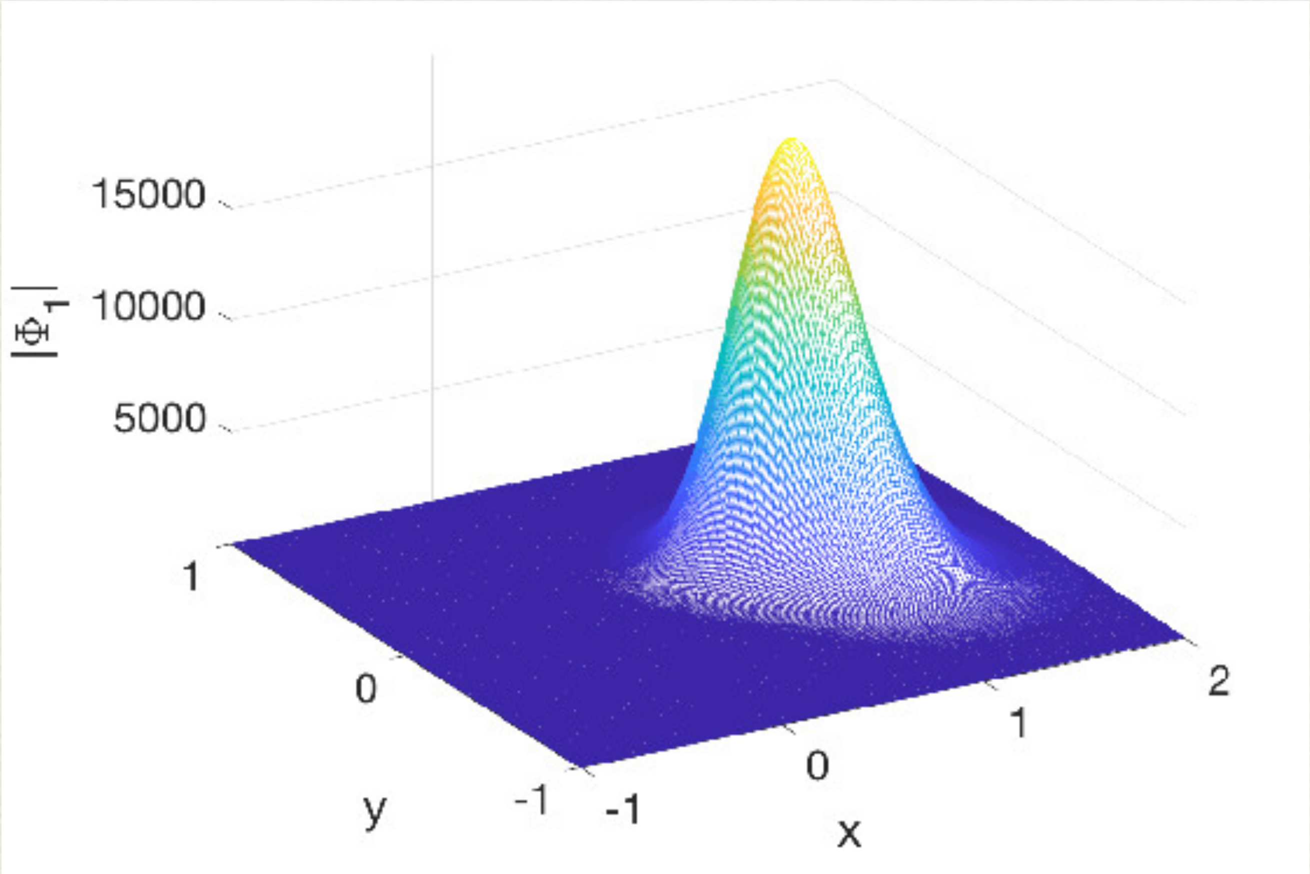
$$q = \exp(-x^2 - 3xy - 5y^2)$$

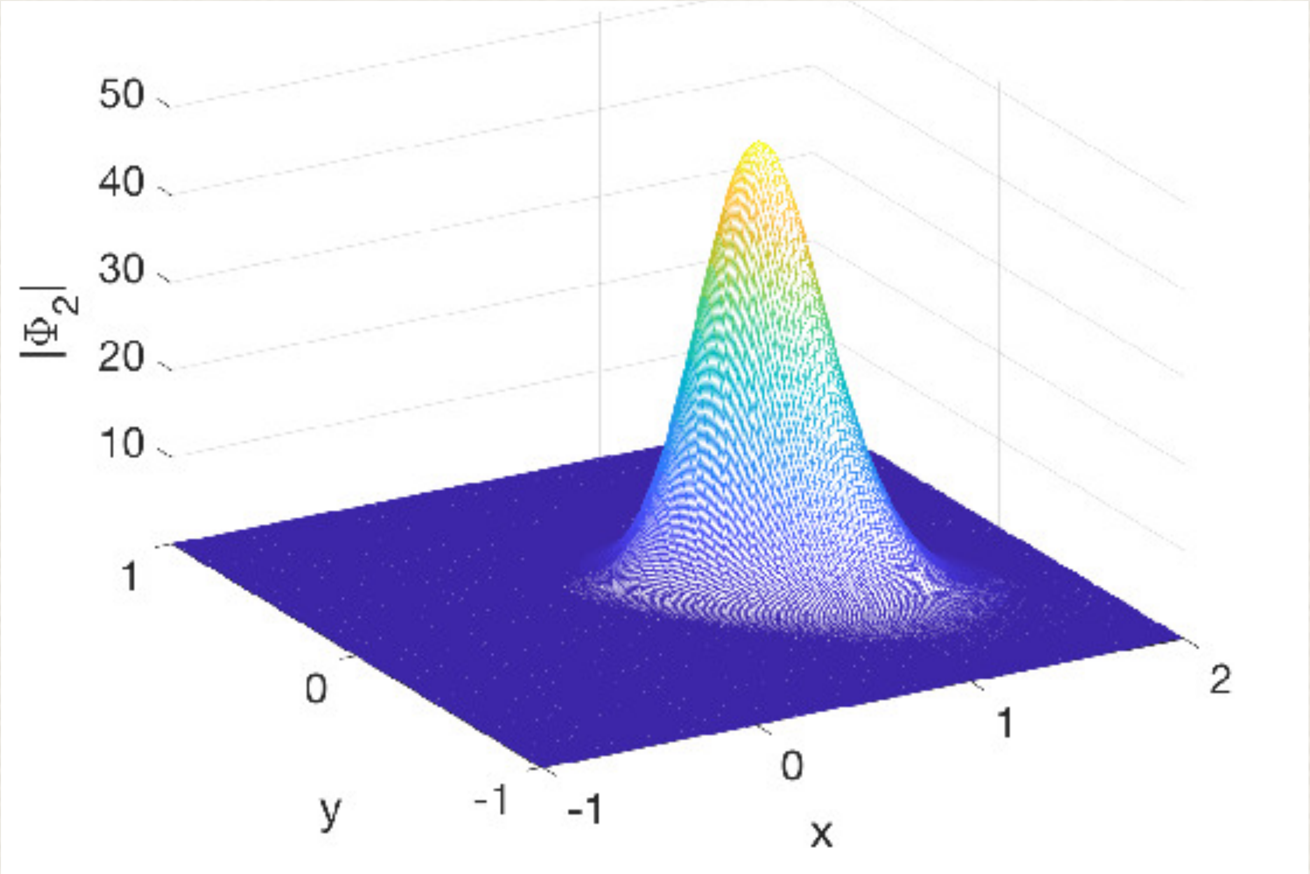
$$k = 1, \quad \epsilon = 1/4$$



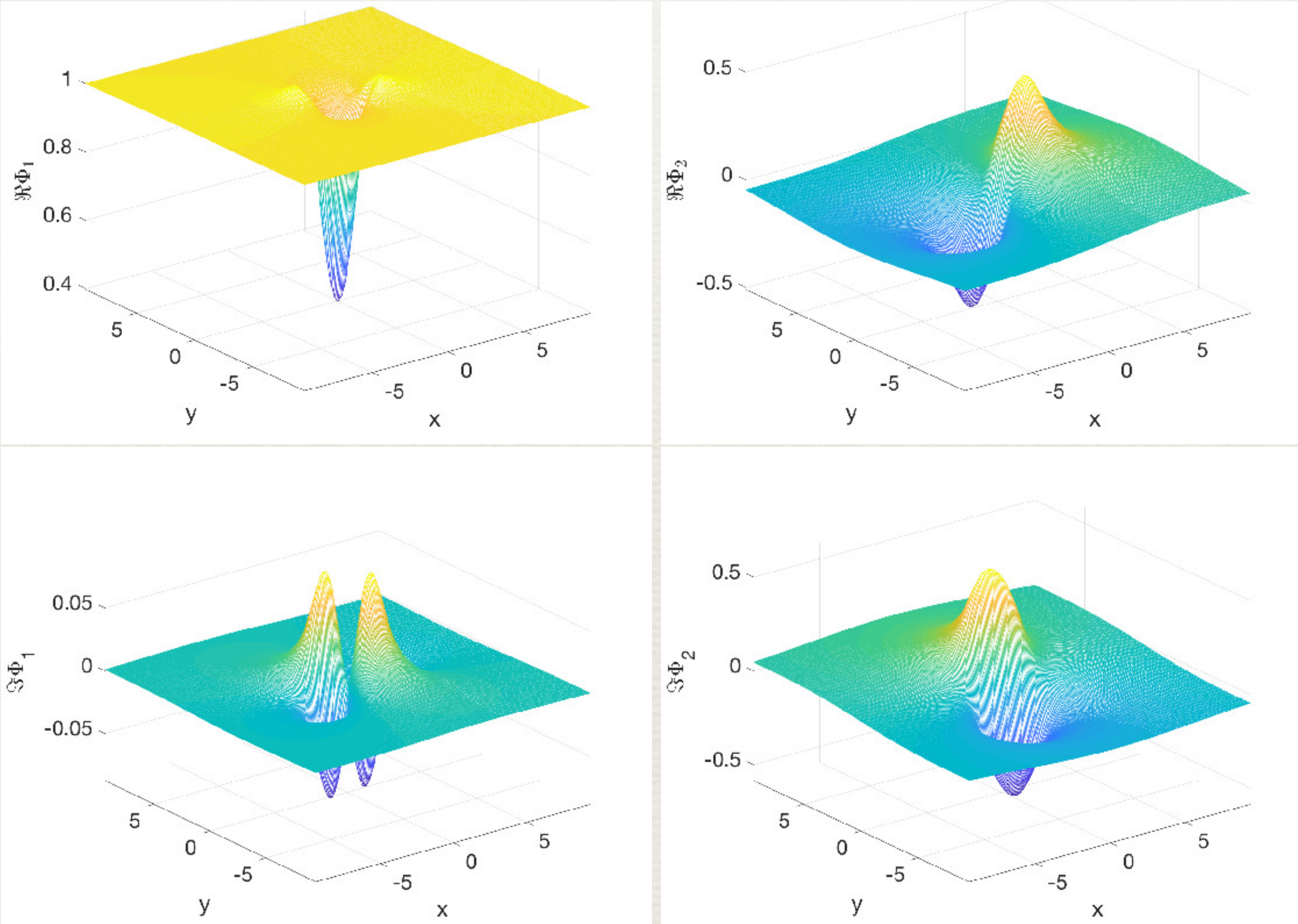


$$k = 1, \quad \epsilon = 1/128$$

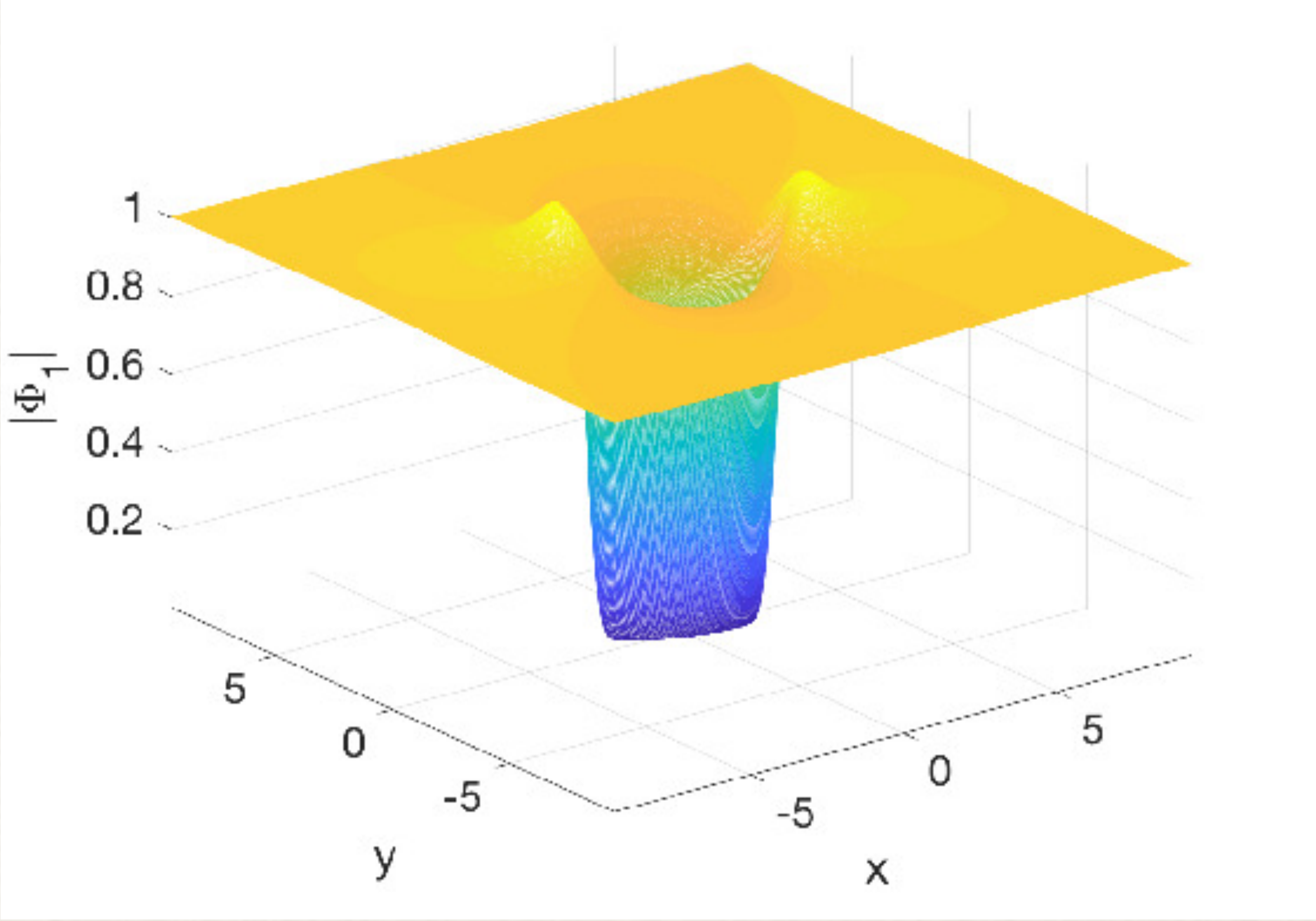


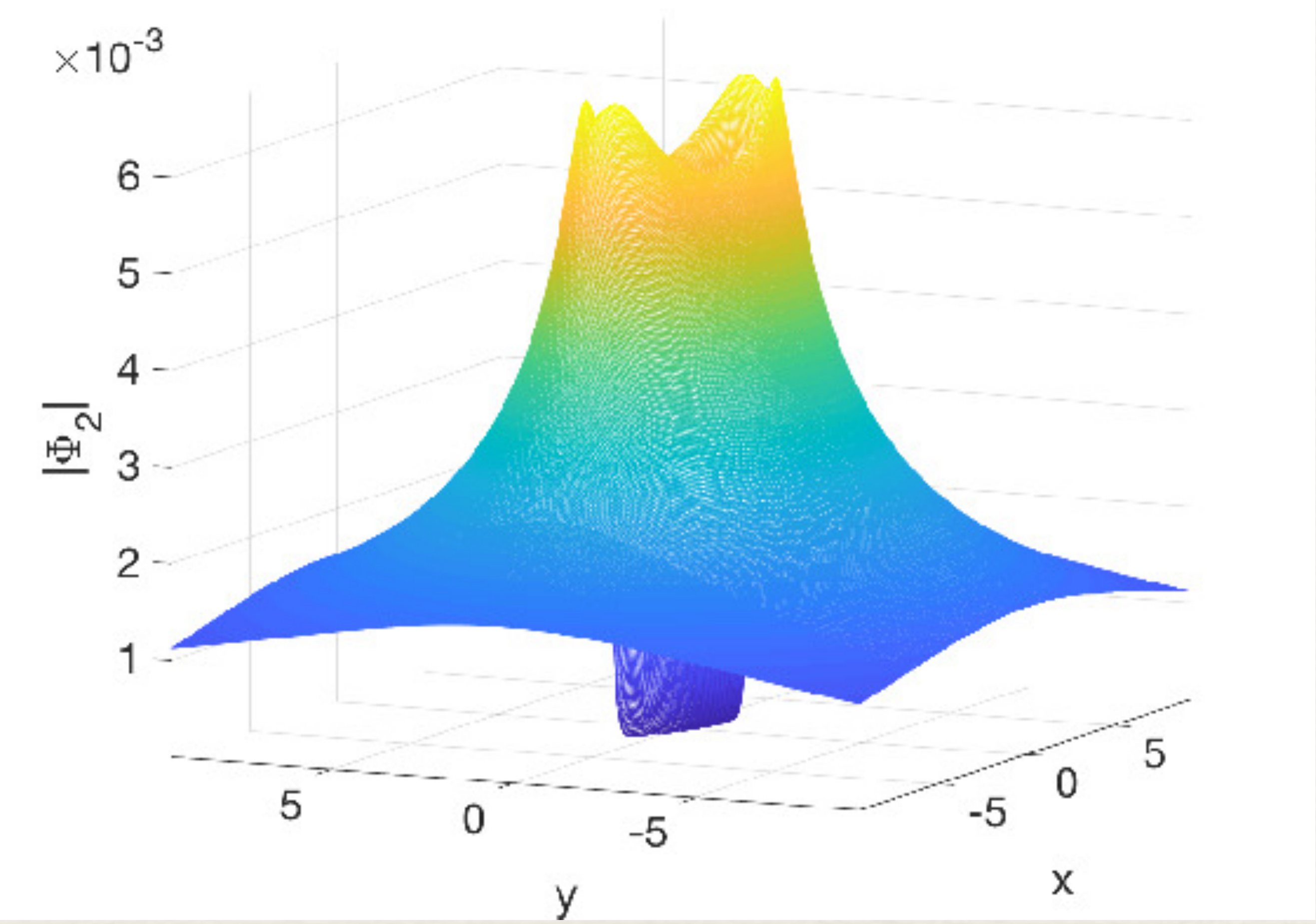


$$k = 0, \quad \epsilon = 1/4$$



$$k = 0, \quad \epsilon = 1/128$$





Compact support

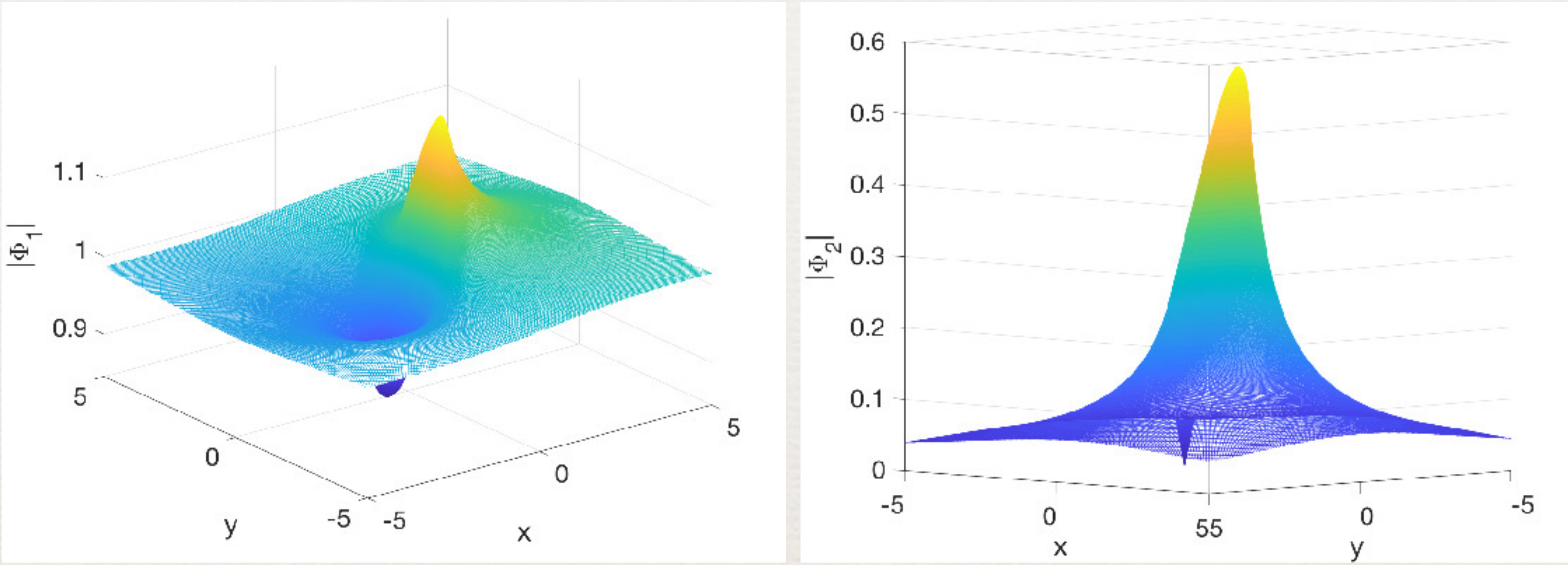
- Potential with compact support on a simply connected domain, assume biholomorphic map to the unit disk (Hyvönen, Päivärinta, Tamminen 2017)

- use polar coordinates $z = re^{i\varphi}$, $k = \kappa e^{i\psi}$

$$\left(\partial_r + \frac{i}{r}\partial_\varphi\right)\Phi_1 = q\exp\left(-2i\kappa r\cos(\varphi-\psi) - i\varphi\right)\Phi_2$$

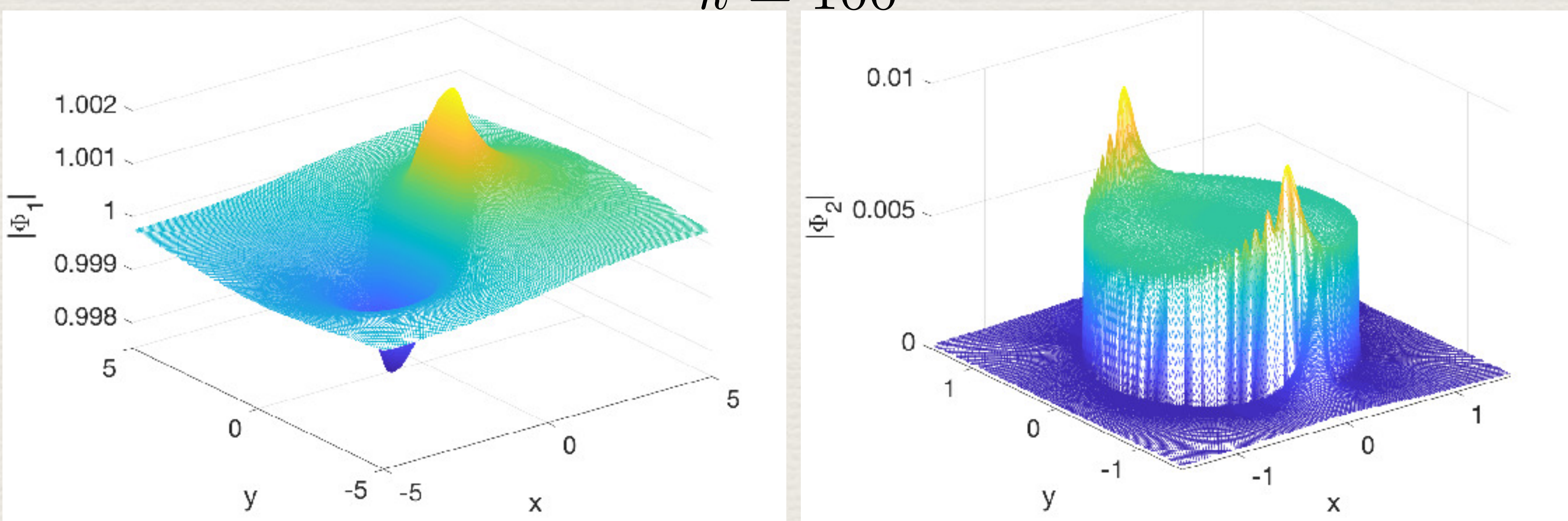
$$\left(\partial_r - \frac{i}{r}\partial_\varphi\right)\Phi_2 = \bar{q}\exp\left(2i\kappa r\cos(\varphi-\psi) + i\varphi\right)\Phi_1$$

$k = 1$



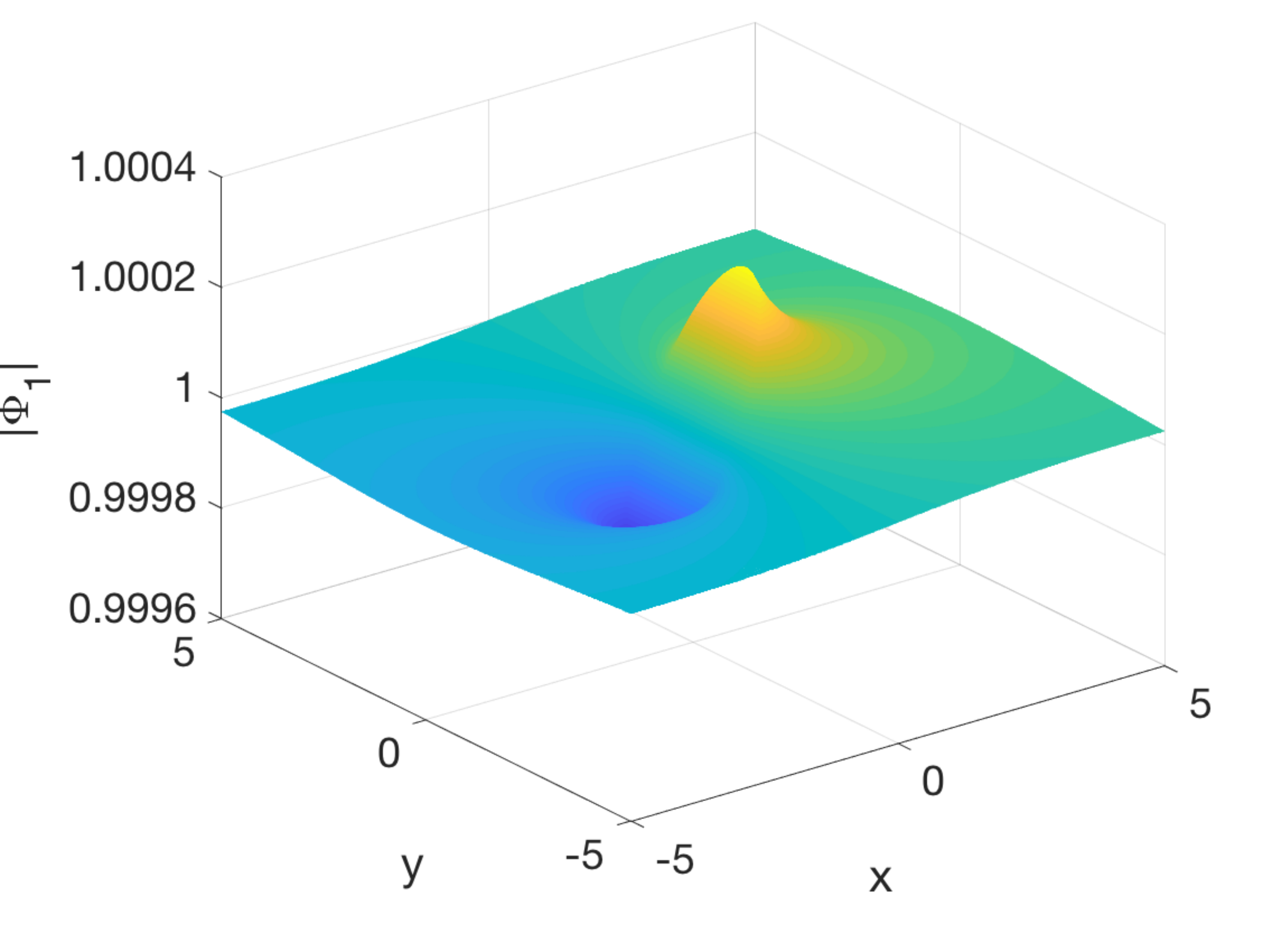
1.1
1
0.9
|Φ1|
5
0
-5
y
-5
0
5
x
0.6
0.5
0.4
0.3
0.2
0.1
0
|Φ2|
-5
0
55
0
-5
x
y

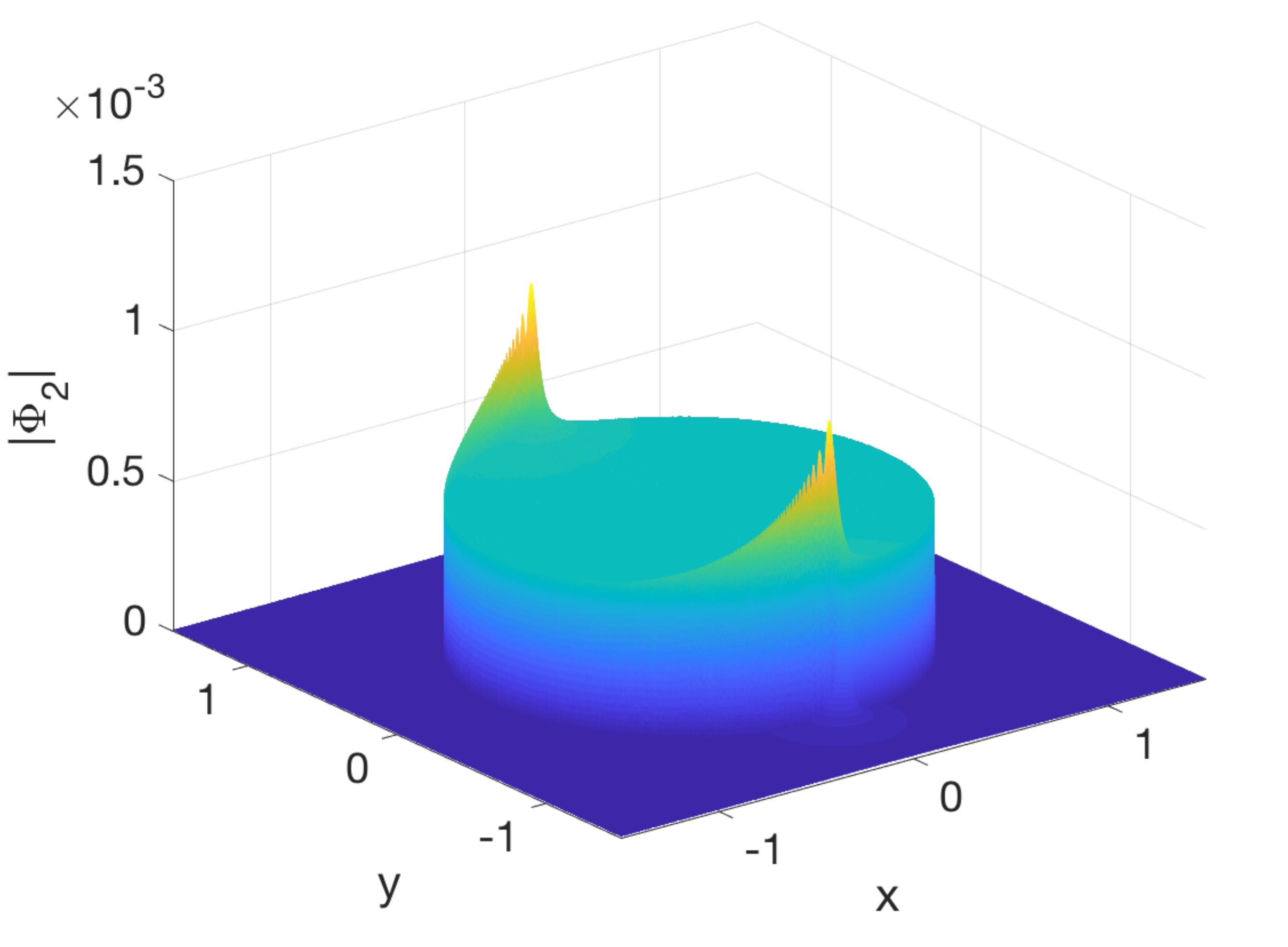
$k = 100$



1.002
1.001
1
0.999
0.998
|Φ1|
5
0
-5
y
-5
0
5
x
0.01
0.005
0
|Φ2|
1
0
-1
y
-1
0
1
x

k = 1000


1.0004
1.0002
1
0.9998
0.9996
$|\Phi_1|$
5
0
-5
y
-5
0
5
x


$\times 10^{-3}$
1.5
1
0.5
0
$|\Phi_2|$
1
0
-1
y
-1
0
1
x

Large |k| asymptotics

- Sjöstrand (2018): iterative solution of the d-bar system, initial iterate $\phi_1^0 = 1$, $\phi_2^0 = 0$

$$\phi_1 - \frac{1}{2}\bar{\partial}^{-1}(qe^{\bar{k}\bar{z}-kz}\phi_2) = \bar{\partial}^{-1}\Psi_1$$
$$\phi_2 - \frac{1}{2}\partial^{-1}(\bar{q}e^{kz-\bar{k}\bar{z}}\phi_1) = \partial^{-1}\Psi_2$$

  of the form $(1-\mathcal{K})\mathbf{\Phi} = \mathbf{\Psi}$

- Hörmander's solution of the d-bar equation with Carleman estimates in weighted L^2 spaces, $h = 1/|k| \ll 1$.

- Operator $\mathcal{K} = \mathcal{O}(1)$, but $\mathcal{K}^2 = \mathcal{O}(1/|k|)$. Therefore

$$(1-\mathcal{K}^2)\mathbf{\Phi} = (1+\mathcal{K})\mathbf{\Psi},$$

  convergence as the geometric series.

Theorem

Let $q \in \langle\cdot\rangle^{-2} H^s$ for some $s \in]1,2]$ and fix $\epsilon \in]0,1]$.
Then $\mathcal{K} = \mathcal{O}(1) : (\langle\cdot\rangle^{\epsilon} L^2)^2 \to (\langle\cdot\rangle^{\epsilon} L^2)^2$,

$$\mathcal{K}^2 = \mathcal{O}(h^{s-1}) : (\langle\cdot\rangle^{\epsilon} L^2)^2 \to (\langle\cdot\rangle^{\epsilon} L^2)^2.$$

For $h_0 > 0$ small enough and $0 < h \leq h_0$,
$1 - \mathcal{K} : (\langle\cdot\rangle^{\epsilon} L^2)^2 \to (\langle\cdot\rangle^{\epsilon} L^2)^2$ has a uniformly bounded inverse.

Characteristic function of a compact domain

Proposition:
Let q be the characteristic function of a strictly convex open set $\Omega \Subset \mathbb{C}$ with smooth boundary and fix $\epsilon \in]0,1]$. Then the conclusions of the theorem hold with $s = 3/2$. In particular

$$\mathcal{K}^2 = \mathcal{O}(h^{1/2}).$$

Computation of an integral

- Leading order contribution to ϕ_2:

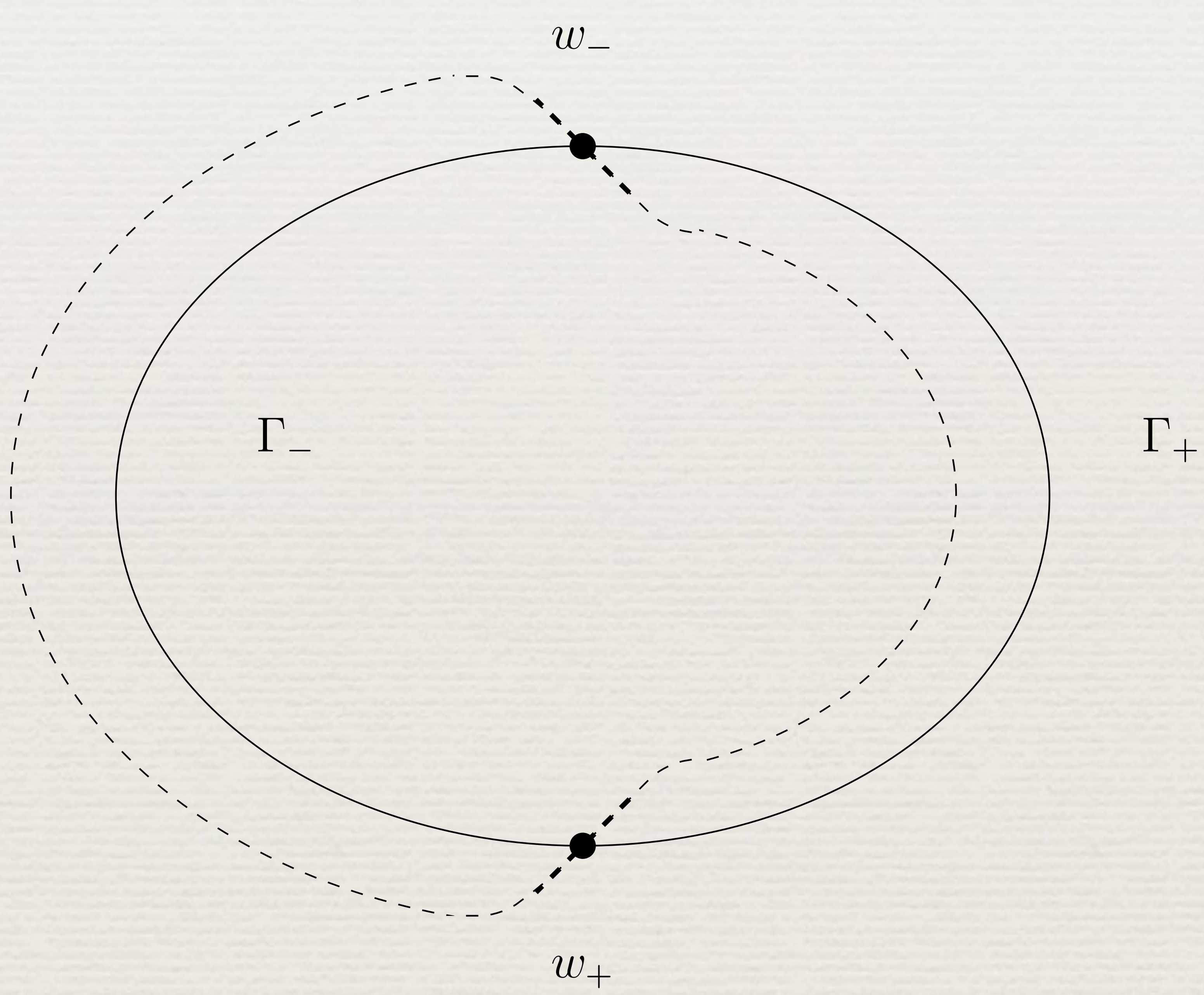
$$f(z,k) = \int_\Omega \frac{1}{z-w} e^{\overline{kw}-kw} L(dw) = \iint_\Omega \frac{e^{\overline{kw}-kw}}{z-w} \frac{d\overline{w}\wedge dw}{2i}, \qquad (1)$$

- Stokes formula

$$\frac{1}{2i\overline{k}} \int_{\partial\Omega} \frac{1}{z-w} e^{\overline{kw}-kw} dw = \iint_\Omega \frac{e^{\overline{kw}-kw}}{z-w} \frac{d\overline{w}\wedge dw}{2i} - \begin{cases} 0 \text{ if } z \notin \Omega, \\ \frac{\pi}{\overline{k}} e^{\overline{kz}-kz}, \text{ if } z \in \Omega. \end{cases}$$

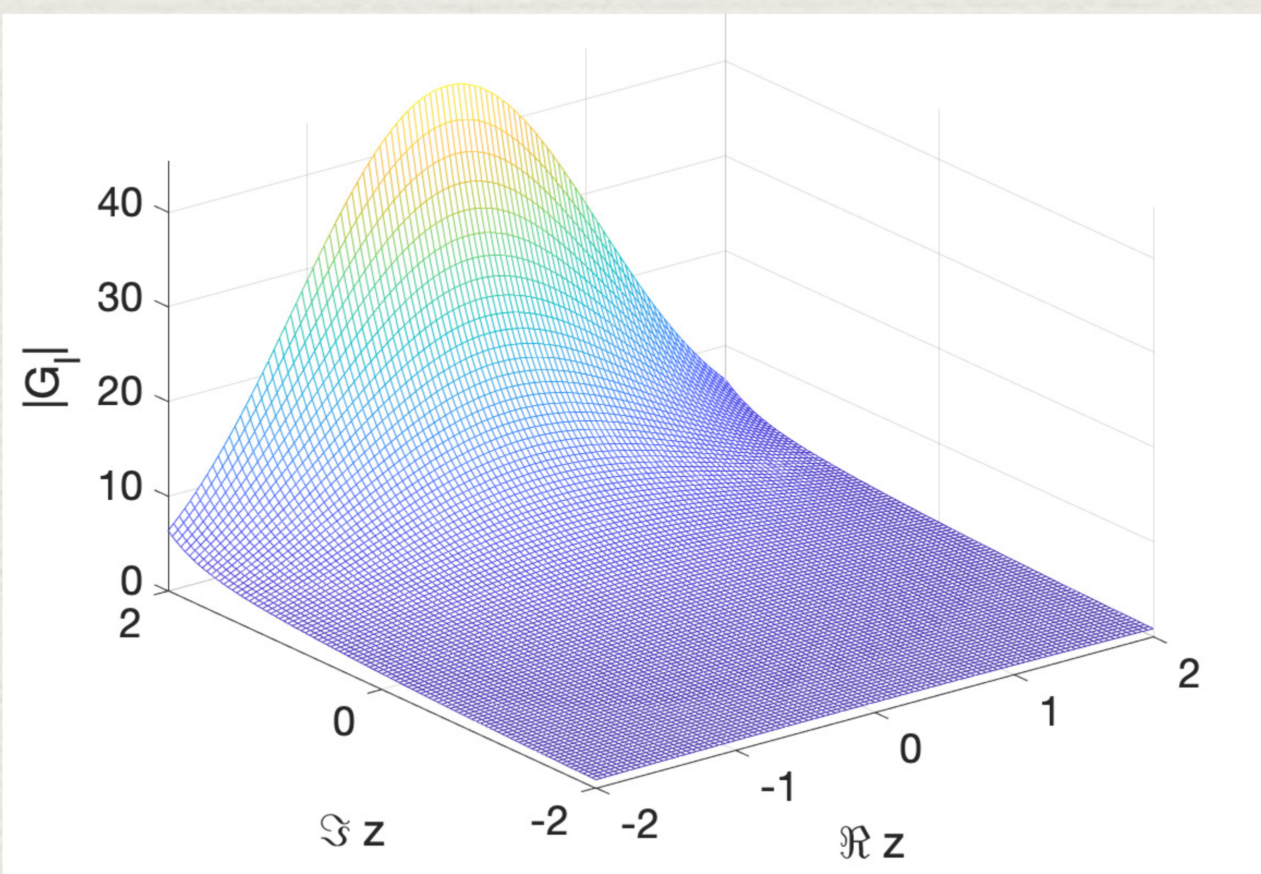
- analytical continuation and deformation of the integration contour, stationary phase approximation.

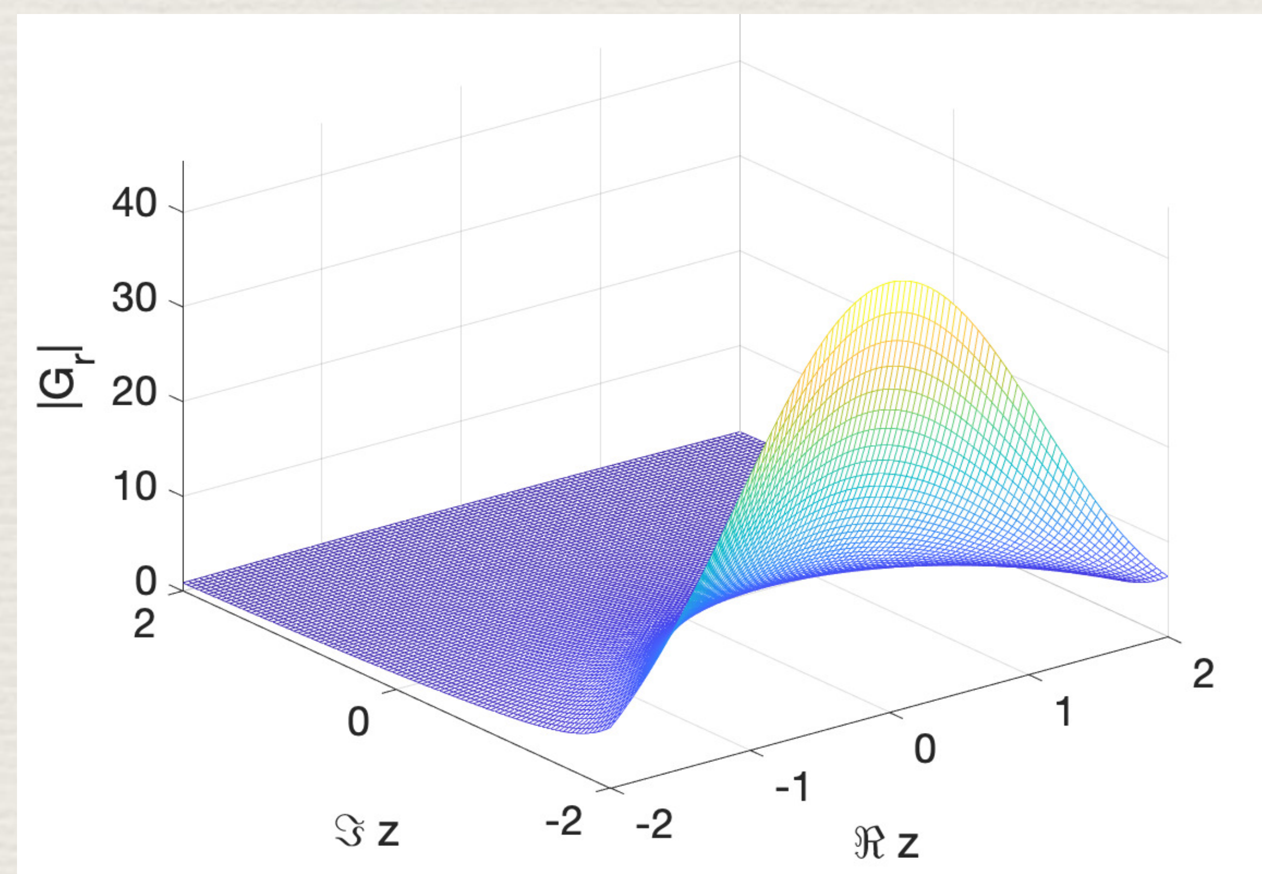
Deformed contour



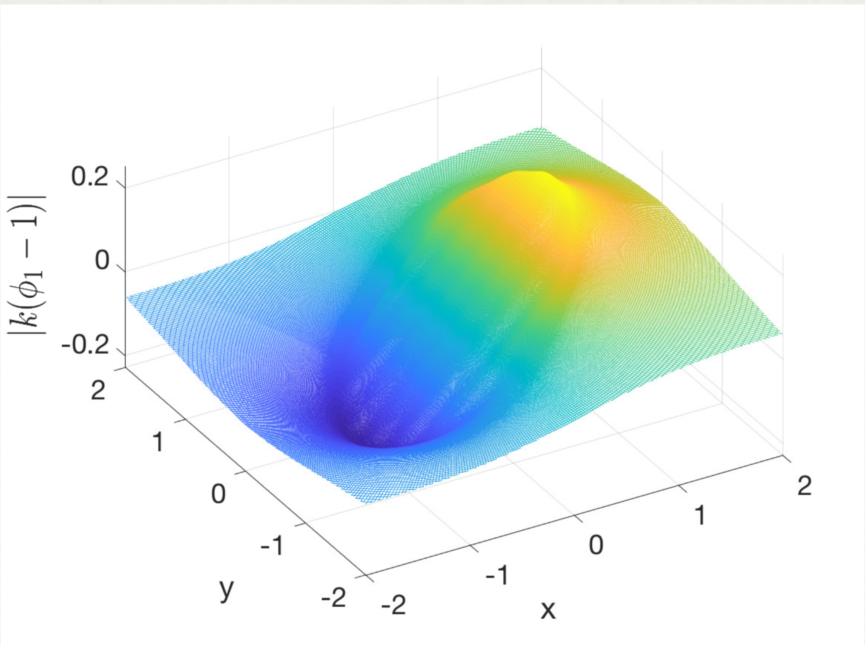
Transcendental function near north and south pole

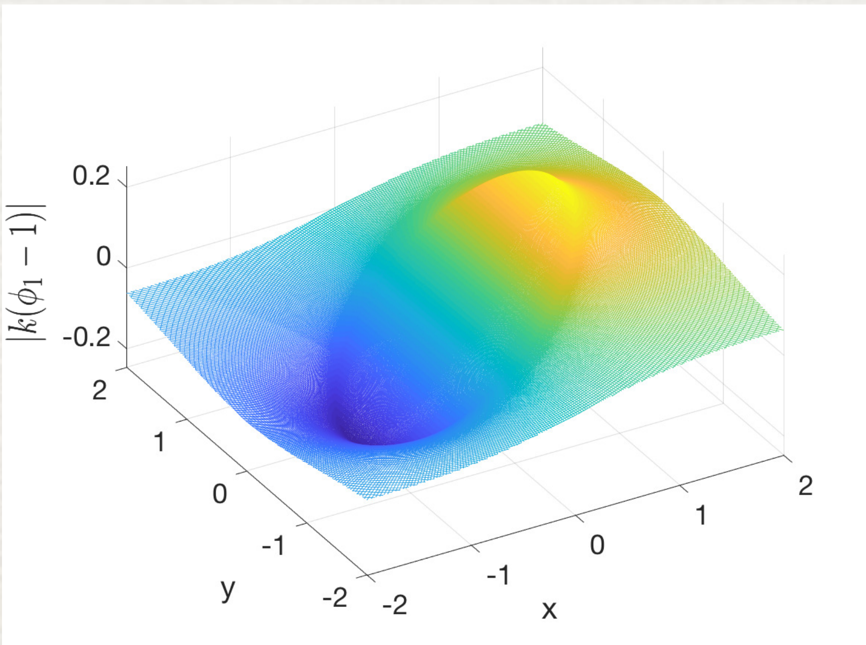
$$G(z) := \int_{-\infty}^{\infty} \frac{e^{-t^2/2}}{z-t} dt$$

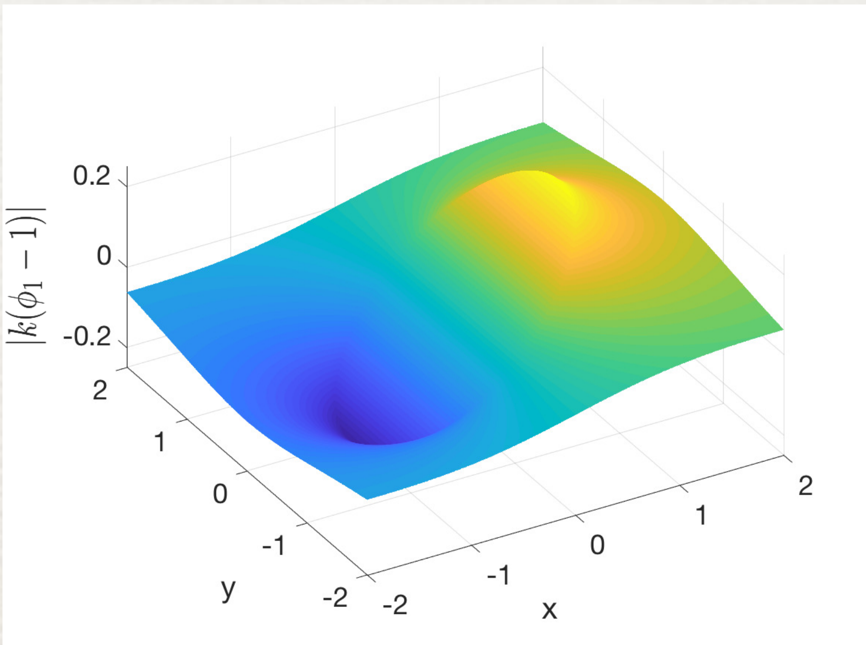


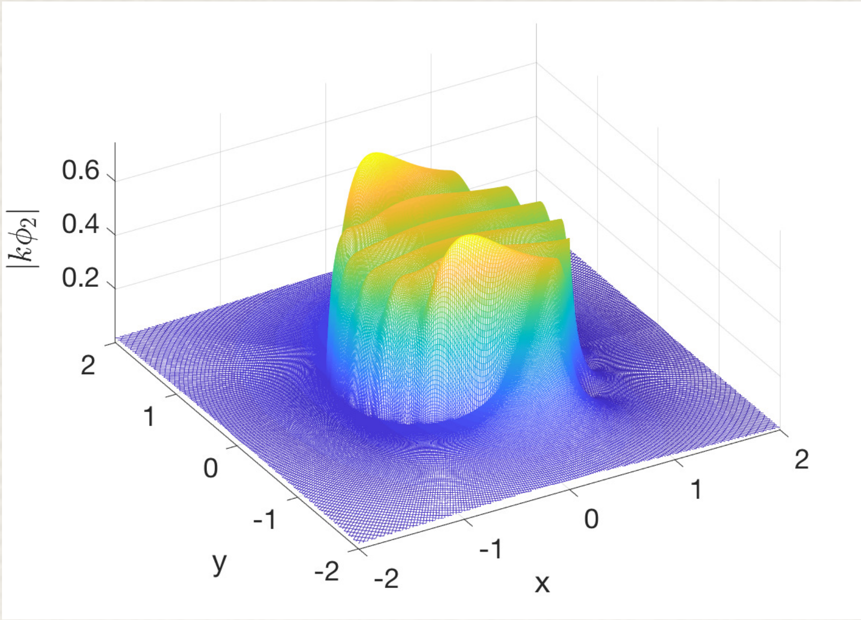


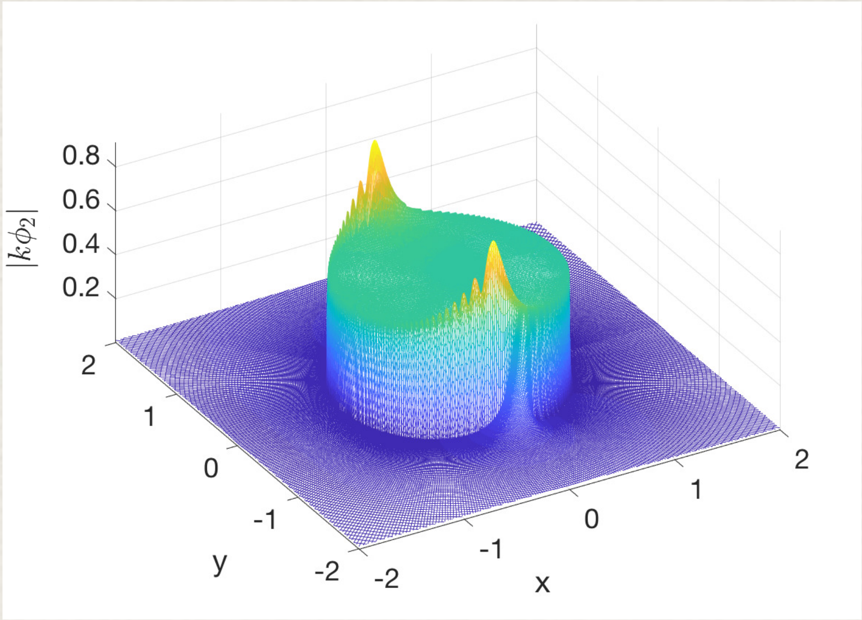
Characteristic function of the disk

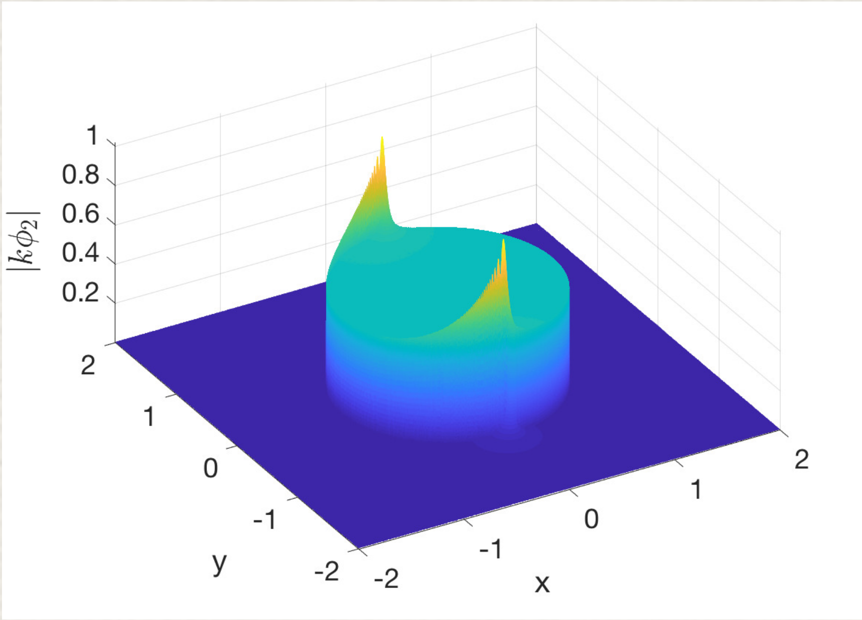




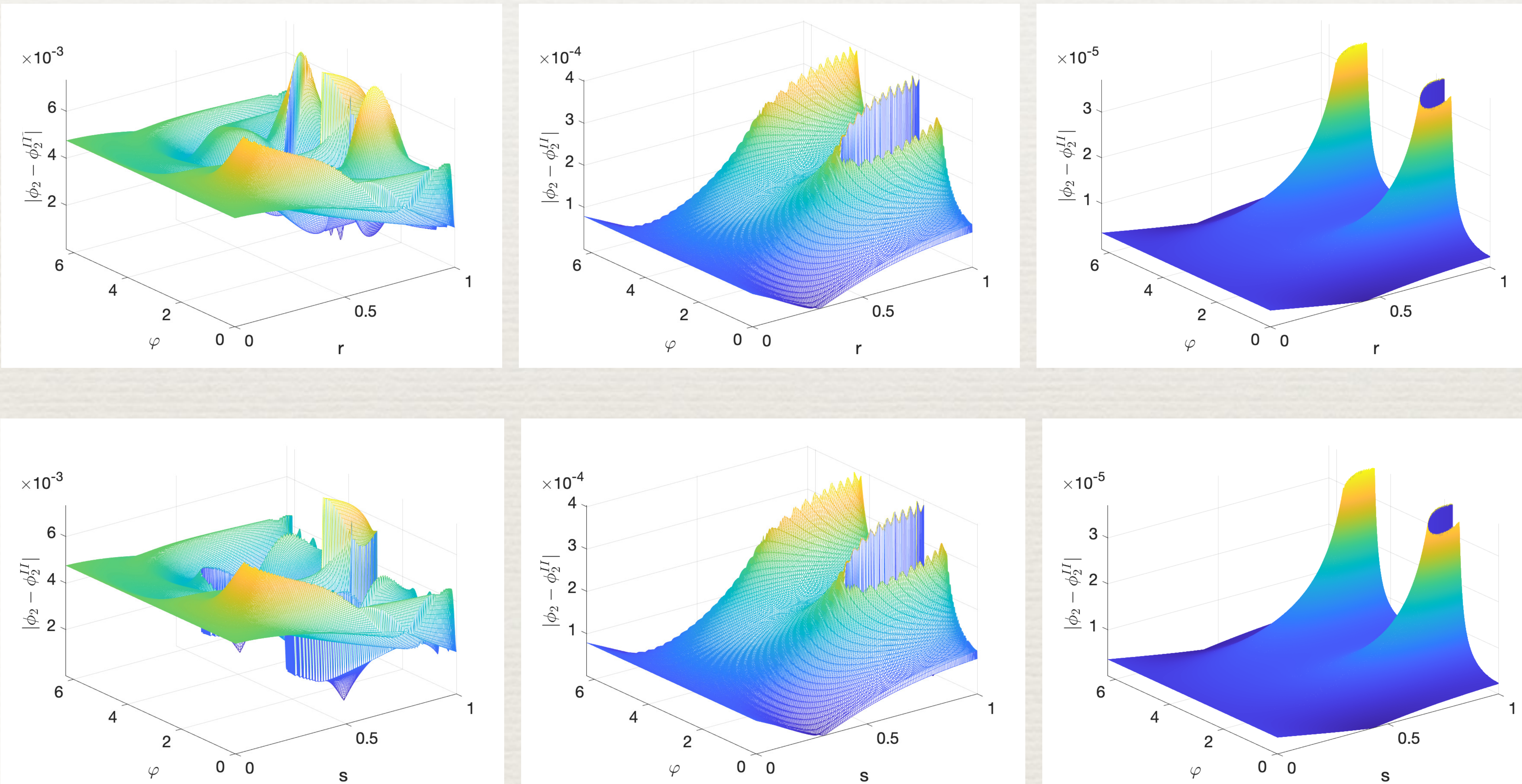








Asymptotic formulae



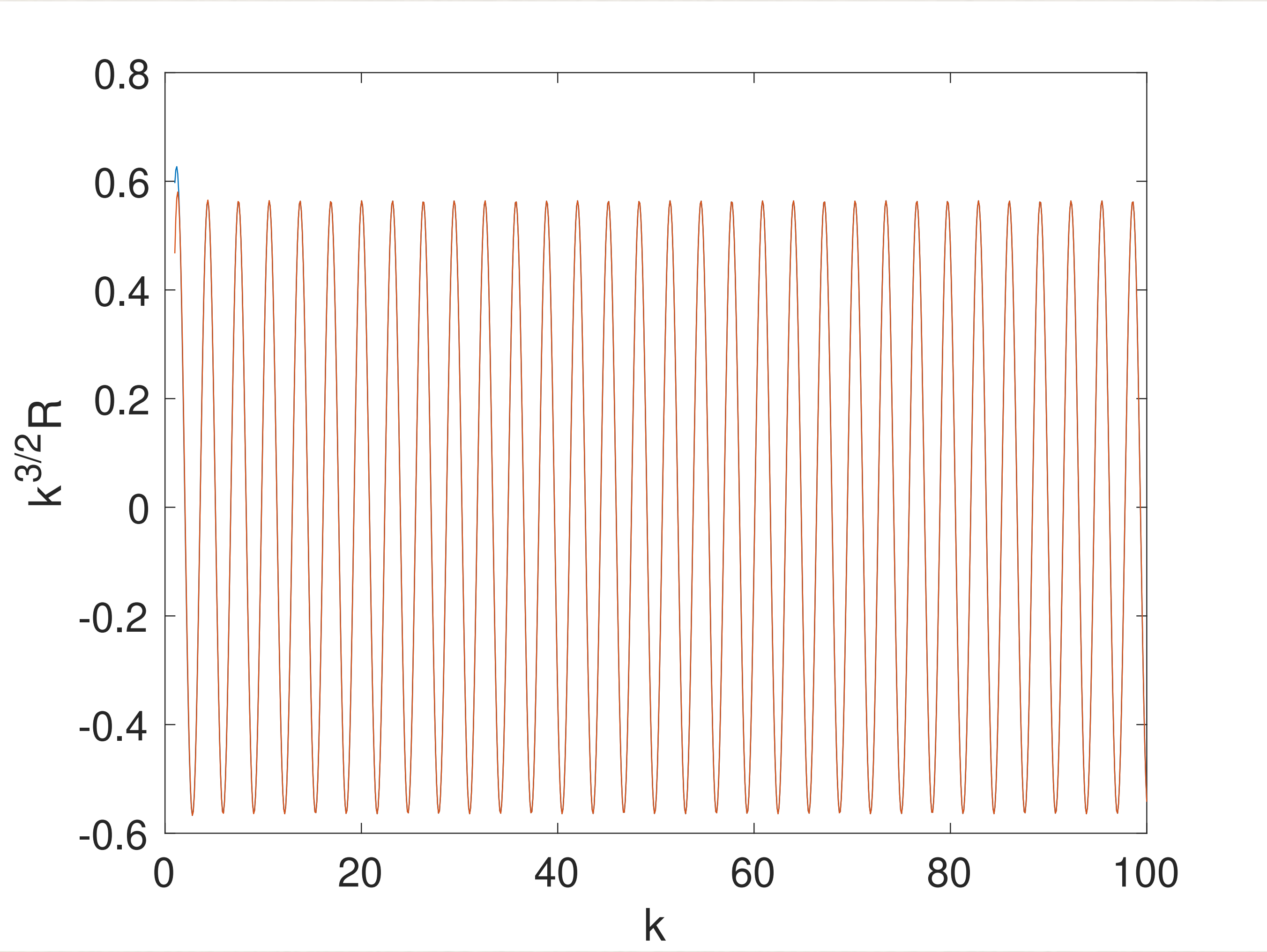
Reflection coefficient

- **Proposition** (Sjöstrand 2021):

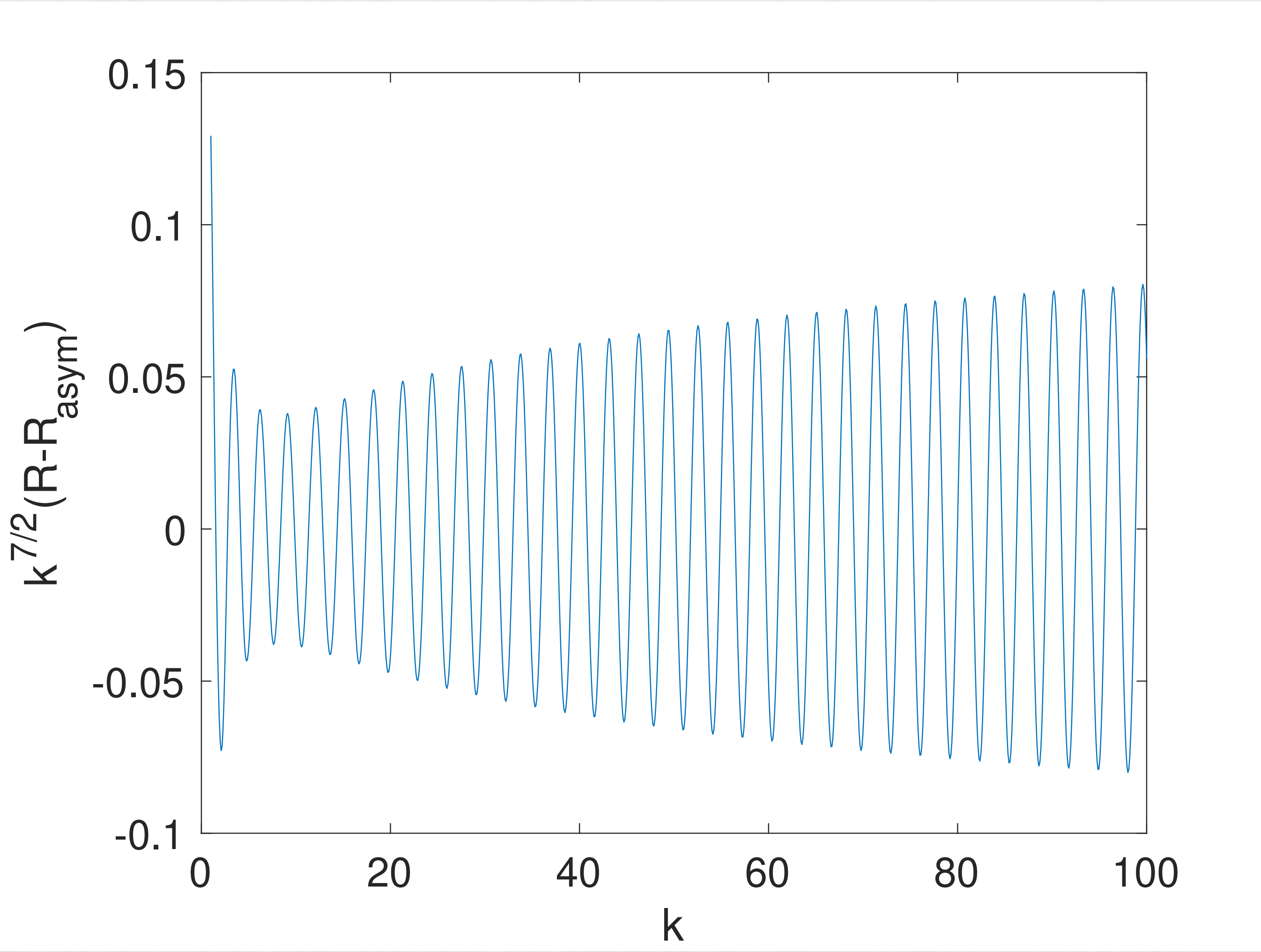
$$\phi_1 = 1 + \frac{\bar{z}}{4k} + \mathcal{O}(|k|^{-2})$$

$$\bar{R} = \frac{2}{\pi}\int_{|z|\le 1} \mathrm{e}^{kw-\bar{k}\bar{w}}\phi_1 d^2w \approx \frac{2}{\pi}\int_{|z|\le 1} \mathrm{e}^{kw-\bar{k}\bar{w}}\left(1+\frac{\bar{w}}{4k}\right) d^2w. \tag{1}$$

$$R \approx R_{asym} := \frac{1}{\sqrt{\pi k^3}}\left(\sin(2k-\pi/4) - \frac{5}{16k}\cos(2k-\pi/4)\right). \tag{2}$$



0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
0
20
40
60
80
100
k
k^3/2R



0.15
0.1
0.05
0
-0.05
-0.1
0
20
40
60
80
100
k
k^7/2(R-R_asym)

Outlook

- conformal transformations of compact domains with analytic boundary to the circle
- d-bar problems for potentials with algebraic decay, hybrid approaches
- DS solution for the disk
- focusing DS, exceptional points
- blow-up in DS I solutions

- C. Klein, J. Sjöstrand, N. Stoilov, Large $|k|$ behavior of d-bar problems for domains with a smooth boundary, http://arxiv.org/abs/2010.04423
- C. Klein, J. Sjöstrand, N. Stoilov, Large $|k|$ behavior of complex geometric optics solutions to d-bar problems, accepted for publication in '"Partial Differential Equations, Spectral Theory, and Mathematical Physics: The Ari Laptev Anniversary Volume' (edited by Pavel Exner, Rupert L. Frank, Fritz Gesztesy, Helge Holden and Timo Weidl), EMS Press, https://arxiv.org/abs/2009.06909
- C. Klein, K. McLaughlin, N. Stoilov, High precision numerical approach for Davey-Stewartson II type equations for Schwartz class initial data, Proc. Royal Soc. A 476, https://doi.org/1098/rspa.2019.0864
- C. Klein, N. Stoilov, Numerical scattering for the defocusing Davey-Stewartson II equation for initial data with compact support, Nonlinearity 32 (2019) 4258-4280
- C. Klein, K. McLaughlin, N. Stoilov, Spectral approach to the scattering map for the semi-classical defocusing Davey-Stewartson II equation, Physica D, doi.org/10.1016/j.physd.2019.05.006 (2019)
- O. Assainova, C. Klein, K. McLaughlin and P. Miller, *A Study of the Direct Spectral Transform for the Defocusing Davey-Stewartson II Equation in the Semiclassical Limit*, Comm. Pure Appl. Math., DOI 10.1002/cpa.21822 (2019)
- C. Klein and N. Stoilov, *A numerical study of blow-up mechanisms for Davey-Stewartson II systems*, Stud. Appl. Math., DOI : 10.1111/sapm.12214 (2018)
- C. Klein and K. McLaughlin, Spectral approach to D-bar problems, Comm. Pure Appl. Math., DOI: 10.1002/cpa.21684 (2017)
- C. Klein and J.-C. Saut, *A numerical approach to Blow-up issues for Davey-Stewartson II type systems*, Comm. Pure Appl. Anal. 14:4, 1443–1467 (2015)
- C. Klein and K. Roidot, *Numerical Study of the semiclassical limit of the Davey-Stewartson II equations*, Nonlinearity 27, 2177-2214 (2014).
- C. Klein, B. Muite and K. Roidot, *Numerical Study of Blowup in the Davey-Stewartson System*, Discr. Cont. Dyn. Syst. B, Vol. 18, No. 5, 1361–1387 (2013).