A Riemann-Hilbert approach to *q*-discrete Painlevé equations

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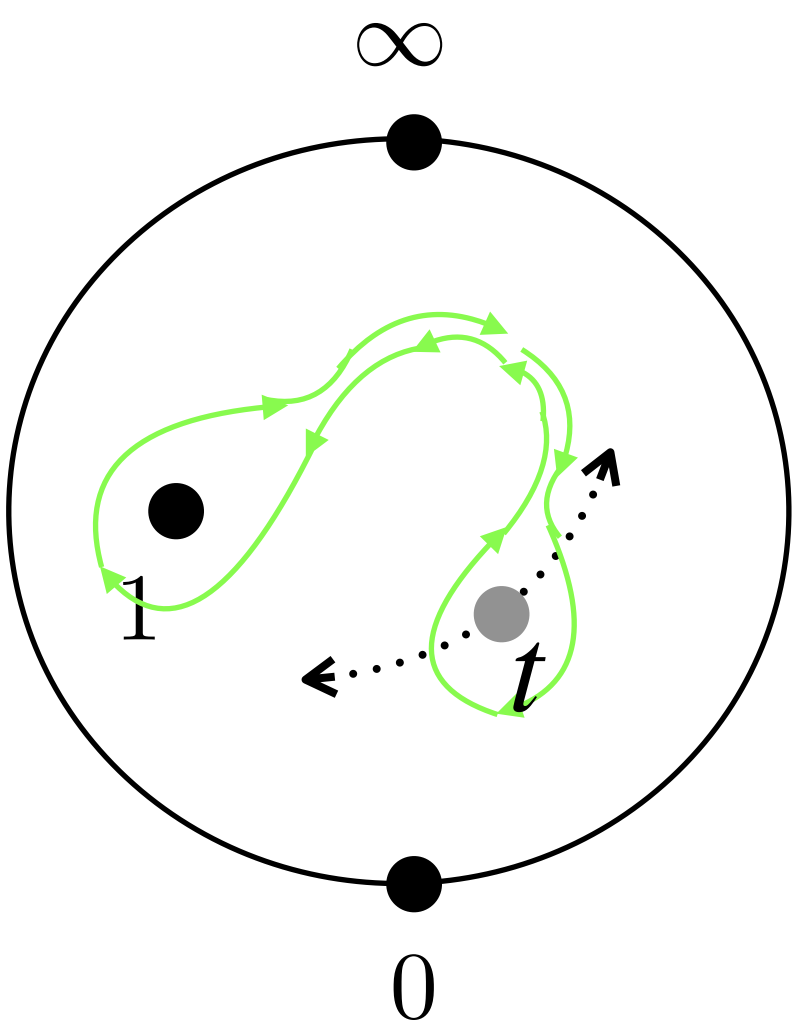
Richard Fuchs

1873 – 1944

Isomonodromy



R. Fuchs 1905

$$\frac{dY}{dz} = \left(\frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t} \right) Y$$

Find the condition under which *monodromy* data of this system stays invariant under deformation of t.

Fuchs equation

$$
\begin{aligned}
w'' = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-t}\right)(w')^2 \\
- \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{w-t}\right) w' \\
+ \quad \frac{w(w-1)(w-t)}{t^2(t-1)^2}\left(\alpha - \frac{\beta t}{w^2}\right. \\
\left. + \quad \frac{\gamma(t-1)}{(w-1)^2} + \delta\,\frac{t(t-1)}{(w-t)^2} + \right)
\end{aligned}
$$

A limiting form

- Take

$$t \mapsto 1 + \epsilon\, t$$
$$\delta \mapsto \frac{\delta}{\epsilon^2}$$
$$\gamma \mapsto \frac{\gamma}{\epsilon} - \frac{\delta}{\epsilon^2}$$

$\epsilon \to 0$

$$\mathrm{P_V}: w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)(w')^2 - \frac{w'}{t} + \frac{(w-1)^2}{t^2}\left(\alpha\, w + \frac{\beta}{w}\right) + \frac{\gamma\, w}{t} + \frac{\delta\, w(w+1)}{w-1}$$

Successive limits

$$\mathrm{P_{IV}}: \quad w'' = \frac{1}{2w}(w')^2 + \frac{3w^3}{2} + 4tw^2 + 2(t^2 - \alpha)w + \frac{\beta}{w}$$

$$\mathrm{P_{III}}: \quad w'' = \frac{1}{w}(w')^2 - \frac{w'}{t} + \frac{1}{t}(\alpha w^2 + \beta) + \gamma\, w^3 + \frac{\delta}{w}$$

$$\mathrm{P_{II}}: \quad w'' = 2w^3 + tw + \alpha$$

$$\mathrm{P_{I}}: \quad w'' = 6w^2 + t$$

Linear Problems ("Lax pairs")

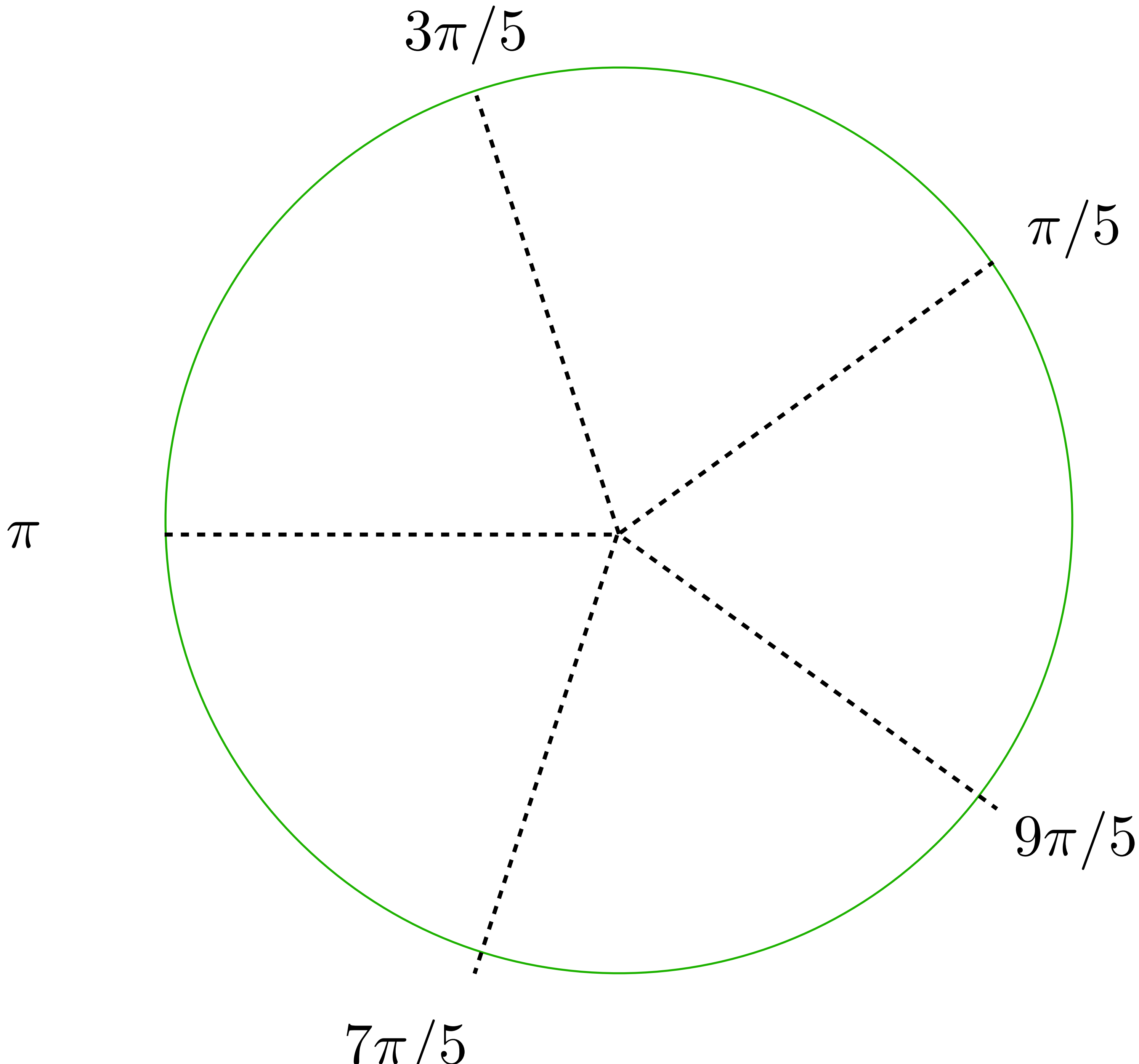
$$\begin{cases} \dfrac{dY}{dz} &= \left(z^2 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + z \begin{pmatrix} 0 & w \\ 4 & 0 \end{pmatrix} + \begin{pmatrix} -v & w^2 + t/2 \\ -4w & v \end{pmatrix} \right) \\ \dfrac{dY}{dt} &= \left(\dfrac{z}{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & w \\ 2 & \end{pmatrix} \right) \end{cases}$$

Jimbo & Miwa 1981

$$\mathrm{P_I}: \quad \frac{dw}{dt} = v, \ \frac{dv}{dt} = 6w^2 + t$$

Other linear problems exist for PI. *Joshi, Kitaev & Treharne 2009*

Monodromy data



Kapaev & Kitaev 1993

$$Y_{k+1}(z) = Y_k(z)\, S_k$$

$$\begin{cases} S_{2l+1} & = \begin{pmatrix} 1 & s_{2l+1} \\ 0 & 1 \end{pmatrix} \\ S_{2l} & = \begin{pmatrix} 1 & 0 \\ s_{2l} & 1 \end{pmatrix} \end{cases}$$

The Stokes parameters s_k are related through symmetries.

Riemann-Hilbert Transform

Linear problem $\mapsto$ monodromy data

holomorphic solutions in D_i

?

Fokas et al 1991
Deift el al 1999
Fokas, Its, Kapaev &
Novokshenov 2006

Monodromy surfaces

$$\mathrm{P}_{\mathrm{VI}}: \quad x\,y\,z - x^2 - y^2 - z^2 + \omega_1\,x + \omega_2\,y + \omega_3\,z + \omega_4 = 0$$

Iwasaki, 2002.

$$\mathrm{P}_{\mathrm{V}}: \quad x\,y\,z + x^2 + y^2 + z^2 - (\omega_1 + \omega_2\omega_3)\,x - (\omega_2 + \omega_1\omega_3)\,y$$
$$-\omega_3\,z + \omega_3^2 + \omega_1\omega_2\omega_3 + 1 = 0$$

$$\mathrm{P}_{\mathrm{IV}}: \quad x\,y\,z + x^2 - (\omega_2^2 + \omega_1\omega_2)\,x - \omega_2^2\,y - \omega_2^2\,z + \omega_2^2 + \omega_1\omega_2^3 = 0$$

$$\mathrm{P}_{\mathrm{III},\mathrm{D}_6}: \quad x\,y\,z + x^2 + y^2 + (1 + \omega_1\omega_2)\,x + (\omega_1 + \omega_2)\,y + \omega_1\omega_2$$
$$-\omega_3\,z + \omega_3^2 + \omega_1\omega_2\omega_3 + 1 = 0$$

$$\mathrm{P}_{\mathrm{II},\mathrm{FN}}: \quad x\,y\,z + x - y + z + \omega_1 = 0$$

$$\mathrm{P}_{\mathrm{I}}: \quad x\,y\,z + x + y + 1 = 0$$

Cubic surfaces

Vanderput & Saito, 2009.

where ω_i are parameters, related to those in P_{J}.

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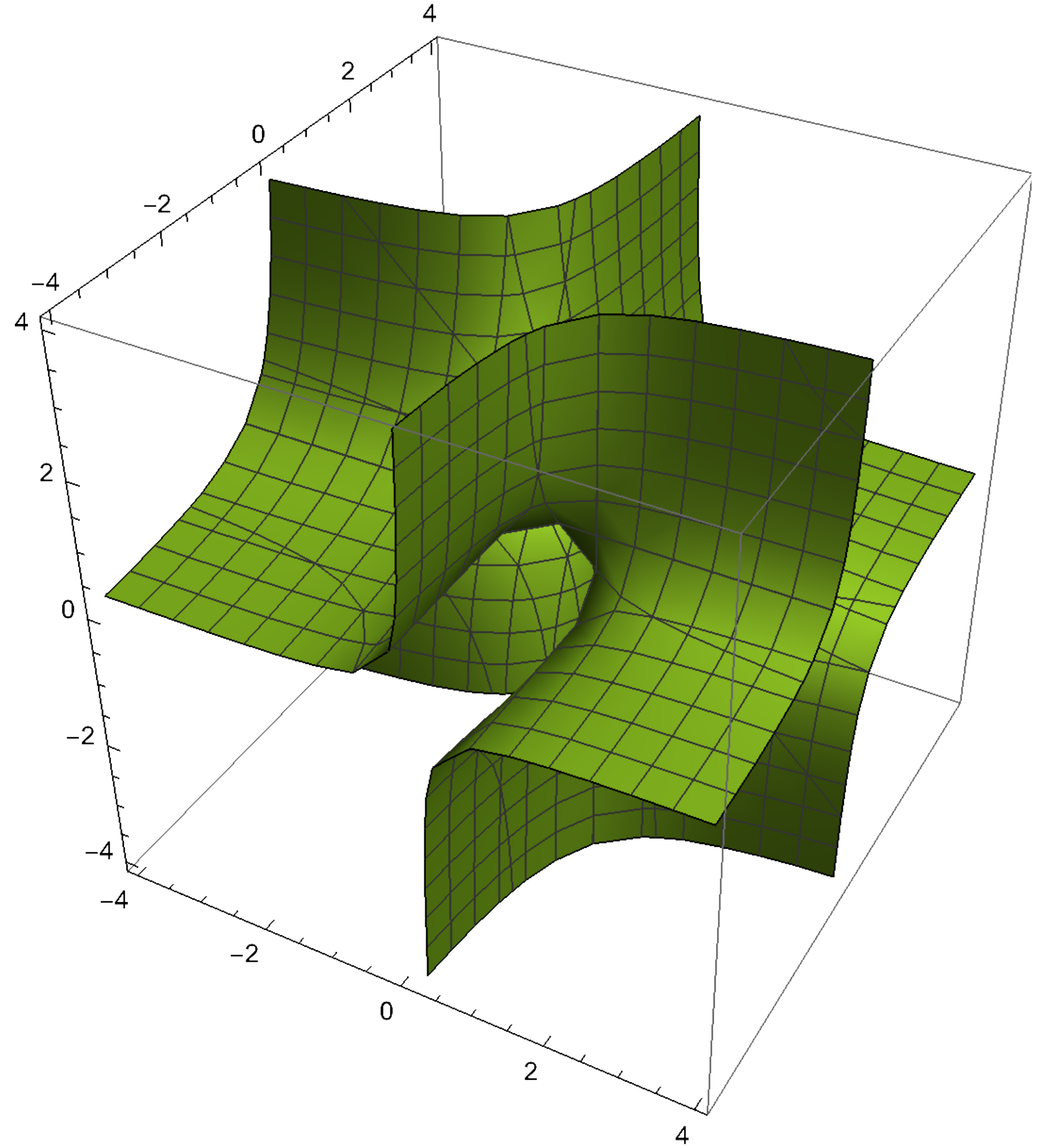
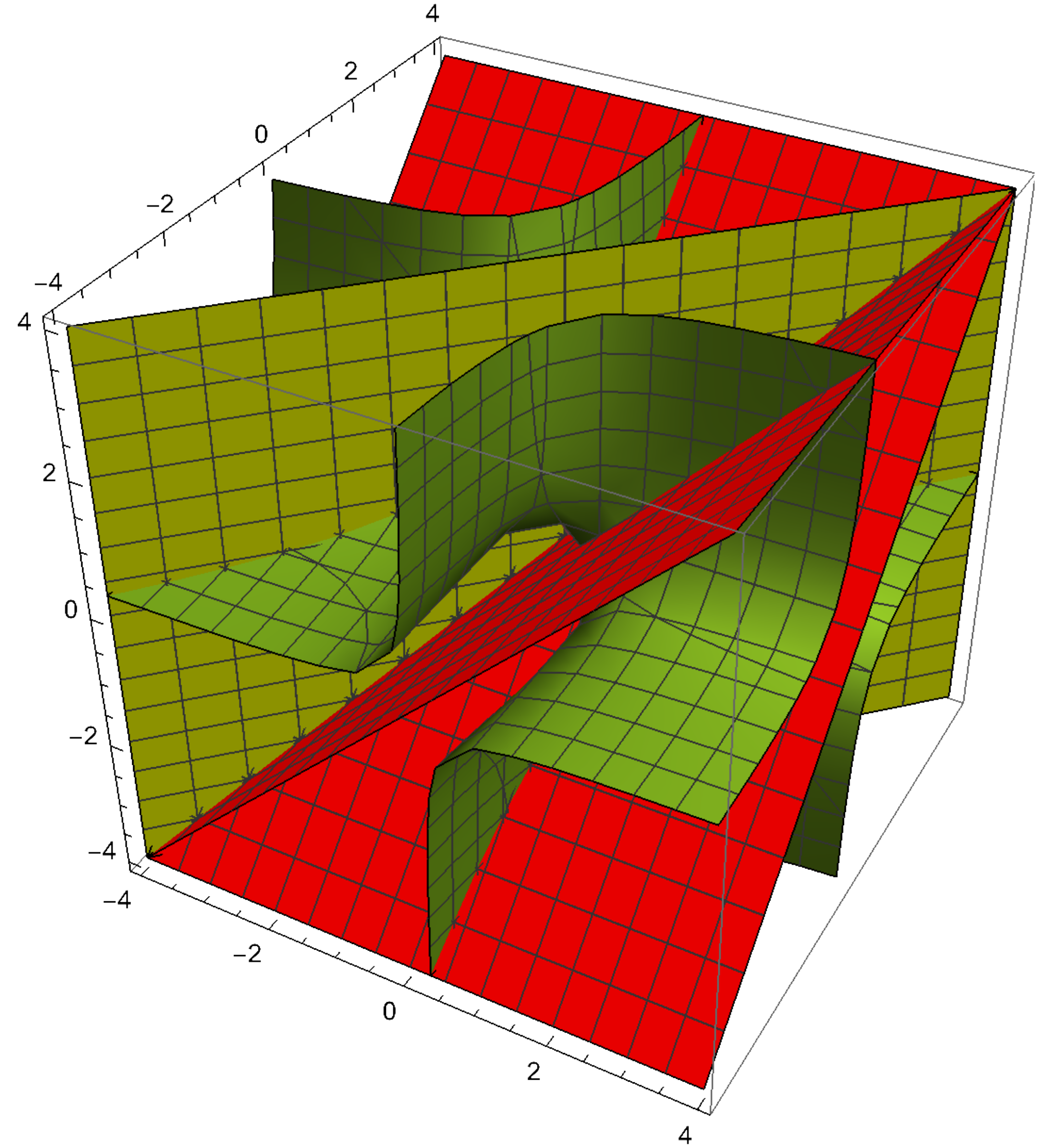




P_{I} where $x = y = z$ gives symmetric solutions.

Are there analogous approaches available for discrete Painlevé equations?

George D. Birkhoff

1884 – 1944

$$\frac{dY(z)}{dz} = A(z)\,Y(z)$$

$$Y(z+1) = A(z)\,Y(z)$$

$$Y(q\,z) = A(z)\,Y(z)$$

G.D. Birkhoff, The generalized Riemann problem for linear differential equations and the allied problems for linear difference and q-difference Equations, *Proc American Academy of Arts and Sciences*, **49** (9) (1913), 521-568.

Robert D. Carmichael

1879 – 1967

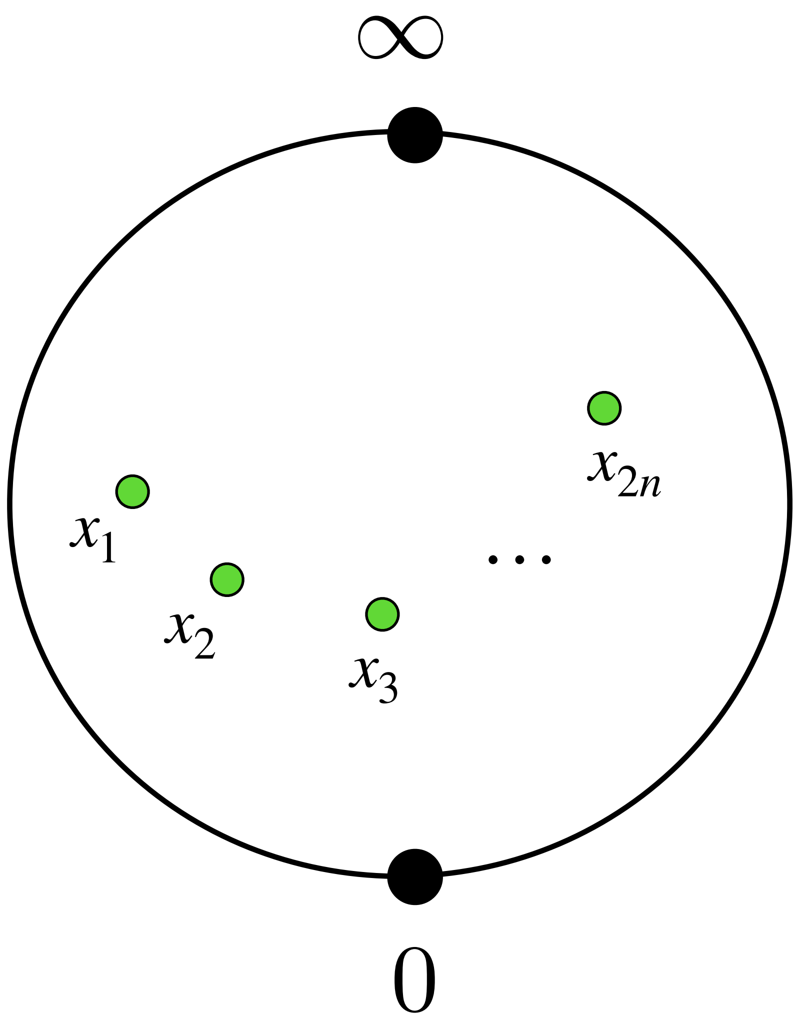
The Jeanne Agnew Papers at the Archives of American Mathematics

Jeanne LeCaine

1917 – 2000

The q-Setting

$$Y(q\,z) = A(z)\,Y(z)$$
$$A(z) = A_0 + A_1\,z + \ldots + A_n\,z^n$$



q-Fuchsian

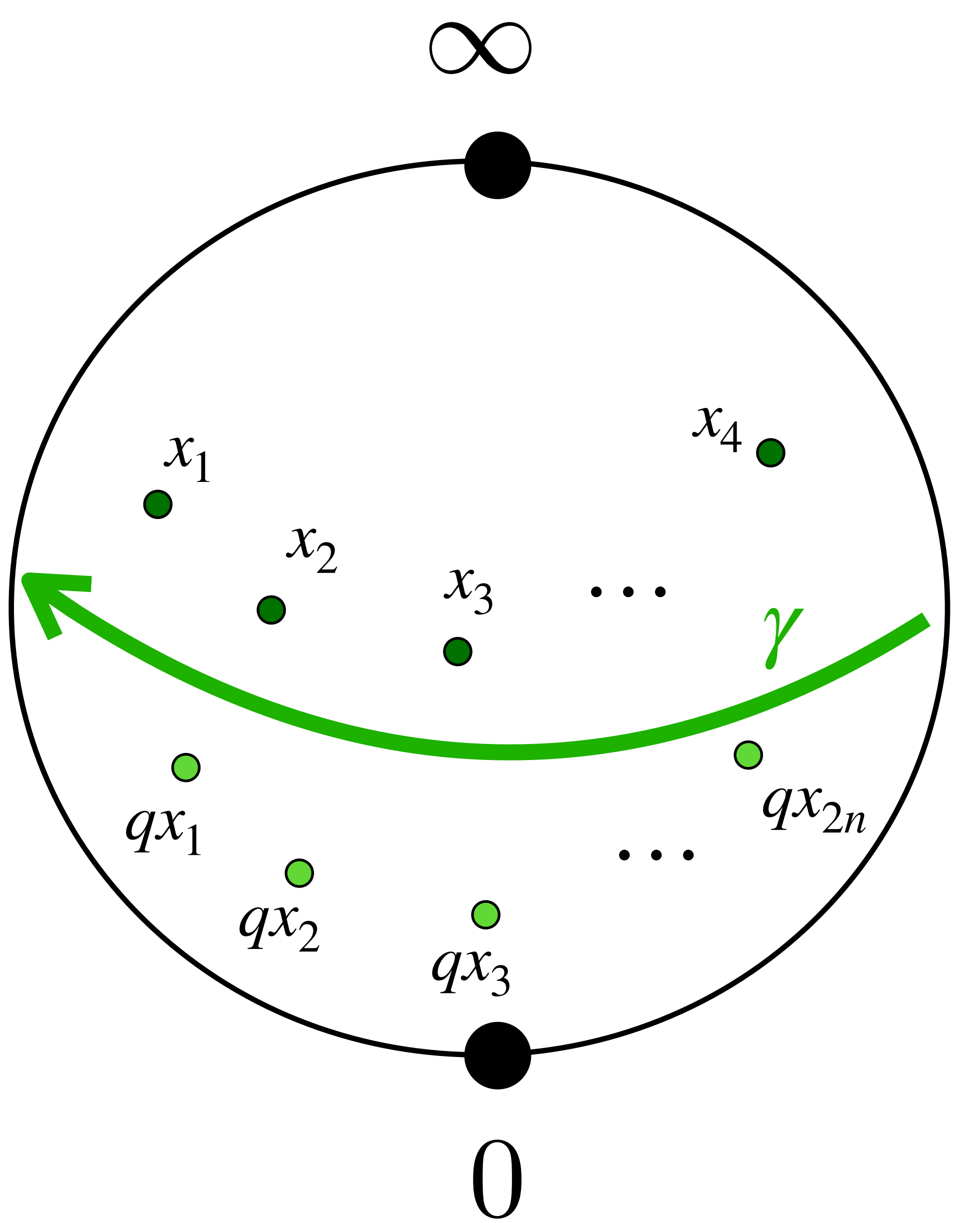
$$Y(q\,z) = A(z)\,Y(z)$$
$$A(z) = A_0 + A_1\,z + \ldots + A_n\,z^n$$

- *Fuchsian*: $\det(A_0) \neq 0$, $\det(A_n) \neq 0$.
- *Critical exponents*: Let
  - the eigenvalues of A_0 be θ_1, θ_2
  - the eigenvalues of A_n be κ_1, κ_2
  - the zeroes of $\det(A)$ be $\{x_1, \ldots, x_{2n}\}$
- *Nonresonant*:

$$\theta_1/\theta_2 \neq q^m, \kappa_1/\kappa_2 \neq q^m, x_i/x_j \neq q^m,$$

for $i \neq j$ for any $m \in \mathbb{Z}$.

q-Iteration

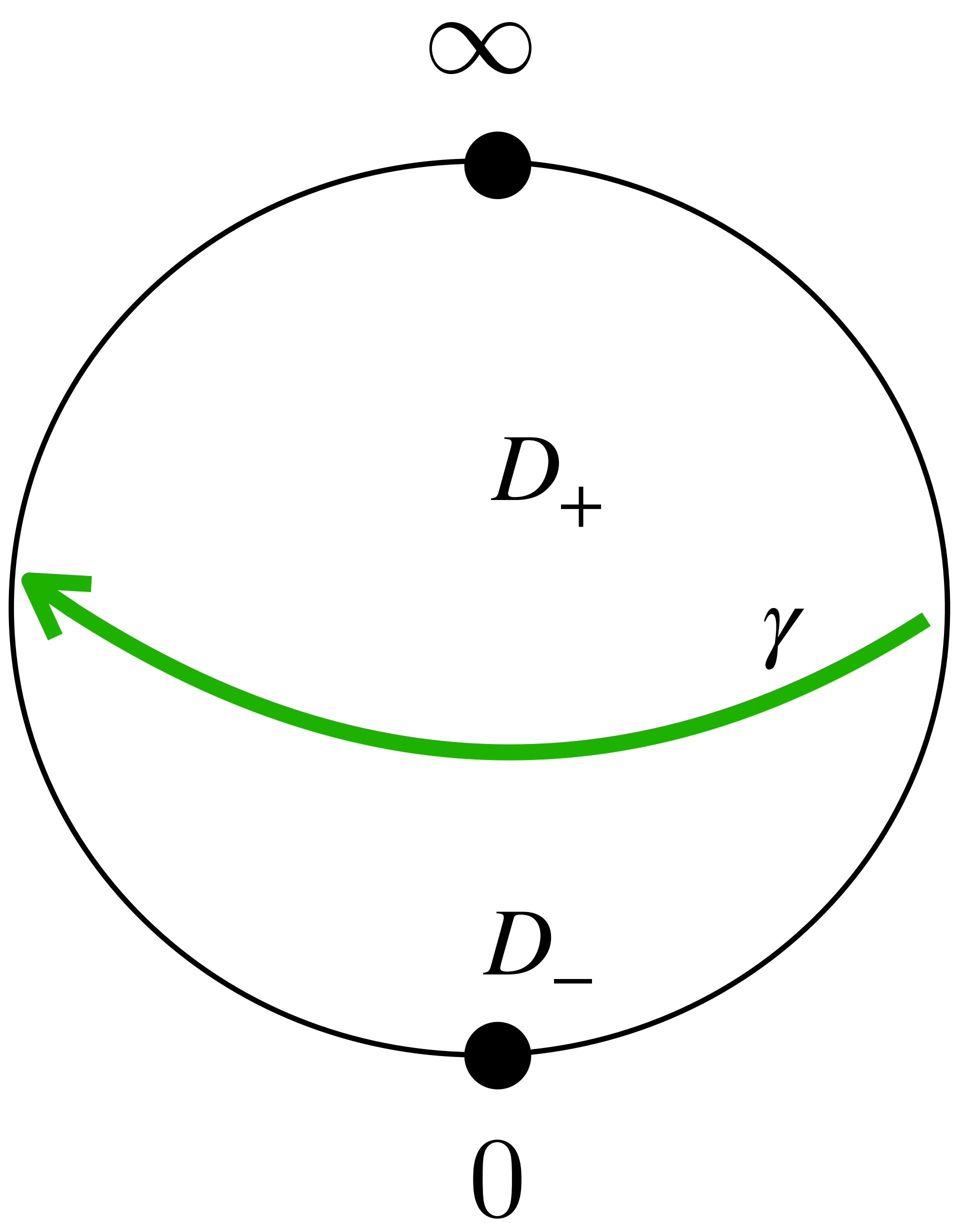


Admissible curve γ:

- Positively oriented, locally analytic Jordan curve
- Bounding $D_\pm$
- With

$$q^k x_i \in \begin{cases} D_-, & \text{if } k > 0 \\ D_+, & \text{if } k \leq 0 \end{cases}$$

q-Connection



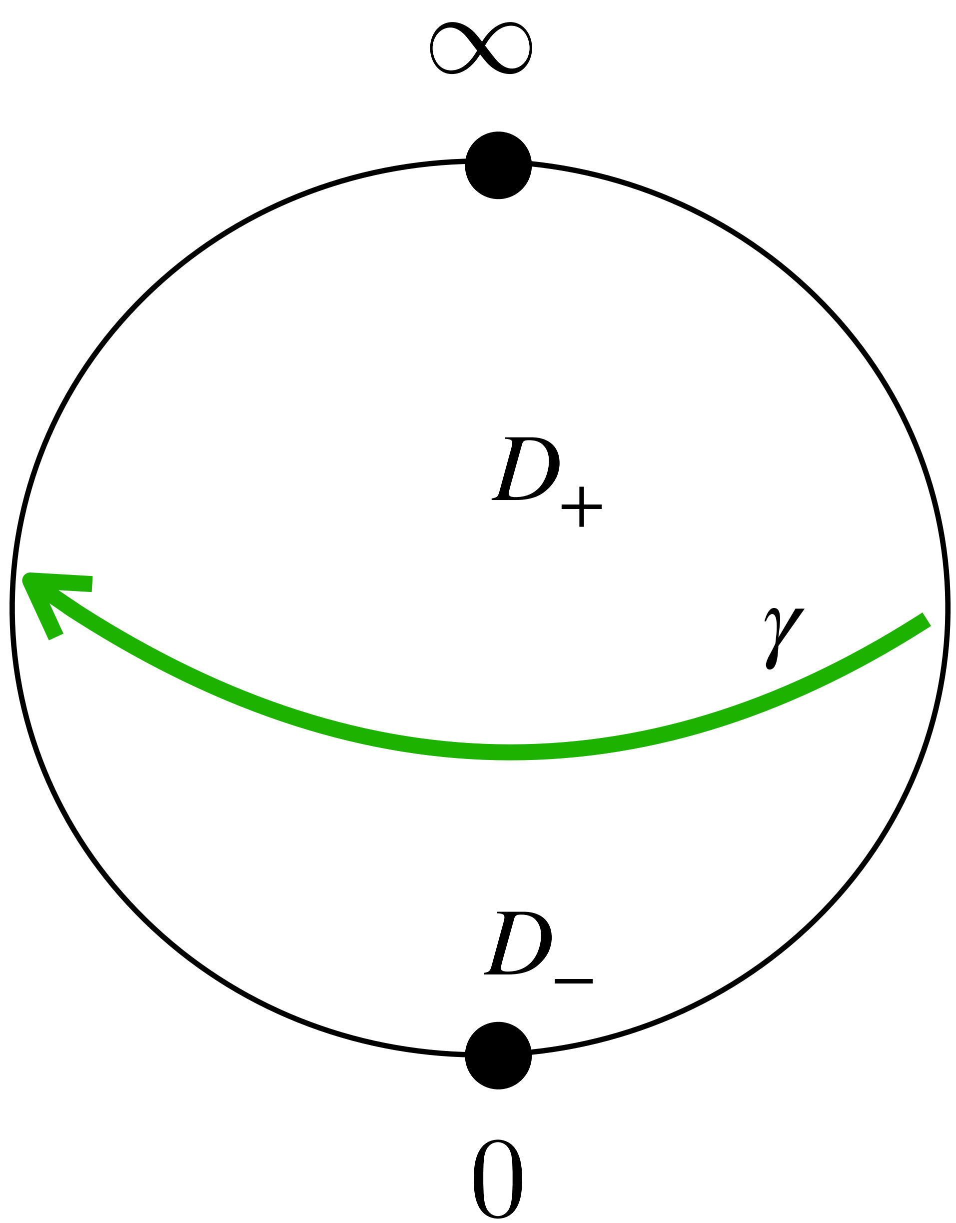
$\exists$ a meromorphic solution $Y_{\pm}(z)$ in $D_{\pm}$, related through a connection matrix

$$Y_+(z) = Y_-(z)C(z)$$

$C(z)$ satisfies certain properties.

Carmichael,1912

q-Riemann-Hilbert problem



Given non-resonant θ_i, κ_i, x_k, a connection matrix $C(z)$ with certain properties, admissible curve γ & $m \in \mathbb{Z}$, a function $Y^{(m)}(z)$ is a solution of the $\mathrm{RHP}^{(m)}(\gamma, C)$ if

(i) $Y^{(m)}(z)$ is analytic on $\mathbb{C} \setminus \gamma$

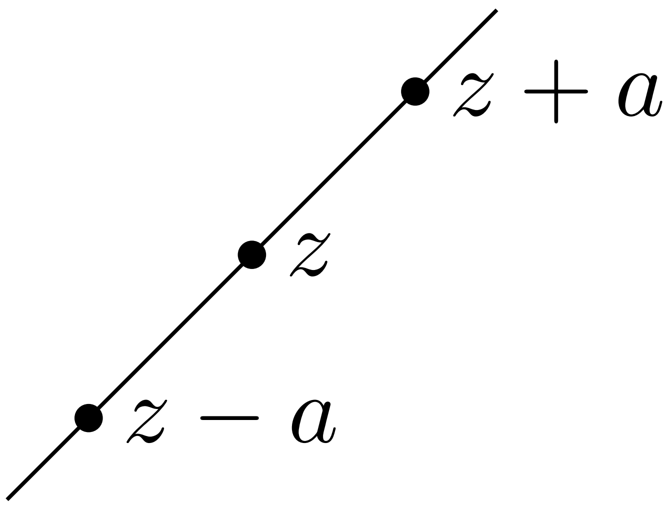
(ii) $Y_+^{(m)}(z) = Y_-^{(m)}(z)C(z)$, $z \in \gamma$

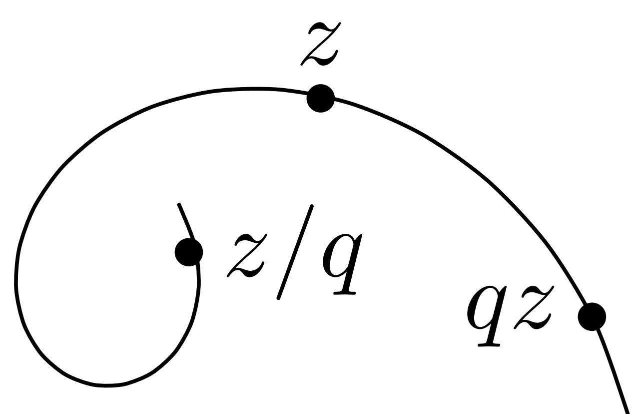
(iii) $Y^{(m)}(z) = (1 + \mathcal{O}(1/z))z^{m\sigma_3}$, for $z \to \infty$.

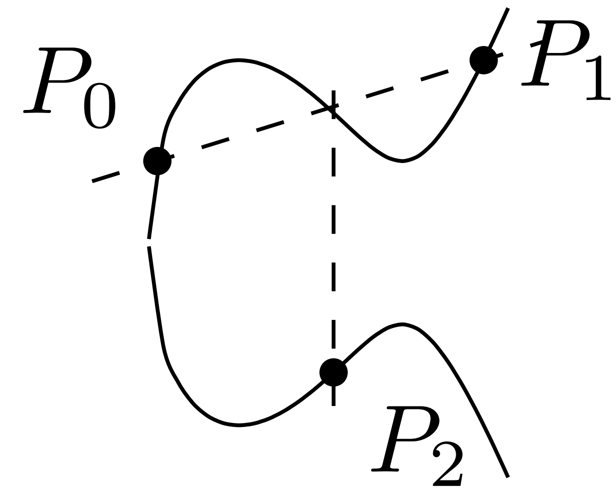
Birkhoff,1913

How are these related to discrete Painlevé equations?

Discrete Painlevé equations







$$|q| \neq 1$$

$$y^2 = x^3 - a\,x - b$$
$$a^3 \neq 27\,b^2$$

Sakai (2001) described 22 classes of such equations,

distinguished by root systems characterising their initial value space or symmetry group.

Each q-discrete Painlevé equation is related to a compatible pair of linear systems

$$\begin{aligned} Y(qz,t) &= A(z,t)Y(z,t), \\ Y(z,qt) &= B(z,t)Y(z,t), \\ A(z,qt)B(z,t) &= B(qz,t)A(z,t) \end{aligned}$$

Jimbo and Sakai (1996) found a 2x2 linear problem for a q-discrete P_{VI} equation, satisfying Carmichael's conditions.

A prototypical linear problem

$$Y(qz,t) = A(z,t)Y(z,t),$$
$$Y(z,qt) = B(z,t)Y(z,t),$$

$$A = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -i\, q \frac{t}{f_2} z & 1 \\ -1 & -\, i\, q \frac{f_2}{t} z \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -bu \\ b^{-1}u^{-1} & 0 \end{pmatrix} + \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix}$$
$$\times \begin{pmatrix} -\, i\, a_0 a_2 \frac{t}{f_0} z & 1 \\ -1 & -\, i\, a_0 a_2 \frac{f_0}{t} z \end{pmatrix}$$
$$\times \begin{pmatrix} -\, i\, a_0 \frac{t}{f_1} z & 1 \\ -1 & -\, i\, a_0 \frac{f_1}{t} z \end{pmatrix} \begin{pmatrix} u^{-1} & 0 \\ 0 & 1 \end{pmatrix}$$

$$A(z,qt)B(z,t) = B(qz,t)A(z,t)$$

Joshi, Nakazono, Proc. Roy Soc. A. 472, 20160696 (2016)

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A prototypical q-Painlevé equation

$$q\mathrm{P}_{\mathrm{IV}} : \begin{cases} \dfrac{\overline{f}_0}{a_0 a_1 f_1} = \dfrac{1 + a_2 f_2 (1 + a_0 f_0)}{1 + a_0 f_0 (1 + a_1 f_1)}, \\ \dfrac{\overline{f}_1}{a_1 a_2 f_2} = \dfrac{1 + a_0 f_0 (1 + a_1 f_1)}{1 + a_1 f_1 (1 + a_2 f_2)}, \\ \dfrac{\overline{f}_2}{a_2 a_0 f_0} = \dfrac{1 + a_1 f_1 (1 + a_2 f_2)}{1 + a_2 f_2 (1 + a_0 f_0)}, \end{cases}$$

$$\overline{f}_j = f_j(qt)$$
$$f_0\, f_1\, f_2 = t^2,$$
$$a_0\, a_1\, a_2 = q$$

Also known as

$$q\mathrm{P}_{\mathrm{IV}}(A_5^{(1)})$$

Initial Value Space

$$q\mathrm{P}_{\mathrm{IV}}\big((A_2 + A_1)^{(1)}\big)$$

Symmetry group

Critical exponents

$$A(z) = A_0(t) + A_1(t)z + A_2(t)z^2 + A_3(t)z^3$$

$$A_0 = \begin{pmatrix} 0 & -u \\ u^{-1} & 0 \end{pmatrix} \qquad \Rightarrow \theta_1 = i, \theta_2 = -i$$

$$A_3 = i\,q\,a_0^2\,a_2 \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \quad \Rightarrow \kappa_1 = iqa_0^2a_2t, \kappa_2 = -iqa_0^2a_2/t$$

$$x_1 = +a_0^{-1}, \qquad x_2 = +a_1/q, \qquad x_3 = +q^{-1},$$
$$x_4 = -a_0^{-1}, \qquad x_5 = -a_1/q, \qquad x_6 = -q^{-1},$$

Carmichael's conditions are satisfied if

$$t^2, \pm a_0, \pm a_1, \pm a_2 \notin q^{\mathbb{Z}}$$

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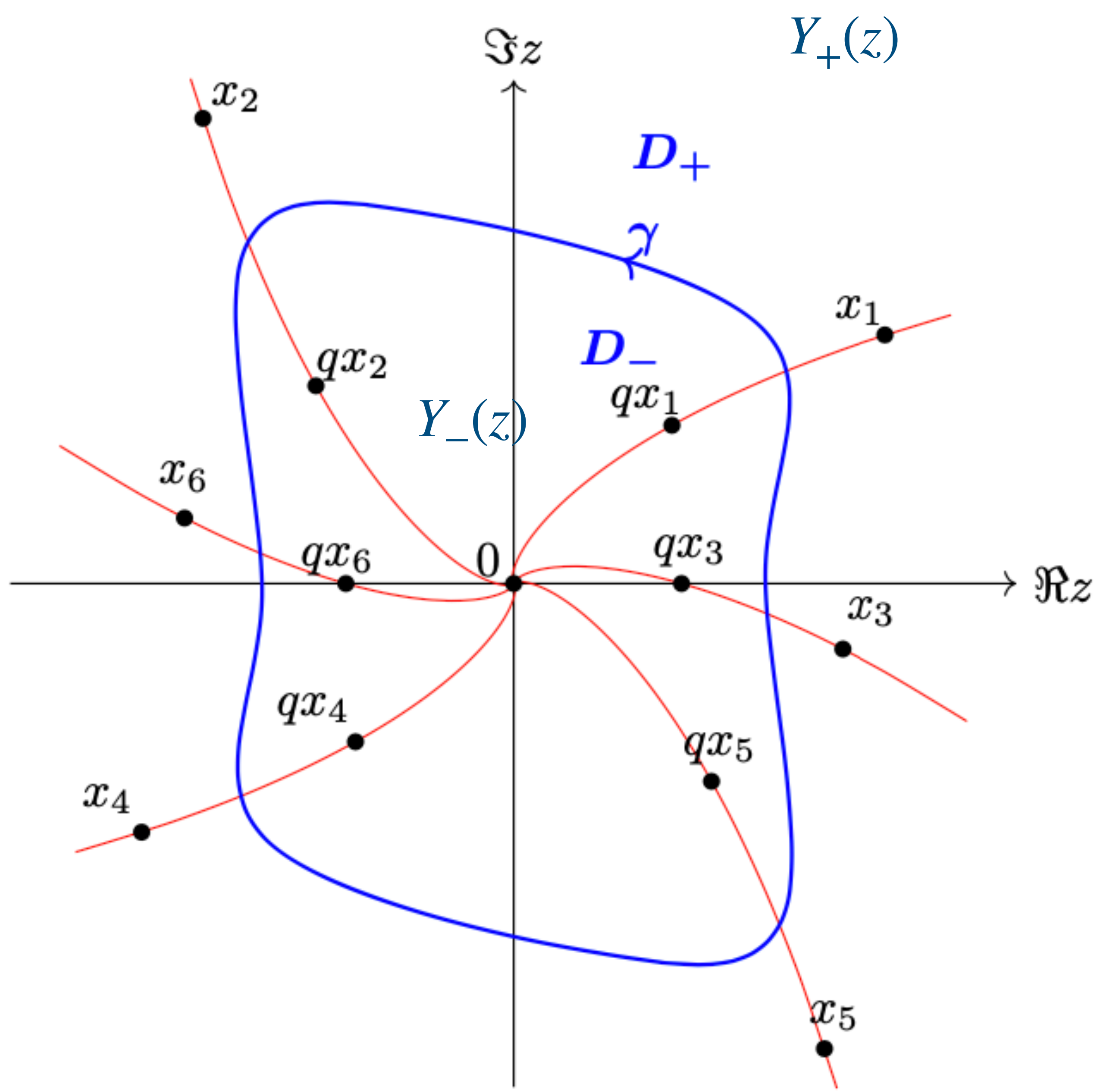
q-Riemann-Hilbert Problem



N. Joshi and P. Roffelsen, *Commun. Math. Phys* (2021)
arXiv:1911.05854

Main results

- A solution $Y^{(m)}(z)$ of $\mathrm{RHP}^{(m)}(\gamma, C)$ exists and is unique.
- $Y^{(m)}(z)$ can be used to reconstruct a matrix polynomial $A(z)$, whose entries give solutions of $\mathrm{qP_{IV}}$.
- $\exists$ a Riemann-Hilbert map, which maps equivalence classes of connection matrices to solutions of $\mathrm{qP_{IV}}$.
- The Riemann-Hilbert mapping is a bijection.

N. Joshi and P. Roffelsen, *Commun. Math. Phys* (2021)
arXiv:1911.05854

Connection matrix

- (c.1) $C(z)$ is analytic in $\mathbb{C}^*$
- (c.2) $C(qz) = \dfrac{1}{qa_0^2a_2} z^{-3}\sigma_3 C(z) t^{-\sigma_3}$
- (c.3) $|C(z)| = c\theta_q(a_0z, -a_0z, a_0a_2z, -a_0a_2z, qz, -qz)$
- (c.4) $C(-z) = -\sigma_1 C(z)\sigma_3$

Note $|C(x_k)| = 0$ so it is not invertible there.

$\Rightarrow$ At each point, R_1 of $C(x_k)$ is a multiple of R_2.

$\Rightarrow$ We obtain a linear system of 3 equations for the ratio R_1/R_2 .

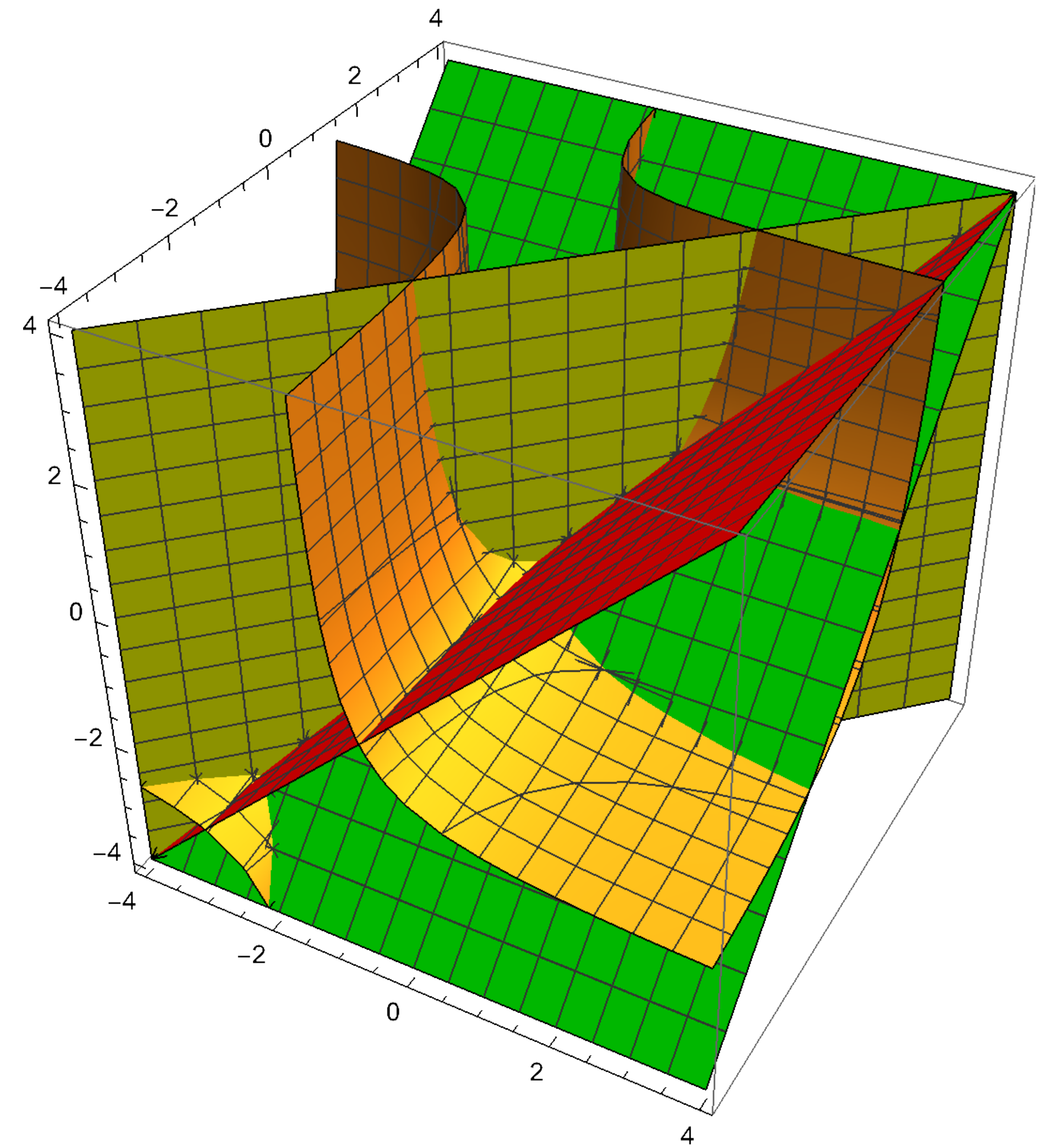
$\Rightarrow$ Determinant of this system vanishes $\Rightarrow$ the *monodromy manifold.*

q-monodromy surface

$$
\begin{aligned}
&\theta_q(+a_0,+a_1,+a_2)\left(\theta_q(\lambda_0)p_1p_2p_3-\theta_q(-\lambda_0)\right)\\
&-\theta_q(-a_0,+a_1,-a_2)\left(\theta_q(\lambda_0)p_1-\theta_q(-\lambda_0)p_2p_3\right)\\
&+\theta_q(+a_0,-a_1,-a_2)\left(\theta_q(\lambda_0)p_2-\theta_q(-\lambda_0)p_1p_3\right)\\
&-\theta_q(-a_0,-a_1,+a_2)\left(\theta_q(\lambda_0)p_3-\theta_q(-\lambda_0)p_1p_2\right)=0
\end{aligned}
$$

A cubic surface in $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$

- Contains lines, indicating "symmetric" solutions of $\mathrm{qP_{IV}}$.


4
2
0
−2
−4
4
2
0
−2
−4
−4
−2
0
2
4

Further results

- $\exists$ Fuchsian and non-resonant q-linear problems for several q-discrete Painlevé equations.
- Surprisingly, corresponding monodromy manifolds are not always cubic.
- They suggest existence of special (transcendental) solutions.
- Solution of q-Riemann-Hilbert problems for q-orthogonal polynomials provide asymptotic estimates of zeroes.

N. Joshi and P. Roffelsen, *Commun. Math. Phys* (2021) arXiv:1911.05854
N. Joshi and T. Lasic Latimer, arXiv:2106.01042
T. Lasic Latimer, arXiv:2011.06270

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