Logarithmic Painlevé functions and Mathieu stability chart

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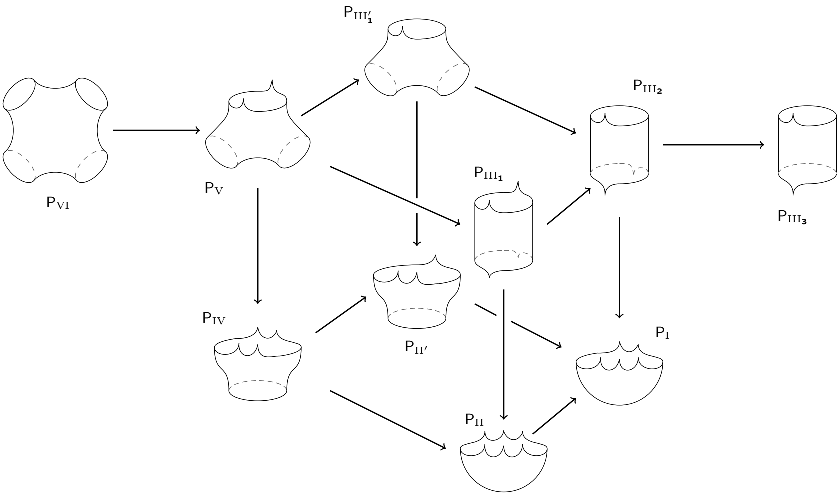
joint work with Pavlo Gavrylenko

- We will be interested in reconstruction of linear ODEs in the complex domain from their monodromy $\to$ inverse monodromy map
- Simplest model cases correspond to ODEs/linear systems for classical special functions (hypergeometric, Bessel, Airy, ...).

  Example: 2nd order ODE with 3 regular singular points $\to$ Gauss
- **NB**: reconstructing an ODE is easier than actually solving it!
- Increasing the order of the equation, the number of singular points or making them more irregular complicates the problem drastically $\to$ accessory parameters
- Next-to-simplest model cases correspond to
  - scalar 2nd order ODEs $\to$ Heun's equation and its degenerations (Mathieu equation, cubic and quartic quantum oscillator, ...)
  - rank 2 linear systems $\to$ Painlevé functions

Degeneration of Painlevé equations [Chekhov, Mazzocco, Rubtsov, '15]



- Inverse monodromy problem for scalar 2nd order ODEs from the upper part of the diagram can be solved using Hill's determinant and continued fractions $\to$ perturbative series for accessory parameter functions
- The same problem for linear systems amounts to solving Painlevé VI, V and III. Solutions are known for generic monodromy (two-parameter dependence on the initial conditions):
  - Fredholm determinants [Cafasso, Gavrylenko, L, '17] (also [Desiraju, '20] for homogeneous PII)
  - combinatorial series
- Also for generic monodromy, the solutions of the inverse monodromy problem for both scalar ODEs and linear systems are related to 2D Virasoro conformal field theory
  - accessory parameter functions $\to$ $c = \infty$ conformal blocks [Zamolodchikov, '86]
  - Painlevé functions $\to$ $c = 1$ conformal blocks [Gamayun, Iorgov, L, '12]

Main question: What can we do for non-generic monodromy?

Mathieu equation

- Classical harmonic oscillator with periodic forcing of the stiffness coefficient

$$\frac{d^2X}{dt^2} + \left(\omega^2 - 2q\cos 2t\right)X = 0$$

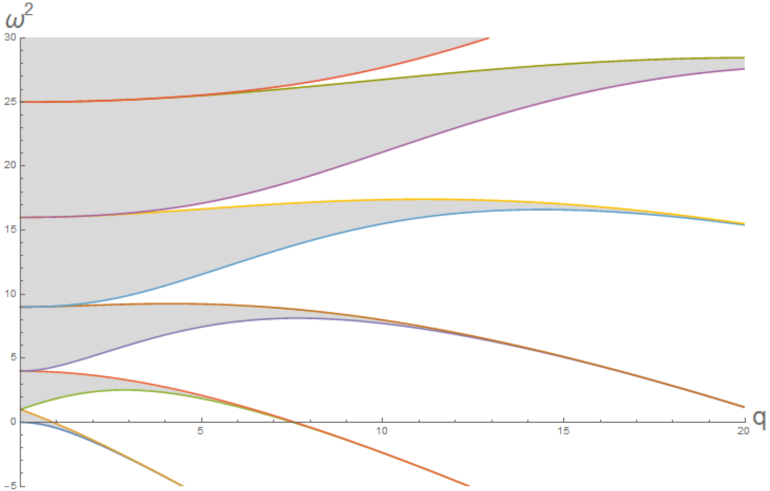
- Schroedinger equation for cosine potential (electron in 1D crystal)

$$\left(-\frac{d^2}{dx^2} + 2r^2\cos 2x\right)\psi = E\psi$$

- Change of variables $\psi(x) = e^{-ix}f\left(r^2e^{2ix}/4\right) \Longrightarrow$ rational potential, two irregular rank 1 singularities at $0, \infty$

$$f''(z) = \left(\frac{t}{z^3} - \frac{\mathcal{E}}{z^2} + \frac{1}{z}\right)f(z)$$

Stability chart [Ince, 1927] $\Longleftrightarrow$ energy bands and gaps



Since $H = -\dfrac{d^2}{dx^2} + 2r^2 \cos 2x$ commutes with translations $x \mapsto x + \pi$, look for solutions in the Floquet form,

$$\psi(x) = e^{i\nu x} \sum_{n \in \mathbb{Z}} a_n e^{2inx}.$$

The coefficients of the Fourier series satisfy the recurrence

$$\left(E - (\nu + 2n)^2\right) a_n = r^2 \left(a_{n+1} + a_{n-1}\right).$$

The existence of a nontrivial solution of this recurrence relation amounts to vanishing of Hill's determinant of the infinite tridiagonal matrix

$$\Delta(\nu) = \det \begin{pmatrix} \ddots & \ddots & 0 & & \\ \ddots & \frac{(\nu+2)^2-E}{2^2-E} & \frac{r^2}{2^2-E} & 0 & \cdots \\ 0 & \frac{r^2}{0^2-E} & \frac{\nu^2-E}{0^2-E} & \frac{r^2}{0^2-E} & 0 \\ & 0 & \frac{r^2}{2^2-E} & \frac{(\nu-2)^2-E}{2^2-E} & \frac{r^2}{2^2-E} \\ & & 0 & \ddots & \ddots \end{pmatrix}$$

There is an surprising relation

$$\Delta(\nu) = \Delta(0) - \frac{\sin^2 \frac{\pi\nu}{2}}{\sin^2 \frac{\pi\sqrt{E}}{2}}$$

The equation determining ν (monodromy) as a function of the eigenvalue E (accessory parameter) and r (time) can therefore be written as

$$\det \begin{pmatrix} \ddots & \ddots & 0 & & \\ \ddots & 1 & \frac{r^2}{2^2-E} & 0 & \cdots \\ 0 & \frac{r^2}{0^2-E} & 1 & \frac{r^2}{0^2-E} & 0 \\ & 0 & \frac{r^2}{2^2-E} & 1 & \frac{r^2}{2^2-E} \\ & & 0 & \ddots & \ddots \end{pmatrix} = \frac{\sin^2 \frac{\pi\nu}{2}}{\sin^2 \frac{\pi\sqrt{E}}{2}}$$

Continued fractions

One can derive a continued fraction equation relating E, ν and $q = r^2$:

$$\frac{q^2}{E-(\nu+2)^2-\frac{q^2}{E-(\nu+4)^2-\frac{q^2}{E-(\nu+6)^2}\cdots}}+\frac{q^2}{E-(\nu-2)^2-\frac{q^2}{E-(\nu-4)^2-\frac{q^2}{E-(\nu-6)^2}\cdots}}=E-\nu^2$$

can be solved by a perturbative series in r with coefficients rational in ν:

$$E(q|\nu)=\nu^2+\frac{q^2}{2(\nu^2-1)}+\frac{(5\nu^2+7)\,q^4}{32(\nu^2-1)^3(\nu^2-4)}+\frac{(9\nu^4+58\nu^2+29)\,q^6}{64(\nu^2-1)^5(\nu^2-4)(\nu^2-9)}+\dots$$

- $E(q|\nu)$ is implemented in Mathematica as **MathieuCharacteristicA[**ν, r^2**]**.
- Coefficients of $E(q|\nu)$ have poles for $\nu \in \mathbb{Z}$
- For $\nu \in \mathbb{Z}$, solutions of different parity and different expansions; e.g.

$$E_{\text{even/odd}}(q|\nu=1)=1\pm q-\frac{q^2}{8}\mp\frac{q^3}{64}-\frac{q^4}{1536}\pm\frac{11q^5}{36864}+\dots$$

Painlevé III_3

- Radial sine-Gordon equation

$$\frac{d^2\varphi}{dr^2} + \frac{1}{r}\frac{d\varphi}{dr} = \sin\varphi$$

  Applications: scaling limit of 2D Ising model, $\mathcal{N} = 2$ 4D SUSY gauge theory, ...

- Canonical form

$$\frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} + \frac{2q^2}{t^2} - \frac{2}{t}.$$

  Reduces to radial sine-Gordon by setting $q\left(2^{-12}r^4\right) = \frac{r^2}{64}\, e^{i\varphi(r)}$

- Painlevé functions solve the inverse monodromy problem for systems of linear ODEs with rational coefficients and prescribed singularity structure

Linear system associated to PIII$_3$:

$$\partial_z\Phi = \Phi\left(\frac{1}{z^2}\begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix} + \frac{1}{qz}\begin{pmatrix} -p & t \\ -q & p \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)$$

- 2nd order poles at $z = 0$ and $z = \infty$ with non-diagonalizable leading contributions
- **Problem**: Find p, q corresponding to given monodromy data (Stokes matrices and connection matrix relating canonical solutions at 0 and ∞)
- $p(t)$, $q(t)$ then *automatically* satisfy

$$\begin{cases} tq_t = 2p + q, \\ tp_t = \frac{2p^2}{q} + p + q^2 - t, \end{cases} \qquad \Longleftrightarrow \qquad \text{Painlevé III}_3$$

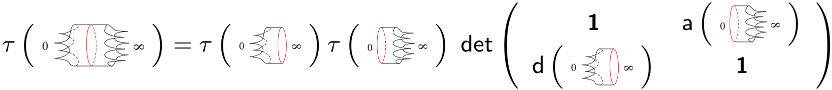
and can be expressed in terms of the tau function defined by

$$t\partial_t \ln\tau = \frac{p^2}{q^2} - q - \frac{t}{q}$$

For example, $q(t) = -\left(t\frac{d}{dt}\right)^2 \ln\tau(t)$

Strategy of solution [Cafasso-Gavrylenko-L, '17]

1. Convert the linear system into a Riemann-Hilbert problem with constant 2×2 jumps.
2. Transform it into a RHP on a circle with the jump matrix determined by solutions $\Phi_{\pm}(z)$ of two auxiliary RHPs (corresponding to linear systems with one 2nd order pole and one regular singularity)
3. Use Widom's variational formula to identify isomonodromic tau function $\tau(t)$ with a Fredholm determinant whose matrix integral kernel is given in terms of $\Phi_{\pm}(z)$.



4. In our case, $\Phi_{\pm}(z)$ can be written explicitly in terms of Bessel functions.
5. Expanding the determinant into a sum of principal minors in the basis of Fourier modes gives a series representation of $\tau(t)$.

Theorem 1. The tau function $\tau(t)$ corresponding to general solution of Painlevé III_3 can be expressed as Fredholm determinant

$$\tau(t) = \mathrm{const} \cdot t^{\left(\sigma+\frac{1}{2}\right)^2} \det\left(\mathbf{1} - K\right), \qquad K = \begin{pmatrix} 0 & \mathsf{a} \\ \mathsf{d} & 0 \end{pmatrix}.$$

Here the operators a, d act on vector-valued functions on a circle C centered at the origin and oriented counterclockwise,

$$(f\mathsf{a})(z) = \frac{1}{2\pi i}\oint_C f(z')\,\mathsf{a}(z',z)\,dz', \qquad (f\mathsf{d})(z) = \frac{1}{2\pi i}\oint_C f(z')\,\mathsf{d}(z',z)\,dz',$$

and the integral kernels $\mathsf{a}(z',z)$, $\mathsf{d}(z',z)$ are explicitly given by

$$\mathsf{a}(z',z) = e^{i\pi(\sigma-2\eta)\sigma_z}\,\frac{\mathbb{J}_\sigma(z',z) - \mathbf{1}}{z-z'}\,e^{i\pi(2\eta-\sigma)\sigma_z},$$

$$\mathsf{d}(z',z) = t^{\left(\sigma+\frac{1}{2}\right)\sigma_z} e^{i\pi\sigma\sigma_z}\sigma_y\,\frac{\mathbf{1} - \mathbb{J}_\sigma\left(\frac{t}{z'},\frac{t}{z}\right)}{z-z'}\,\sigma_y e^{-i\pi\sigma\sigma_z} t^{-\left(\sigma+\frac{1}{2}\right)\sigma_z},$$

$$\mathbb{J}_\sigma(z',z) = \frac{\pi}{\sin 2\pi\sigma}\begin{pmatrix} z' j_{\sigma+\frac{1}{2}}(z) j_{-\sigma}(z') - j_\sigma(z) j_{-\sigma-\frac{1}{2}}(z') & iz' j_{-\sigma-\frac{1}{2}}(z) j_{-\sigma}(z') - iz j_{-\sigma}(z) j_{-\sigma-\frac{1}{2}}(z') \\ i j_{\sigma+\frac{1}{2}}(z) j_\sigma(z') - i j_\sigma(z) j_{\sigma+\frac{1}{2}}(z') & z j_{-\sigma}(z) j_{\sigma+\frac{1}{2}}(z') - j_{-\sigma-\frac{1}{2}}(z) j_\sigma(z') \end{pmatrix}$$

with $j_\sigma(z) = z^{-\sigma} I_{2\sigma}\left(2\sqrt{z}\right) = \dfrac{{}_0F_1(2\sigma+1;z)}{\Gamma(2\sigma+1)}$.

Theorem 2. The tau function $\tau(t)$ can be written as a Fourier transform

$$\tau(t) = \mathrm{const} \cdot \sum_{n \in \mathbb{Z}} e^{4i\pi n\eta} C(\sigma + n)\, \mathcal{B}(t|\sigma + n),$$

where $\mathcal{B}(t|\sigma)$ admits the following combinatorial series representation:

$$\mathcal{B}(t|\sigma) = \sum_{\mathsf{Y}_+, \mathsf{Y}_- \in \mathbb{Y}} \frac{t^{\sigma^2 + |\mathsf{Y}_+| + |\mathsf{Y}_-|}}{\prod_{s,s'=\pm 1} Z(\sigma(s - s') \,|\, \mathsf{Y}_{s'}, \mathsf{Y}_s)},$$

$$Z(\sigma \,|\, \mathsf{Y}_+, \mathsf{Y}_-) = \prod_{\square \in \mathsf{Y}_+} (\sigma + 1 + a_{\mathsf{Y}_+}(\square) + l_{\mathsf{Y}_-}(\square)) \prod_{\square \in \mathsf{Y}_-} (\sigma - 1 - a_{\mathsf{Y}_-}(\square) - l_{\mathsf{Y}_+}(\square)),$$

$$C(\nu) = [G(1 + 2\sigma)\, G(1 - 2\sigma)]^{-1}.$$

Notations:

- $\mathbb{Y}$ is the set of all Young diagrams
- $a_{\mathsf{Y}}(\square)$ and $l_{\mathsf{Y}}(\square)$ are the arm- and leg-length of the box $\square$ in the Young diagram Y
- $|\mathsf{Y}|$ is the total number of boxes in Y
- $G(\sigma)$ is the Barnes G-function satisfying $G(1 + \sigma) = \Gamma(\sigma)\, G(\sigma)$.

Painlevé III_3 tau function

$$\tau(t) = \sum_{n\in\mathbb{Z}} e^{4i\pi n\eta}\mathcal{B}(t|\sigma+n)$$

Building block $\mathcal{B}(t|\sigma)$

- small t expansion absolutely convergent in the entire complex plane for $2\sigma \notin \mathbb{Z}$ [Its-L-Prokhorov, '14]
- few first terms are explicitly given by

$$\mathcal{B}(t|\sigma) = t^{\sigma^2}\left[1 + \frac{t}{2\sigma^2} + \frac{(8\sigma^2+1)\,t^2}{4\sigma^2(4\sigma^2-1)^2} + \frac{(8\sigma^4-5\sigma^2+3)\,t^3}{24\sigma^2(4\sigma^2-1)^2(\sigma^2-1)^2} + \dots\right]$$

Leading terms for $|\Re\sigma| < \frac{1}{2}$:

$$\tau(t) = t^{\sigma^2}\left[\left(1 + \frac{t}{2\sigma^2} + \frac{(8\sigma^2+1)\,t^2}{4\sigma^2(4\sigma^2-1)^2} + \ldots\right)\right.$$
$$- e^{4\pi i\eta}\frac{\Gamma^2(1-2\sigma)}{\Gamma^2(1+2\sigma)}\left(1 + \frac{t}{2(\sigma+1)^2} + \ldots\right)\frac{t^{1+2\sigma}}{4\sigma^2(1+2\sigma)^2}$$
$$\left. - e^{-4\pi i\eta}\frac{\Gamma^2(1+2\sigma)}{\Gamma^2(1-2\sigma)}\left(1 + \frac{t}{2(\sigma-1)^2} + \ldots\right)\frac{t^{1-2\sigma}}{4\sigma^2(1-2\sigma)^2} + \ldots\right].$$

Singular as $\sigma \to 0$ but finite limit if simultaneously $\eta \to 0$, η/σ finite!

Problem: Show that the limit exists in all orders and compute the corresponding logarithmic expansion.

Conjecture: The Painlevé III_3 tau functions corresponding to $\sigma = 0$ and $\sigma = \frac{1}{2}$ have asymptotic expansions

$$\tau_{\sigma=0}(t) = \sum_{n=0}^{\infty} P_n(\Omega)\, t^n, \qquad \tau_{\sigma=\frac{1}{2}}(t) = \sum_{n=0}^{\infty} Q_n(\Omega)\, t^{n+\frac{1}{4}},$$

where P_n and Q_n are polynomials in $\Omega = \ln t + \kappa$ and κ is a constant parameterizing the initial conditions. Moreover,

$$\deg\ P_n = 2\lfloor\sqrt{n}\rfloor, \qquad \deg\ Q_n = 2\lfloor\sqrt{n+\tfrac{1}{4}} - \tfrac{1}{2}\rfloor + 1.$$

One readily finds

$$\begin{aligned}
&P_0(\Omega) = 1, && Q_0(\Omega) = \Omega,\\
&P_1(\Omega) = -\Omega^2 + 4\Omega - 6, && Q_1(\Omega) = 2\Omega - 8,\\
&P_2(\Omega) = -\frac{\Omega^2}{2} + 3\Omega - \frac{7}{4}, && Q_2(\Omega) = \frac{1}{4}\left(-\Omega^3 + 10\Omega^2 - 37\Omega + 47\right),\\
&\ldots && \ldots
\end{aligned}$$

Theorem. For $p, q \in \mathbb{Z}_{\geq 0} + \frac{1}{2}$, define 2×2 matrices

$$\tilde{a}_{-q}^{\ \ p} = \frac{\begin{pmatrix} -2H_{q-\frac{1}{2}} - \frac{1}{p+q} & -1 \\ \left(2H_{p-\frac{1}{2}} + \frac{1}{p+q}\right)\left(2H_{q-\frac{1}{2}} + \frac{1}{p+q}\right) + \frac{1}{(p+q)^2} & 2H_{p-\frac{1}{2}} + \frac{1}{p+q} \end{pmatrix}}{\left(p-\frac{1}{2}\right)!^2 \left(q-\frac{1}{2}\right)!^2 (p+q)}.$$

Further introduce

$$\tilde{b}_{-q}^{\ \ p} = Q^{-1} \tilde{a}_{-q}^{\ \ p} Q, \qquad \tilde{c}^{-q}_{\ \ p} = t^{p+q} \tilde{a}_{-p}^{\ \ q},$$

with $Q = \begin{pmatrix} -1 & 0 \\ \Omega & 1 \end{pmatrix}$ and $\Omega = \ln t + \kappa$. Then the tau function $\tau_{\sigma=0}(t)$ admits the following determinant representation:

$$\tau_{\sigma=0}(t) = \det \left(\mathbf{1} - \begin{pmatrix} \cdot & \tilde{c}^{-\frac{1}{2}}_{\ \ \frac{3}{2}} & \tilde{c}^{-\frac{1}{2}}_{\ \ \frac{1}{2}} \\ \cdot & \cdot & \tilde{c}^{-\frac{3}{2}}_{\ \ \frac{1}{2}} \\ \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} \cdot & \cdot & \cdot \\ \tilde{b}_{-\frac{1}{2}}^{\ \ \frac{3}{2}} & \cdot & \cdot \\ \tilde{b}_{-\frac{1}{2}}^{\ \ \frac{1}{2}} & \tilde{b}_{-\frac{3}{2}}^{\ \ \frac{1}{2}} & \cdot \end{pmatrix} \right) =$$

$$= 1 - \left(\Omega^2 - 4\Omega + 6\right) t - \left(\frac{\Omega^2}{2} - 3\Omega + \frac{7}{4}\right) t^2 - \left(\frac{\Omega^2}{4} - \frac{43\Omega}{18} + \frac{311}{54}\right) t^3 + \ldots$$

Linear system $\partial_z \Phi = \Phi A(z)$ associated to PIII_3:

$$A(z) = \frac{1}{z^2}\begin{pmatrix} 0 & 0 \\ q & 0 \end{pmatrix} + \frac{1}{qz}\begin{pmatrix} -p & t \\ -q & p \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{1}{z^2}\begin{pmatrix} * & * \\ q-z & * \end{pmatrix}$$

The components of the first column of the fundamental matrix, $f(z) = \frac{\Phi_{k1}(z)}{\sqrt{A_{21}(z)}}$ satisfy the equation $f'' = V(z) f$ with

$$V(z) = -\det A + A'_{11} - A_{11}\frac{A'_{21}}{A_{21}} + \frac{3}{4}\left(\frac{A'_{21}}{A_{21}}\right)^2 - \frac{1}{2}\frac{A''_{21}}{A_{21}}.$$

For $A(z)$ as above we have

$$V(z) = \frac{t}{z^3} + \frac{\frac{p^2}{q^2} - \frac{t}{q} - q}{z^2} + \frac{1}{z} + \frac{3}{4(z-q)^2} - \frac{z-p}{z^2(z-q)}.$$

One may hope for a reduction to Mathieu if $q = \infty$ (i.e. at movable poles of q) or $q = 0$ (zeros). Local analysis of PIII_3 shows that near a zero $t = s$ we indeed have

$$q = \frac{(t-s)^2}{s} + \frac{E}{3s^3}(t-s)^4 + O\left((t-s)^5\right),$$

$$\frac{p^2}{q^2} - \frac{t}{q} - q = \left(E + \tfrac{1}{4}\right) + O\left((t-s)^2\right),$$

$$V(z) = \frac{s}{z^3} + \frac{E}{z^2} + \frac{1}{z} + O(t-s),$$

Conclusion: The position s of zero of $q(t)$ is related to Mathieu coupling constant by $s = \frac{r^4}{16}$, while the logarithmic derivative of the tau function $t\partial_t \tau(t)\big|_{t=s}$ coincides with $E/4$.

Backlund transformations

If $q(t)$ solves PIII$_3$, then $\tilde{q}(t) = \frac{t}{q(t)}$ solves it as well. Moreover, the tau function transforms as

$$\partial_t \ln \frac{\tilde{\tau}}{\tau} = \frac{1}{2} \partial_t \ln \frac{q}{\sqrt{t}} \qquad \Longleftrightarrow \qquad \tilde{\tau} = \mathrm{const} \cdot \frac{q^{\frac{1}{2}} \tau}{t^{\frac{1}{4}}}$$

and is characterized by monodromy data $(\tilde{\sigma}, \tilde{\eta}) = \left(\sigma + \frac{1}{2}, \eta\right)$.

At every zero $t = s$ of q (which is necessarily of 2nd order):

- associated tau function τ is non-vanishing,
- Backlund transformed solution $\tilde{q}$ has a 2nd order pole,
- Backlund transformed tau function $\tilde{\tau}$ has a simple zero.

Mathieu characteristic values

- Consider $\tau(t) = 0$ as an equation for Ω (monodromy!) in terms of t
- It has an infinite number of solutions, both for $\sigma = 0$ and $\sigma = \frac{1}{2}$ which differ by their asymptotic behavior as $t \to 0$, for instance,

$$\Omega_0(t) = 8t - \frac{111t^2}{4} + \frac{30740t^3}{243} - \frac{108164207t^4}{165888} + \frac{6502377283t^5}{1800000} + \ldots,$$

$$\Omega_{1,a}(t) = \frac{1}{\sqrt{t}} + 2 - \frac{5\sqrt{t}}{4} + \frac{t}{2} + \frac{61t^{3/2}}{72} + \frac{89t^2}{216} - \frac{4085t^{5/2}}{20736} + \ldots,$$

$$\Omega_{2,a}(t) = \frac{2}{t} + \frac{61}{9} - \frac{20917t}{2592} + \frac{32478977t^2}{1166400} - \frac{106234716511t^3}{839808000} + \ldots,$$

$$\ldots$$

- Compute $t\partial_t\tau(t)$ and substitute the above expressions for $\Omega \to$ Mathieu eigenvalues $a_n(q)$, $b_n(q)$

Conclusions

1. One-parameter logarithmic solutions of Painlevé III_3 admit both Fredholm determinant and combinatorial series representations
2. Together with generic 2-parameter family, this covers all Painlevé III_3 solutions ($\sigma = \frac{1}{2}$ is obtained from $\sigma = 0$ by Bäcklund transformations)
3. Series expansions of Mathieu characteristic values $a_n(q)$, $b_n(q)$ are related to zeros of logarithmic tau functions $\tau_{\sigma=0}(t)$ and $\tau_{\sigma=\frac{1}{2}}(t)$

THANK YOU!