Initial data for Airy kernel determinant solutions of the Korteweg-de Vries equation

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Based on joint work with Mattia Cafasso and Giulio Ruzza

Two important sources of inspiration for this talk:

- ✓ Boris Dubrovin. On Hamiltonian perturbations of hyperbolic systems of conservation laws. II. Universality of critical behaviour. Comm. Math. Phys. 267 (2006), no. 1, 117-139.
- ✓ Boris Dubrovin and Alexander Minakov. On a class of compact perturbations of the special pole-free joint solution of KdV and P2I. ArXiv:1901.07470 (2019).

Outline

1. Airy kernel Fredholm determinant solutions of the Korteweg-de Vries (KdV) equation.
2. Integro-differential Painlevé II, singular initial data and the KPZ equation.
3. Riemann-Hilbert problem and scattering data.

If y solves the PAINLEVÉ II EQUATION

$$y''(\xi) = \xi y(\xi) + 2y(\xi)^3,$$

then u defined as

$$u(x,t) = \frac{x}{2t} - t^{-2/3} y^2\left(-xt^{-1/3}\right)$$

solves the KDV EQUATION

$$\partial_t u + 2u\partial_x u + \frac{1}{6}\partial_x^3 u = 0.$$

The $t \to 0$ ASYMPTOTICS *of the self-similar KdV solution are not well-defined if* $\lim_{x\to+\infty} y(x) = 0$*:*

$$u(x,t) = \frac{x}{2t} + o(t^{-2/3}) \text{ for } x < -\delta.$$

If y *is the* HASTINGS-MCLEOD SOLUTION *of the Painlevé II equation,*

$$y(\xi) \sim \mathrm{Ai}(\xi) \text{ as } \xi \to +\infty, \quad y(\xi) \sim \sqrt{\frac{-\xi}{2}} \text{ as } \xi \to -\infty,$$

we have the $t \to 0$ *asymptotics*

$$u(x,t) = \frac{1}{8x^2} + o(x^{-2}) \;\text{ for } x > \delta.$$

Transition between singular and regular part of initial data takes place for $x = \mathcal{O}(t^{1/3})$.

If y is the Hastings-McLeod solution, $u(x,t)$ can be written in terms of the AIRY KERNEL FREDHOLM DETERMINANT:

$$u\left(x,t\right) := \partial_x^2 \log\det\left(1 - \chi_{[-xt^{-1/3},\infty]} K^{\mathrm{Ai}}\right),$$

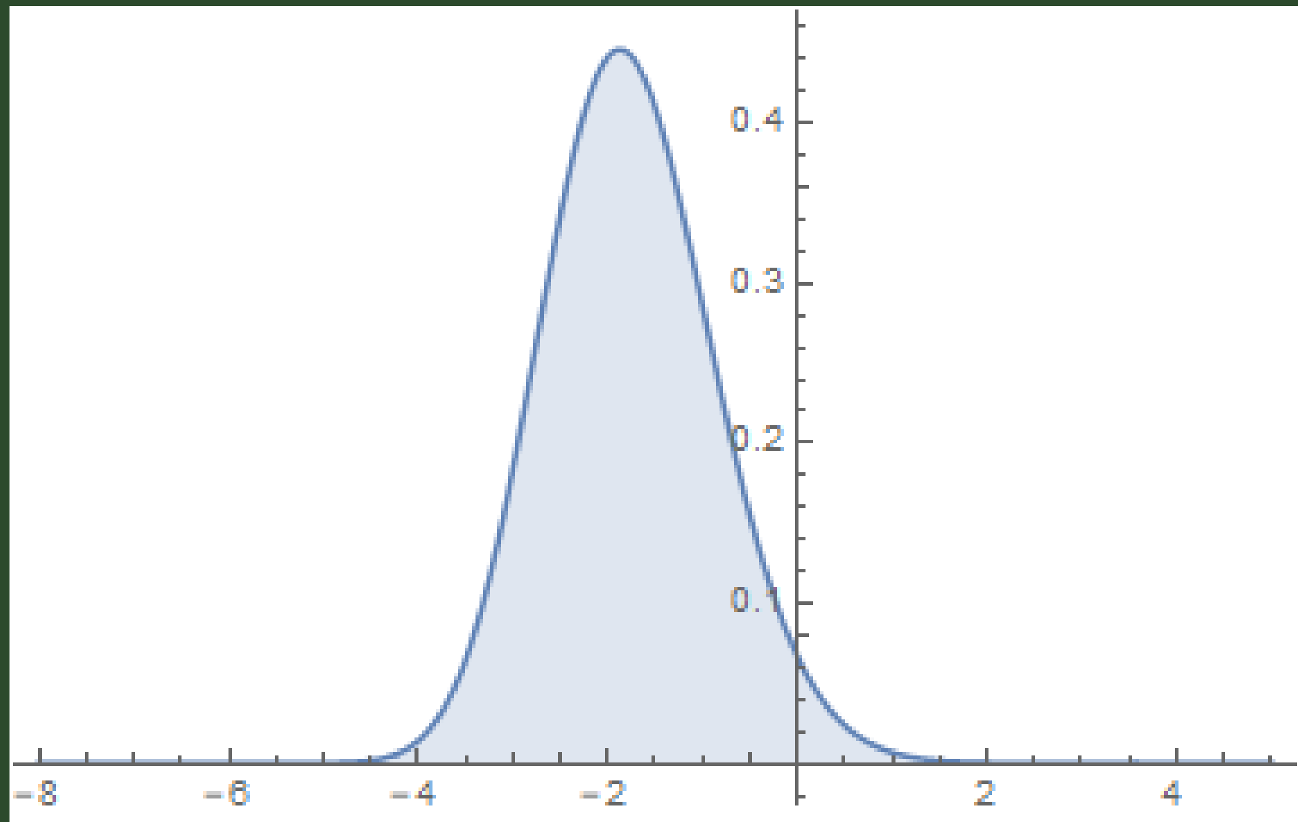
where

$$K^{\mathrm{Ai}}(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y} = \int_0^{+\infty} \mathrm{Ai}(u+r)\mathrm{Ai}(v+r)dr,$$

and with the Fredholm determinant given by

$$\det(1-\phi K) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \int_{\mathbb{R}^k} \det\left(\phi(x_i)K(x_i,x_j)\right)_{i,j=1,\ldots,k} dx_1 \ldots dx_k.$$

The Airy kernel determinant $\det\left(1-\chi_{[-xt^{-1/3},\infty]}K^{\mathrm{Ai}}\right)$ *is the* TRACY-WIDOM DISTRIBUTION (*TRACY-WIDOM '93*).



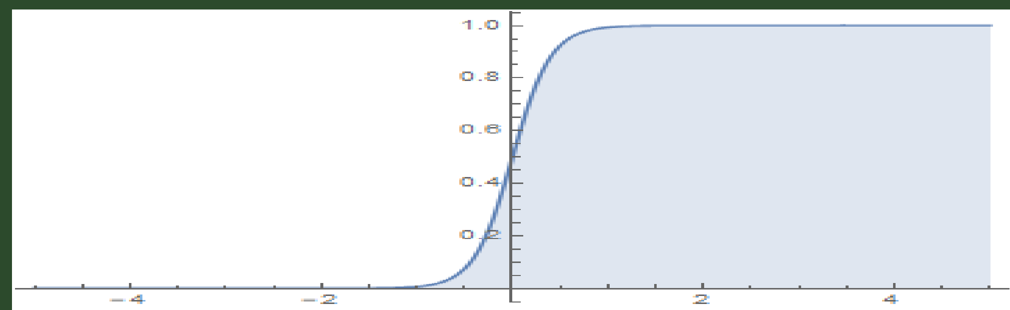
Density of the ($\beta = 2$) *Tracy-Widom distribution*

Deformations of the Tracy-Widom distribution appear in models of FINITE-TEMPERATURE FREE FERMIONS *(or the MNS matrix model):*

$$F_\tau(s) = \det\left(1 - \chi_{[s,\infty]} K_\tau^{\mathrm{Ai}}\right) = \det(1 - \sigma(\tau^{-1}(u+s))K^{\mathrm{Ai}}(u,v))$$

$$K_\tau^{\mathrm{Ai}}(u,v) = \int_{-\infty}^{+\infty} \sigma(\tau^{-1}r)\mathrm{Ai}(u+r)\mathrm{Ai}(v+r)dr$$

$\sigma(r) = \frac{1}{1+e^{-r}}$



(Moshe-Neuberger-Shapiro '94, Johansson '07, Liechty-Wang '20, Dean-Le Doussal-Majumdar-Schehr '14-'20).

The FINITE TEMPERATURE TRACY-WIDOM DISTRIBUTION appears also in the context of the KPZ EQUATION (KARDAR-PARISI-ZHANG '86), which is the stochastic PDE

$$\partial_T \mathcal{H}(T,X) = \frac{1}{2}\partial_X^2 \mathcal{H}(T,X) + \frac{1}{2}(\partial_X \mathcal{H}(T,X))^2 + \xi(T,X)$$

where $\xi(T,X)$ is "space-time white noise".

Physical interpretation:

relaxation + nonlinear slope-dependent growth + random forcing

Allows to model various types of RANDOM INTERFACE GROWTH:

bacterial growth, coffee stains, forest fires, burning paper ...

One of the physically relevant KPZ solutions is the one with NARROW WEDGE INITIAL DATA, *which means formally that*

$$\mathcal{H}(0,X) = \log Z(0,X) \text{ with } Z(0,X) = \delta_{X=0}.$$

$\mathcal{H}(T,X)$ *typically behaves roughly like a parabola* $-\frac{X^2}{2T}$ *becoming narrow as* $T \to 0$.

This solution is characterized (via a Laplace transform and the Cole-Hopf transformation) by the finite temperature Tracy-Widom distribution (Amir-Corwin-Quastel, Sasamoto-Spohn, Dotsenko, Calabrese-Le Doussal-Rosso '10, Borodin-Gorin '16).

More general results now: Fredholm determinant expressions for KPZ fixed point (Matetski-Quastel-Remenik '17).

Define

$$Q(x,t) := \det(1-\sigma_{x,t}(u)K^{\mathrm{Ai}}(u,v)), \quad \sigma_{x,t}(u) = \sigma(t^{-2/3}u - x/t),$$

where

1. $\sigma : \mathbb{R} \to [0,1]$ is non-decreasing and piecewise C^∞;
   $\gamma = \lim_{r\to+\infty} \sigma(r) \in [0,1]$,
2. there exist $c_1, c_2, c_3 > 0$ such that

$$\left|\sigma(y) - \gamma\chi_{(0,+\infty)}(y)\right| \leq c_1 \mathrm{e}^{-c_2|y|}, \quad |\sigma'(y)| \leq \frac{c_3}{|y|^2+1}.$$

Theorem (Cafasso-C-Ruzza '20)

The function $u_\sigma(x,t) := \partial_x^2 \log Q_\sigma(x,t) + \frac{x}{2t}$ *solves the KdV equation*

$$\partial_t u_\sigma + 2u_\sigma \partial_x u_\sigma + \frac{1}{6}\partial_x^3 u_\sigma = 0, \text{ and}$$

$$u_\sigma(x,t) = -\frac{1}{t}\int_{\mathbb{R}} \phi_\sigma^2(r;x,t)\, d\sigma(r) + \frac{x}{2t},$$

where ϕ_σ *solves the Schrodinger equation with potential* $2u_\sigma$,

$$\partial_x^2 \phi_\sigma(z;x,t) = (z - 2u_\sigma(x,t))\,\phi_\sigma(z;x,t),$$

and has asymptotic behavior

$$\phi_\sigma(z;x,t) \sim t^{1/6}\,\mathrm{Ai}\left(t^{2/3}z - xt^{-1/3}\right) \text{ as } z \to \infty \text{ with } |\arg z| < \pi - \delta.$$

Consequence

$$\partial_x^2 \log Q_\sigma(x,t) = -\frac{1}{t}\int_{\mathbb{R}} \phi_\sigma^2(r;x,t)\,d\sigma(r),$$

where ϕ_σ *satisfies the* INTEGRO-DIFFERENTIAL PAINLEVÉ II EQUATION

$$\partial_x^2 \phi_\sigma(z;x,t) = \left(z - \frac{x}{t} + \frac{2}{t}\int_{\mathbb{R}} \phi_\sigma^2(r;x,t)\,d\sigma(r)\right)\phi_\sigma(z;x,t).$$

This reproduces a result of AMIR-CORWIN-QUASTEL '10.

KPZ, KdV and KP

The fact that Airy kernel determinants yield KdV solutions goes back to Poppe-Sattinger '88.

More general relation between KPZ fixed point and the KP hierarchy recently obtained by Quastel-Remenik '19 and Le Doussal '20.

Fredholm determinants of a similar nature connected to integrable hierarchies Baik-Liu-Silva '20, Krajenbrink '20, Liechty-Nguyen-Remenik '20, Bothner-Cafasso-Tarricone '21.

Can we understand the small t behavior of the KdV solutions $u_\sigma(x,t)$? This encodes information about the tail asymptotics for the KPZ solution with narrow wedge initial data.

Theorem (Cafasso-C-Ruzza '20)

1. *For any* $t_0 > 0$, *there exist* $M, c > 0$ *such that*

$$u_\sigma(x,t) = \frac{x}{2t} + \mathcal{O}\left(\mathrm{e}^{-c\frac{|x|}{t^{1/3}}}\right) \quad x \leq -Mt^{1/3}, 0 < t < t_0.$$

2. *There exists* $\epsilon > 0$ *such that for any* $M > 0$,

$$u_\sigma(x,t) = \frac{x}{2t} - t^{-2/3} y_\gamma^2\left(-xt^{-1/3}\right) + \mathcal{O}(1), \quad |x| \leq Mt^{1/3}, 0 < t < \epsilon,$$

where y_γ *is the Ablowitz-Segur solution of Painlevé II.*

3. *If* $\gamma = 1$, *there exist* $\epsilon, M > 0$ *such that for any* $K > 0$,

$$u_\sigma(x,t) = v_\sigma(x)\left(1 + \mathcal{O}\left(x^{-1}t^{1/3}\right)\right), \quad Mt^{1/3} \leq x \leq K, 0 < t < \epsilon,$$

where $v_\sigma(x)$ *is a function of* $x > 0$, *independent of* t.

Painlevé equations

Ablowitz-Segur solution of Painlevé II equation characterized by

$$y_\gamma''(s) = s y_\gamma(s) + 2 y_\gamma(s)^3, \; y_\gamma(s) \sim \sqrt{\gamma}\,\mathrm{Ai}(s), \; s \to +\infty.$$

v_σ satisfies an integro-differential Painlevé V equation, and has asymptotics

$$v_\sigma(x) = \frac{1}{8x^2} + \frac{1}{2}\int_{\mathbb{R}} \left(\chi_{(0,+\infty)}(r) - \sigma(r)\right) dr + \mathcal{O}\left(x^2\right), \qquad \text{as } x \to 0.$$

Direct and inverse scattering theory is understood (among others) for KdV solutions decaying at $\pm\infty$ and also for certain classes of unbounded solutions (Its-Sukhanov '20, Dubrovin-Minakov '19), but not for the solutions under consideration.

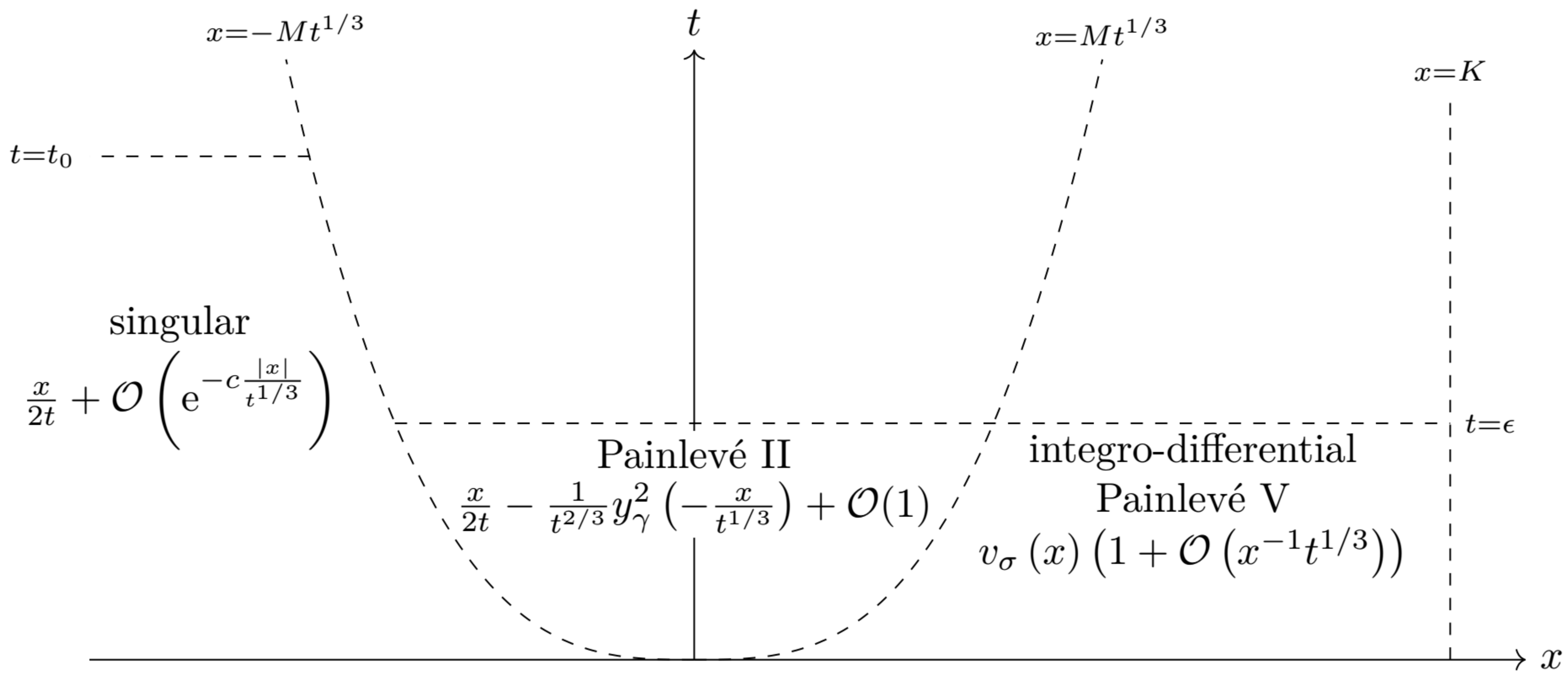


Figure 1: Phase diagram showing the different types of small t asymptotic behavior for $u_\sigma(x,t)$.

By the JACOBI IDENTITY, *we obtain*

$$\partial_t \log\det(1-\sigma_t K) = -\mathrm{Tr}\,(1-\sigma_t K)^{-1}\partial_t\sigma_t K = \mathrm{Tr}\,\frac{\partial_t\sigma_t}{1-\sigma_t}\widetilde{K}_{\sigma_t}$$

with

$$\widetilde{K}_{\sigma_t} = (1-\sigma_t)K(1-\sigma_t K)^{-1}.$$

Probabilistically, this is a relation between an AVERAGE MULTIPLICATIVE STATISTIC *in the Airy point process and an* AVERAGE LINEAR STATISTIC *in another point process:*

$$\partial_t \log \mathbb{E}_{\mathrm{Ai}} \prod_{j=1}^{\infty}(1-\sigma_t(x_i)) == \widetilde{\mathbb{E}} \sum_{j=1}^{\infty} \frac{\partial_t\sigma_t(x_j)}{1-\sigma_t(x_j)}.$$

The Its-Izergin-Korepin-Slavnov method allows to express $\widetilde{K}_\sigma$ in terms of the solution of a $k \times k$ Riemann-Hilbert (RH) problem.

(a) $\Psi : \mathbb{C}\backslash\mathbb{R} \to \mathbb{C}^{2\times 2}$ is analytic
(b) Jump relation

$$\Psi_+(\zeta) = \Psi_-(\zeta) \begin{pmatrix} 1 & 1-\sigma(u) \\ 0 & 1 \end{pmatrix}, \; \zeta \in \mathbb{R}.$$

(c) Asymptotic behavior as $\zeta \to \infty$

$$\Psi(\zeta) = \left(I + \mathcal{O}\left(\frac{1}{\zeta}\right)\right) \frac{\zeta^{\frac{1}{4}\sigma_3}}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \mathrm{e}^{\left(-\frac{2}{3}t\zeta^{3/2} + x\zeta^{1/2}\right)\sigma_3} \times \begin{cases} I, \\ \begin{pmatrix} 1 & 0 \\ \mp 1 & 1 \end{pmatrix}. \end{cases}$$

Consequences

KdV initial data encode precise estimates for KPZ tail probabilities (cf. Corwin-Ghosal '18, Tsai '19, Le Doussal '20, Lin-Tsai '20).

KdV initial data encode large gap asymptotics in models for finite temperature free fermions.

KdV initial data encode asymptotics for averages of multiplicative statistics in the Airy point process.

Further questions

Uniform $t \to 0$ asymptotics of $u_\sigma(x,t)$ for x large? In progress Charlier-C-Ruzza.

Scattering theory for a class of perturbations of self-similar KdV solutions $u_\gamma(x,t) = \frac{x}{2t} - t^{-2/3} y_\gamma^2\left(-xt^{-1/3}\right)$?

In analogy to

1. *classical scattering theory for perturbations of zero solution* $u(x,t) = 0$,
2. *scattering theory for perturbations of the exact KdV solution* $u(x,t) = \frac{x}{2t}$ (Its-Sukhanov '20),
3. *scattering theory for perturbations of the* P_I^2 *solution of KdV* (Dubrovin-Minakov '19).

The end

Thank you for your attention!