Hirota equations for extended NLS hierarchy

Guido Carlet

Dijon

June 2021

Based on a joint work with van de Leur, Posthuma and Shadrin:

“Higher genera Catalan numbers and Hirota equations for extended nonlinear Schrödinger hierarchy”,

published in the LMP Dubrovin Memorial issue.

My motivation for this work goes back to my first paper with Boris...

The Extended Toda hierarchy

The Lax representation of the Toda hierarchy

$$L = \Lambda + v + e^u \Lambda^{-1}$$

$\Lambda f(x) = f(x+\varepsilon)$

$$\epsilon \frac{\partial L}{\partial q_l^i} = [(A_l^i)_+, L]$$

where for $i = 1$, $l \geqslant 0$ we have the classical flows with

$$A_l^1 = \frac{L^{l+1}}{(l+1)!}$$

admits an extension by a second infinite sequence of commuting flows for $i = 2$, $l \geqslant 0$ with

$$A_l^2 = 2\frac{L^l}{l!}\left(\underline{\log L} - \mathfrak{h}(l)\right).$$

Notice that here the spatial variable x coincides with q_0^2.

[C-Dubrovin-Zhang]

$\log L$

We define a logarithm of the difference operator L via dressing operators and find that

$$\log L = \sum_{k\in\mathbb{Z}} w_k \Lambda^k$$

where the coefficients w_k are uniquely differential polynomials in

$$\mathcal{A} := \mathbb{C}[u, v, e^{\pm u}][v_k, u_k; k \geqslant 0][[\epsilon]]_0.$$

[C-Dubrovin-Zhang]

Toda conjecture

[Eguchi-Yang, Getzler, Zhang, Okounkov-Pandharipande]

The Gromov-Witten potential of $\mathbb{C}P^1$ is a tau function of the Extended Toda hierarchy.

...via Virasoro constraints [Givental, Dubrovin-Zhang]

...via Givental total descendent potential and Hirota equations [Milanov]

Extended NLS

If we choose as spatial variable X the variable q_0^1 we obtain a hierarchy with Lax formulation given in terms of pseudodifferential operators

$$\mathcal{L} = \epsilon\partial_X + \rho(\epsilon\partial_X - \phi)^{-1},$$

$$\epsilon\frac{\partial\mathcal{L}}{\partial q_\ell^i} = [-(\tilde{A}_\ell^i)_-, \mathcal{L}]$$

$$\tilde{A}_\ell^1 = \frac{\mathcal{L}^{\ell+1}}{(\ell+1)!}, \qquad \tilde{A}_\ell^2 = \frac{2}{\ell!}(\log\mathcal{L} - \mathfrak{h}(\ell))\mathcal{L}^\ell,$$

where here the operator $\log\mathcal{L}$ is defined indirectly by change of variables in the operator $\log L$.

[C-Dubrovin-Zhang]

Aim

Our aim is to prove that the topological partition function associated to a $d = -1$ two-dimensional Frobenius manifold is the string tau-function of the extended NLS hierarchy.

We follow the approach used by Givental, Milanov, Tseng in a number of cases (simple singularities, orbifold $\mathbb{C}P^1$ GW theory) to show that the total descendent potential satisfies Hirota equations, and then derive Lax formulations.

Our general philosophy is to avoid superpotential / singularity theory arguments to focus on the structures intrinsically existing on the Dubrovin-Frobenius manifold.

This is the first step of a work in progress to generalize these claims to rationally constrained KP hierarchies, cf. Liu-Zhang-Zhou conjecture.

From Frobenius to Hirota and Lax

2D Frobenius manifold $\underline{d = -1}$

$$\downarrow$$

- Deformed flat connection $\rightarrow$ S, R
- Fuchsian system $\rightarrow$ $I_a^{(l)}$ period vector

$$\downarrow$$

$$\text{quantization} \begin{cases} \text{of quadratic Hamiltonians} \rightarrow \hat{S}, \hat{R} \\ \text{of linear Hamiltonians} \rightarrow \Gamma^a = e^{\hat{f}_a} \end{cases}$$

vertex ops

$$\downarrow$$

$$\text{potentials} \begin{cases} \text{ancestor} \rightarrow \mathcal{A} = \hat{\Psi}\underline{\hat{R}}(\tau_{KW} \otimes \tau_{KW}) \\ \text{descendent} \rightarrow \mathcal{D} = C\underline{\hat{S}}^{-1}\mathcal{A} \end{cases}$$

$$\downarrow$$

Hirota equations

$$\begin{cases} \text{ancestor} \rightarrow \mathcal{N} \otimes \mathcal{N} \left(\sum_{a\in O} c_a \Gamma^a \otimes \Gamma^{-a}\right) \underline{\mathcal{A}} \otimes \underline{\mathcal{A}}\, d\lambda \\ \text{descendent} \rightarrow \mathcal{N}_\infty \otimes \mathcal{N}_\infty \left(\sum_{a\in O} c_a^\infty \Gamma^a_\infty \otimes \Gamma^{-a}_\infty\right) \underline{\mathcal{D}} \otimes \underline{\mathcal{D}}\, d\lambda \end{cases}$$

$$\downarrow$$

Lax formulation

The Dubrovin-Frobenius manifold

We consider the $d = \underline{-1}$ Frobenius manifold $M = \mathbb{C} \times \mathbb{C}^*$ with potential in flat coordinates t^1, t^2 given by

$$F(t^1, t^2) = \frac{1}{2}(t^1)^2 t^2 + \frac{1}{2}(t^2)^2 \log t^2,$$

with the standard antidiagonal flat metric $\eta_{\alpha\beta} = \delta_{\alpha+\beta,3}$ and the unit and Euler vector fields

$$e = \partial_{t^1}, \qquad E = t^1 \partial_{t^1} + 2t^2 \partial_{t^2}.$$

[Dubrovin]

$$\eta = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

The deformed flat connection and the S matrix

The deformed flatness equations for a matrix valued function Y on $M \times \mathbb{C}$ take the form

$$-z\frac{\partial Y}{\partial z} = (\mu + \frac{\mathcal{U}}{z})Y, \qquad z\frac{\partial Y}{\partial t^\alpha} = C_\alpha Y,$$

in our case $\mu = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}, \mathcal{U} = \begin{pmatrix} t^1 & 2 \\ 2t^2 & t^1 \end{pmatrix}$.

E.

The normal form of the solution near $z = \infty$ is

$$Y(t,z) = S(t,z)z^{-\mu}z^{-\tilde{R}}, \qquad S = \sum_{k\geqslant 0} S_k z^{-k}$$

with $\tilde{R} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$. The matrix S is uniquely determined by the choice of the constant ψ in $(S_1)_{1,2} = \psi + \log t^2$ (calibration).

The deformed flat connection and the R matrix

In the normalized canonical frame the (z part of the) deformed flatness equation takes the form

$$-z\frac{\partial \tilde{Y}}{\partial z} = (V + \frac{U}{z})\tilde{Y},$$

$z \sim 0$

and there exists a unique formal solution of this equation of the form

$$\tilde{Y}(t,z) = R(t,z)e^{U/z},$$

where $R = \sum_{k\geqslant 0} R_k z^k$, $R_0 = \mathbf{1}$.

In our case we find

$$R_k = \frac{\left(\frac{1}{2}\right)_{k-1}\left(\frac{1}{2}\right)_k}{(k)!}\begin{pmatrix} \frac{(-1)^{k+1}}{2} & k\mathrm{i} \\ (-1)^{k+1}k\mathrm{i} & -\frac{1}{2}\end{pmatrix}(u_2 - u_1)^{-k}.$$

canonical coords.

Ancestor and descendent potentials

The total ancestor potential is

KdV τ-fn

$$\mathcal{A}(\{q_a^1, q_a^2\}_{a\geqslant 0}) := \hat{\Psi}\hat{R}\prod_{i=1}^{2}\tau_{KW}(\{Q_a^i\}_{a\geqslant 0}),$$

Kontsevich Witten

where $\hat{\Psi}$ changes the variables from Q_a^i (normalized canonical frame) to q_a^i (flat frame).

The total descendent potential

$$\mathcal{D} = C\hat{S}^{-1}\mathcal{A},$$

where $\log C(t^1, t^2) = -\frac{1}{16}\log t^2$.

$\mathcal{D}$ has zero derivatives with respect to the point of M.

[Givental]

Symplectic loop space and quantization (1)

Given $(V, (,))$ with basis $\{\phi_i\}$ we set up a formal loop space $V((z))$ with symplectic structure given by a residue

$$\Omega(\mathfrak{f}, \mathfrak{g}) = \mathrm{Res}_z(\mathfrak{f}(-z), \mathfrak{g}(z))dz$$

on which we have Darboux coordinates q_k^i, $p_{i,k}$, $k \geqslant 0$.

As explained by Givental two days ago we consider quantization of linear and quadratic Hamiltonians on the loop space, with the usual quantization rules.

Weyl quant

$q \longrightarrow q$

$p \longrightarrow \frac{\partial}{\partial q}$

Linear Hamiltonian: $f \mapsto \hat{f}$

Quadratic Hamiltonians: $\begin{cases} S = e^s \mapsto \hat{S} = e^{\hat{s}} \\ R = e^r \mapsto \hat{R} = e^{\hat{r}} \end{cases}$

$qp \longrightarrow q \frac{\partial}{\partial q}$

[Givental]

Symplectic loop space and quantization (2)

The linear Hamiltonian $h_{\mathfrak{f}}(\cdot) = \Omega(\mathfrak{f}, \cdot)$ associated with a constant vector field $\mathfrak{f} = \sum_l I^l(-z)^l \in \mathcal{V}$ is quantized as

$$\hat{\mathfrak{f}} = (h_{\mathfrak{f}})\hat{} = \sum_{l\geqslant 0} \left[\epsilon(-1)^{l+1} (I^l)^i \frac{\partial}{\partial q_l^i} + \frac{1}{\epsilon} (I^{-(l+1)})_i q_l^i \right] .$$

The quadratic Hamiltonian $h_{\mathfrak{m}}(\mathfrak{f}) = \frac{1}{2}\Omega(\mathfrak{m}\mathfrak{f}, \mathfrak{f})$ associated with an infinitesimal symplectic vector field of the form $\mathfrak{f} \mapsto \mathfrak{m}\mathfrak{f}$ where $\mathfrak{m} = \sum_{\ell\geqslant 1} m_\ell z^{-\ell}$ or $\mathfrak{m} = \sum_{\ell\geqslant 1} m_\ell z^{\ell}$ with $m_\ell \in \mathrm{End}(V)$ is quantized as

S R

$$\hat{s} = (h_{\mathfrak{m}})\hat{} = \frac{1}{2\epsilon^2} \sum_{a,b\geqslant 0} (-1)^{b+1} q_a^i q_b^j (\underbrace{m_{a+b+1}}_{=s})_i^k \eta_{kj} - \sum_{a\geqslant 0, \ell\geqslant 1} q_{a+\ell}^i \frac{\partial}{\partial q_a^j} (\underbrace{m_\ell}_{=s})_i^j .$$

$S = e^s$ $\hat{S} = e^{\hat{s}}$

$$\hat{r} = (h_{\mathfrak{m}})\hat{} = \frac{\epsilon^2}{2} \sum_{a,b\geqslant 0} (-1)^a \frac{\partial}{\partial q_a^i} \frac{\partial}{\partial q_b^j} (\underbrace{m_{a+b+1}}_{r})_k^i \eta^{kj} - \sum_{a\geqslant 0, \ell\geqslant 1} q_a^i \frac{\partial}{\partial q_{a+\ell}^j} (\underbrace{m_\ell}_{r})_i^j .$$

$R = e^r$ $\hat{R} = e^{\hat{r}}$

For $M = \exp(\mathfrak{m})$ we define $\hat{M} = \exp(\hat{\mathfrak{m}})$.

[Givental]

Period vectors

The period vectors $I_{e_i}^{(l)}$ are uniquely defined on $(M \times \mathbb{C}) \setminus \Delta_\lambda$ as holomorphic multivalued solutions of the Fuchsian system, explicitly

$$(\mathcal{U} - \lambda)\frac{\partial I^{(l)}}{\partial \lambda} = (\mu + l + \frac{1}{2})I^{(l)}$$

with asymptotic conditions at the $\lambda \sim u^i$ $i=1,2$

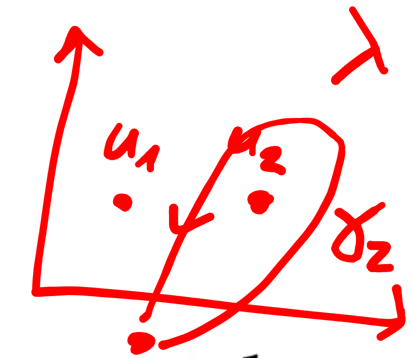
$$I_{e_i}^{(l)} = \frac{(-1)^l}{\sqrt{2\pi}}\Gamma(l+1/2)(\lambda - u_i)^{-l-\frac{1}{2}}(\Psi^{-1}e_i + O(\lambda - u^i))$$

and such that the analytic continuation of $I_{e_i}^{(l)}$ along a small path γ_i surrounding u^i is equal to $-I_{e_i}^{(l)}$.

For $a = a_1e_1 + a_2e_2 \in \mathbb{C}^2$ we denote $I_a^{(l)} = a_1I_{e_1}^{(l)} + a_2I_{e_2}^{(l)}$.

[Dubrovin]

Monodromy of period vectors (1)



Let $\pi = \pi_1(\mathbb{C} \setminus \{u^1, u^2\})$ be the fundamental group of the pointed λ-plane with base point λ_0.

The flat pencil of metric of M induces on $\mathbb{C}^2$ a degenerate metric

$$G = -\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Let us denote by γ_i the reflection along e_i w.r.t. G, and define the group homomorphism $\pi \to GL(\mathbb{C}^2)$ by sending the loop γ_i to the reflection γ_i.

For each $\gamma \in \pi$ we have $\gamma^* I_a^{(l)} = I_{\gamma a}^{(l)}$.

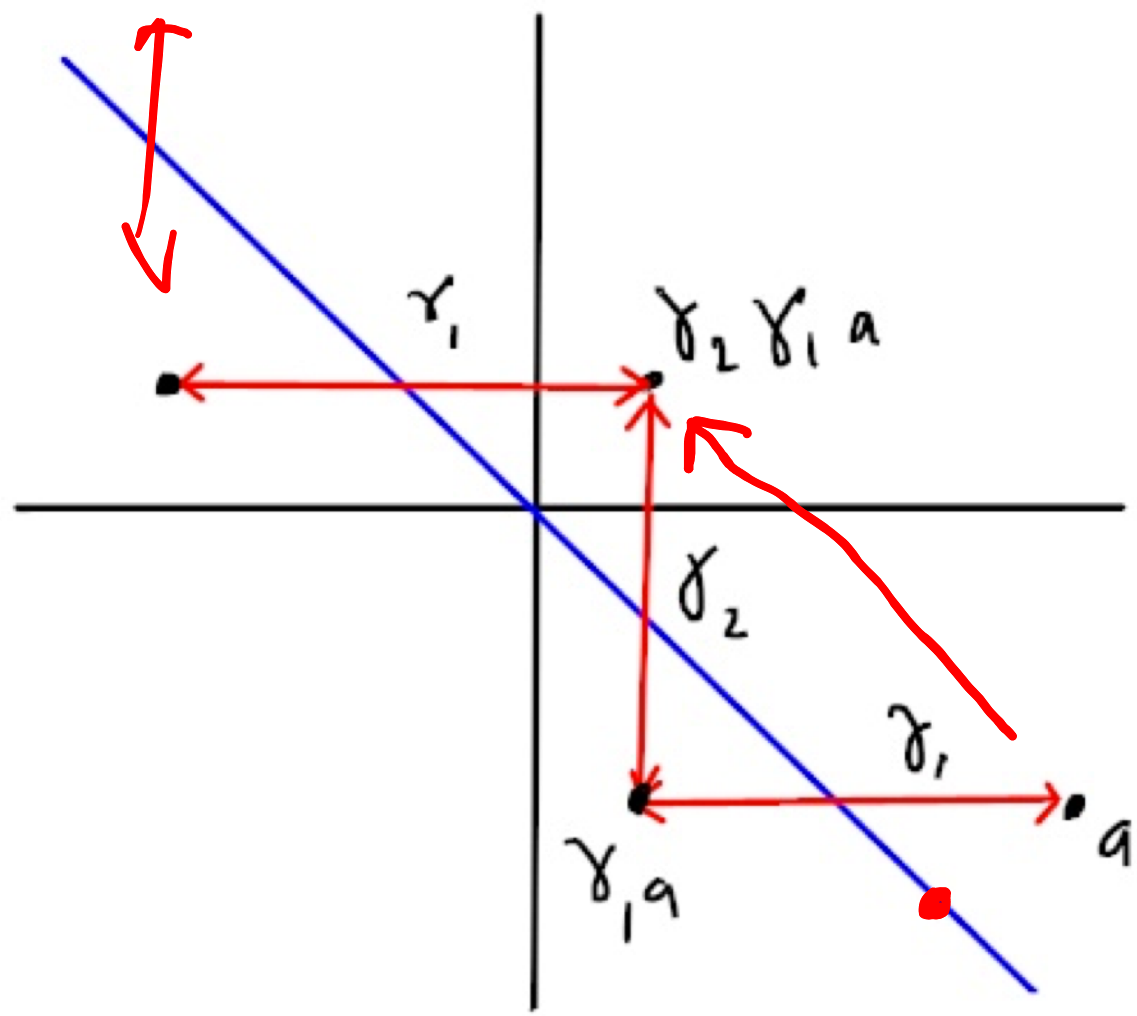
[Dubrovin]

analytic continuation

Monodromy of period vectors (2)

Explicitly the monodromy action is given by

$$\gamma_1 e_1 = -e_1, \qquad \gamma_1 e_2 = e_2 - 2e_1,$$
$$\gamma_2 e_1 = e_1 - 2e_2, \qquad \gamma_2 e_2 = -e_2.$$



Vertex operators

Given $a \in \mathbb{C}^2$ a vertex operator Γ^a is constructed by quantization of the linear Hamiltonian defined by the period vectors

$$\mathfrak{f}_a(t, \lambda, z) = \sum_{l \in \mathbb{Z}} I_a^{(l)}(t, \lambda)(-z)^l$$

where the quantization procedure gives

$$\Gamma^a = \widehat{e^{\mathfrak{f}_a}} = e^{\widehat{\mathfrak{f}}_{a-}} e^{\widehat{\mathfrak{f}}_{a+}}.$$

Hirota equations and monodromy

Before proceeding let's have a look at the expected form of the ancestor Hirota equations for example as they appear in the case of Gelfand-Dickey hierarchy, or A_n singularity. [Givental]

We have a multivalued one-form ω_{Hir} over the λ-plane given by

$$\left(\sum_{a\in O} c_a(t,\lambda)\Gamma^a \otimes \Gamma^{-a}\right) \mathcal{A} \otimes \mathcal{A}\, d\lambda.$$

The first requirement is that this one-form is single-valued.

Because the period vectors and consequently the vertex operators and the functions c_a are covariant under the action of π, this follows if O is an orbit of the reflection group over $\mathbb{C}^n$.

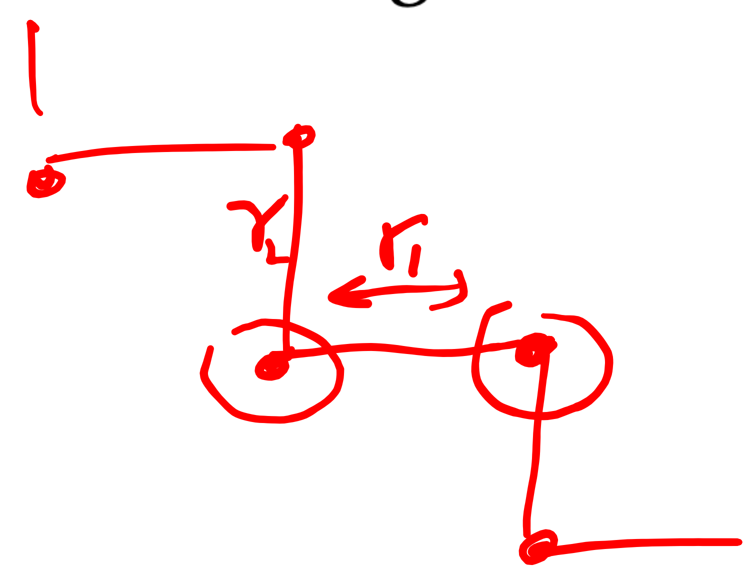
In such case we can say that ancestor Hirota equations are satisfied if ω_{Hir} is entire.

Infinite orbits...

In our case the non-trivial orbits are infinite, therefore it is not possible to average over an orbit to get a single-valued ω_{Hir}.

We can choose however a partial orbit $O' = \{a^+, a^-\}$, with $a^+ = a_1 e_1 + a_2 e_2$ and $\gamma_1 a^\pm = a^\mp$.

In such case the monodromy of ω'_{Hir} over a "large circle" γ_∞ is nontrivial.



...and Bakalov operator

However if we define the operator

$$\mathcal{N} = \exp\left(- \sum_{(j,\ell)\neq(2,0)} \frac{(I_o^{(-\ell-1)})_j q_\ell^j}{(I_o^{(-1)})_2} \frac{\partial}{\partial q_0^2} \right)$$

then we can prove that if $b^2 \in \mathbb{Z}$ and $\frac{b}{\epsilon}(q_0^2 - \bar{q}_0^2) \in \mathbb{Z}$ then

$$\mathcal{N} \otimes \mathcal{N} \omega'_{Hir}$$

is a single-valued function of λ.

More explicitly the expression above is

$$\mathcal{N} \otimes \mathcal{N} \left(c_{a^+} \Gamma^{a^+} \otimes \Gamma^{-a^+} + c_{a^-} \Gamma^{a^-} \otimes \Gamma^{-a^-} \right) (\mathcal{A} \otimes \mathcal{A})\, d\lambda.$$

Ancestor Hirota equation

Let $b^2 = 1$. Then the ancestor potential $\mathcal{A}$ satisfies the ancestor Hirota quadratic equation

$\mathcal{A} = \hat{\Psi}\hat{R}\,\tau_{KW}$

$$\mathcal{N} \otimes \mathcal{N} \left(\sum_{a \in O'} c_a \Gamma^a \otimes \Gamma^{-a} \right) \mathcal{A} \otimes \mathcal{A} \, d\lambda.$$

To prove this fact we need to show that this expression has no poles at $\lambda = u^i$, $i = 1, 2$. We can prove using BCH formula that

$$\hat{R}^{-1} e^{\widehat{\Psi \mathfrak{f}_{ce_i}}} \hat{R} = e^{\frac{c^2}{2}\int_{u^i}^{\lambda}\left(\mathcal{W}_{e_i,e_i} - \frac{1}{2}\frac{1}{\rho - u^i}\right)d\rho}\, e^{c\,\widehat{\mathfrak{f}_{KdV}(\lambda - u^i, z)}e_i} \qquad \lambda \sim u^i,$$

Γ_{KdV}

and this implies that the polar part of the ancestor Hirota equation at $\lambda \sim u^i$ is proportional to Hirota for KdV, which is satisfied by τ_{KW}.

Periods at $\lambda \sim \infty$

For any $a \in \mathbb{C}^2$, the asymptotic behaviour of $\mathfrak{f}_a$ for $|\lambda| \sim \infty$, $\arg \lambda \neq \pi/2$ is given by

$$\mathfrak{f}_a(t, \lambda, z) \sim S(t, z)\, \mathfrak{f}_{a,\infty}(\lambda, z)$$

where

$I_{a\infty}$

$$\mathfrak{f}_{e_1,\infty}(\lambda, z) := \sum_{l \in \mathbb{Z}} \partial_\lambda^l \begin{pmatrix} \partial_\lambda(\log \lambda + \frac{\psi}{2}) \\ \frac{1}{2} \end{pmatrix} (-z)^l,$$

$$\mathfrak{f}_{e_2,\infty}(\lambda, z) := \mathfrak{f}_{e_1,\infty}(\lambda, z) + \sum_{l \in \mathbb{Z}} \partial_\lambda^{l+1} \begin{pmatrix} \pi \mathrm{i} \\ 0 \end{pmatrix} (-z)^l,$$

and for $a = (a_1, a_2) \in \mathbb{C}^2$ let $\mathfrak{f}_{a,\infty} = a_1 \mathfrak{f}_{e_1,\infty} + a_2 \mathfrak{f}_{e_2,\infty}$.

The equivalent $I_{e_i}^{(l)} \sim \sum_{k=0}^{\infty} (-1)^k S_k I_{e_i,\infty}^{(l+k)}$ is actually convergent and can be used to give an alternative definition of periods.

[Milanov]

Descendent Hirota equation (1)

The descendent Hirota equation, which should be understood as a formal series near $\lambda \sim \infty$, is obtained by substituting the period vectors $I_a^{(\ell)}$ with $I_{a,\infty}^{(\ell)}$ defined by the expansion of $\mathfrak{f}_{a,\infty}$

$$\mathcal{N}_\infty \otimes \mathcal{N}_\infty \left(\sum_{a \in O'} c_a^\infty \Gamma_\infty^a \otimes \Gamma_\infty^{-a} \right) \mathcal{D} \otimes \mathcal{D} \, d\lambda.$$

In this case only the single-valuedness near ∞, namely the covariance under the action of a big circle, is necessary.

The descendent Hirota equation is verified if it is regular at $\lambda \sim \infty$, namely if it is polynomial in λ.

Descendent Hirota equation (2)

To prove that the total descendent potential $\mathcal{D} = C\widehat{S^{-1}}\mathcal{A}$ satisfies the descendent Hirota equation we show that the $\hat{S}$ conjugation of the descendent Hirota equation is proportional to the $\lambda \sim \infty$ expansion of the ancestor Hirota equation which vanishes as proved before.

This essentially follows from the following result that we can prove using BCH.

For $a \in \mathbb{C}^2$ we have

$$\hat{S}\Gamma^a_\infty \hat{S}^{-1} = e^{\frac{c(t)}{2} + \frac{1}{2}\int_\lambda^\infty (\mathcal{W}_{a,a} - \mathcal{W}^\infty_{a,a})d\rho}\Gamma^a,$$

with $c(t) = \frac{(a_1+a_2)^2}{4}(\log t^2 + \psi)$.

Descendent Hirota equation (3)

For any value of the calibration parameter ψ the total descendent potential $\mathcal{D}$ satisfies the following equations:

$$
\begin{aligned}
0 = \operatorname{Res}_{\lambda=\infty} \lambda^{n-1} d\lambda \Big[\\
\lambda^k e^{\frac{k\psi}{2}} \exp\Big(\frac{1}{\epsilon}\sum_{\ell\geqslant 0} \frac{1}{2}\frac{\lambda^{\ell+1}}{(\ell+1)!}(q^1_\ell - \bar{q}^1_\ell) - \frac{1}{\epsilon}\sum_{\ell\geqslant 1}\frac{\lambda^\ell}{\ell!}\mathfrak{h}(\ell)(q^2_\ell - \bar{q}^2_\ell)\Big)\times \\
\mathcal{D}\Big(\{q^1_\ell - \epsilon\tfrac{\ell!}{\lambda^{\ell+1}}\}_{\ell\geqslant 0},\ \underline{q^2_0 - \tfrac{\epsilon}{2} - \textstyle\sum_{\ell\geqslant 1}\tfrac{\lambda^\ell}{\ell!}q^2_\ell},\ \{q^2_\ell\}_{\ell\geqslant 1}\Big)\times \\
\mathcal{D}\Big(\{\bar{q}^1_\ell + \epsilon\tfrac{\ell!}{\lambda^{\ell+1}}\}_{\ell\geqslant 0},\ \bar{q}^2_0 + \tfrac{\epsilon}{2} - \textstyle\sum_{\ell\geqslant 1}\tfrac{\lambda^\ell}{\ell!}\bar{q}^2_\ell,\ \{\bar{q}^2_\ell\}_{\ell\geqslant 1}\Big) \\
-\lambda^{-k} e^{-\frac{k\psi}{2}} \exp\Big(\tfrac{-1}{\epsilon}\textstyle\sum_{\ell\geqslant 0}\tfrac{1}{2}\tfrac{\lambda^{\ell+1}}{(\ell+1)!}(q^1_\ell - \bar{q}^1_\ell) + \tfrac{1}{\epsilon}\sum_{\ell\geqslant 1}\tfrac{\lambda^\ell}{\ell!}\mathfrak{h}(\ell)(q^2_\ell - \bar{q}^2_\ell)\Big)\times \\
\mathcal{D}\Big(\{q^1_\ell + \epsilon\tfrac{\ell!}{\lambda^{\ell+1}}\}_{\ell\geqslant 0},\ q^2_0 + \tfrac{\epsilon}{2} - \textstyle\sum_{\ell\geqslant 1}\tfrac{\lambda^\ell}{\ell!}q^2_\ell,\ \{q^2_\ell\}_{\ell\geqslant 1}\Big)\times \\
\mathcal{D}\Big(\{\bar{q}^1_\ell - \epsilon\tfrac{\ell!}{\lambda^{\ell+1}}\}_{\ell\geqslant 0},\ \bar{q}^2_0 - \tfrac{\epsilon}{2} - \textstyle\sum_{\ell\geqslant 1}\tfrac{\lambda^\ell}{\ell!}\bar{q}^2_\ell,\ \{\bar{q}^2_\ell\}_{\ell\geqslant 1}\Big)\Big]\Big|_{q^2_0 - \bar{q}^2_0 = k\epsilon}
\end{aligned}
$$

for any $k \in \mathbb{Z}$ and for any $n \geqslant 0$.

Lax, again (1)

From the explicit form of the Hirota equation proceeding as for a rational reduction of KP is quite straightforward as long as we do not consider the ”logarithmic” times q_k^2.

As Hirota equations look like extended Toda Hirota can proceed as in this case by exchanging q_0^1 and q_0^2 and deriving a Lax formulation with space variable $x = q_0^2$.

We can the apply the procedure in our original work with Dubrovin and Zhang to obtain a Lax formulation in the natural space variable $X = q_0^1$.

Lax, again (2)

We rewrite the descendent Hirota equation in terms of pseudodifferential operator form (using the so-called fundamental lemma):

$$\Big[P(q,\epsilon\partial_X)\exp\Big(\frac{1}{\epsilon}\sum_{\ell\geqslant 0}\frac{(\epsilon\partial_X)^{\ell+1}}{(\ell+1)!}(q^1_\ell-\bar q^1_\ell)-\frac{2}{\epsilon}\sum_{\ell\geqslant 1}\frac{(\epsilon\partial_X)^{\ell}}{\ell!}\mathfrak{h}(\ell)(q^2_\ell-\bar q^2_\ell)\Big)\times$$

$$\times\exp\Big(\sum_{\ell\geqslant 1}\frac{(\epsilon\partial_X)^{\ell}}{\ell!}(q^2_\ell-\bar q^2_\ell)\partial_x\Big)e^{-(k-1)\epsilon\partial_x}(\epsilon\partial_X)^{n+k-1}\tilde P(\bar q,-\epsilon\partial_X)^*\Big]_- =$$

$$= e^{-k\psi}\,\mathrm{Res}_\lambda\Big[Q(q)e^{\epsilon\partial_x}\tilde P(q,\lambda)\exp\Big(\sum_{\ell\geqslant 1}\frac{\lambda^\ell}{\ell!}(q^2_\ell-\bar q^2_\ell)\partial_x\Big)e^{-(k+1)\epsilon\partial_x}\lambda^{n-k-1}\times$$

$$\times(\epsilon\partial_X)^{-1}P(\bar q,\lambda)Q^{-1}(\bar q)e^{\epsilon\partial_x}\Big]d\lambda$$

for $n\geqslant 0$ and $k\in\mathbb{Z}$.

We then proceed to obtain Sato and Lax for extended NLS.

Notice that here we don't have two different dressing operators like in ETH to define the logarithm.

...and where is Catalan?

The generalized Catalan number[1] $C_{g;k_1,\dots,k_n}$ enumerates genus g graphs with $n \geqslant 1$ ordered vertices of indices $k_1, \dots, k_n$ connected by edges, with a fixed cyclic order of half-edges attached to each vertex, and with one distinguished half-edge at each vertex.

Assume $\psi = 0$ and fix the point of expansion for $\mathcal{D}$ to be $(t^1, t^2) = (0, 1)$. Using CEO topological recursion we prove that

$$\log \mathcal{D}\Big|_{\substack{t_0^2=1 \\ t_a^2=0, a\geqslant 1}} = \sum_{g=0}^{\infty} \sum_{n=1}^{\infty} \frac{\epsilon^{2g-2}}{n!} \sum_{k_1,\dots,k_n \geqslant 0} C_{g,k_1+1,\dots,k_n+1} \prod_{i=1}^{n} \frac{t^1_{k_i}}{(k_i+1)!}.$$

See also [Dunin-Barkowski-et-al, Andersen-et-al]

[1] also studied under the names of strictly monotone Hurwitz numbers, enumerations of ribbon graphs, (rooted) maps on surfaces, Grothendieck's dessins d'enfants for strict Belyi functions, lattice points in the moduli spaces of curves, etc...