A threefold way to integrable probabilistic models

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Integrable systems around the world
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In collaboration with

Figure 1: Jinho Baik - UMichigan.

and partially based on arXiv:1808.02419 as well as arXiv:2008.01694.

This talk's theme

We will discuss certain integrable probabilistic systems:

1. Finding concise and exact formulæ for the systems' interesting observables
2. Analyzing the systems' scaling limits
3. Identifying universality classes

We will throughout emphasize the above threefold way, in particular when discussing our recent work with J. Baik in the second part of this talk.

Gaussian universality in a nutshell

Flip a fair coin n times and consider

$$X_n = \#\{\text{heads in } n \text{ fair coin tosses}\}$$

Counting leads to the exact formula

$$\mathrm{Prob}(X_n = k) = \binom{n}{k} 2^{-n}, \quad k = 0, 1, 2, \ldots, n,$$

and we naturally expect that as n grows,

$$X_n \approx \frac{n}{2}.$$

This is an instance of a limit law, found by Jakob Bernoulli (1713).

Let's look at a histogram

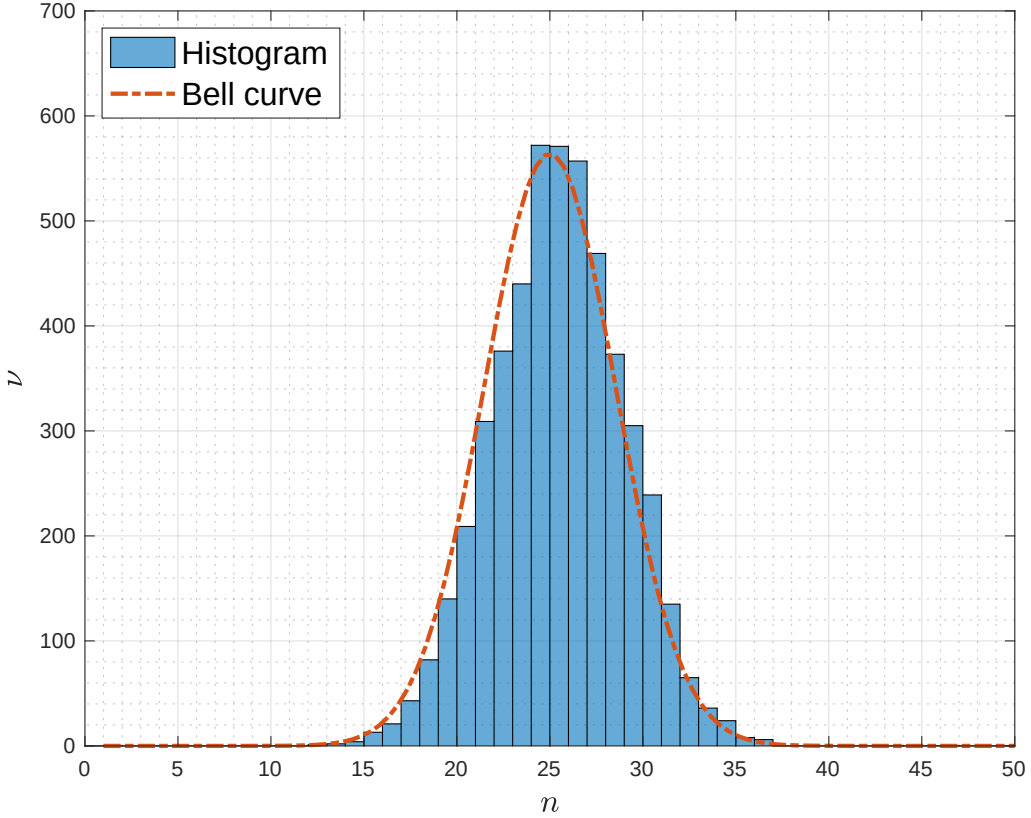


Figure 2: 50 fair coin tosses repeated 5000 times.

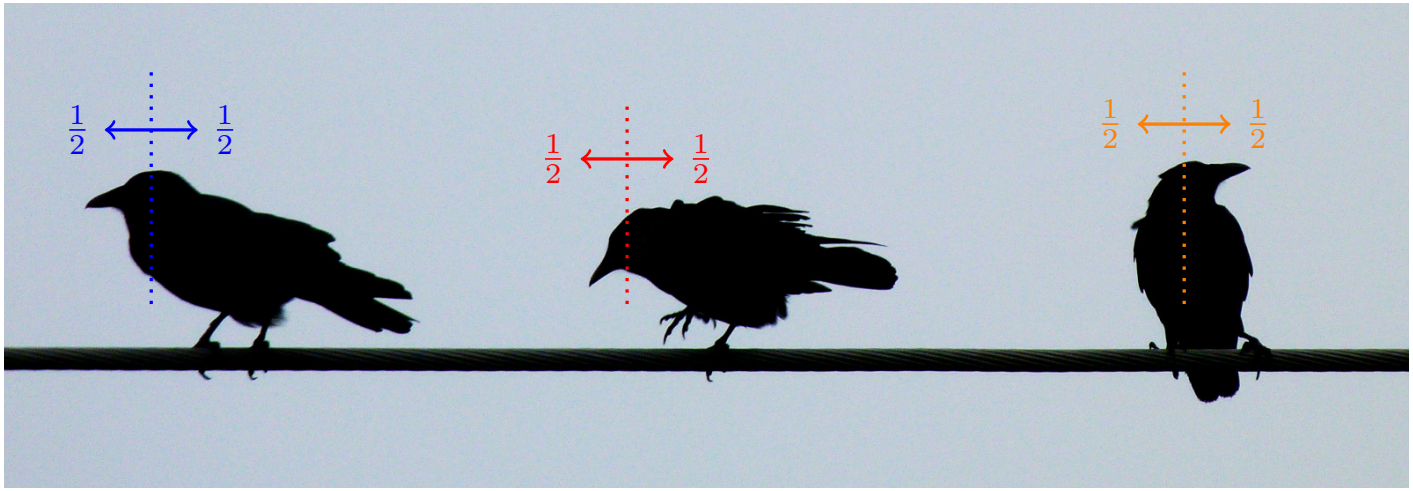
The larger n, the better the hump aligns with the orange curve.

This observation goes back to Abraham de Moivre (1738), Carl Gauss (1809), Pierre-Simon Laplace (1812) and is quantified as

$$X_n \approx \frac{n}{2} + \frac{1}{2}\sqrt{n}\,Z, \qquad Z \sim N(0,1); \tag{1}$$

i.e. the number of heads varies randomly (with standard normal distribution) around the value $\frac{n}{2}$ in the scale $\sqrt{n}$.

The de Moivre-Laplace limit theorem (1) has a *weak* universal flavor since many problems can be mapped to coin flips:



If $B_n = \#\{\text{of birds (out of } n\text{) not seen by any direct neighboring bird}\}$, then

$$B_n \approx \frac{n}{4} + \frac{1}{4}\sqrt{3n}\,Z, \qquad Z \sim N(0,1).$$

The *strong* universality of the Gaussian distribution was only realized around 1900 by Pafnuty Chebyshev, Andrey Markov and Aleksandr Lyapunov: the exact nature of coin flipping is immaterial

The central (after Pólya) limit theorem

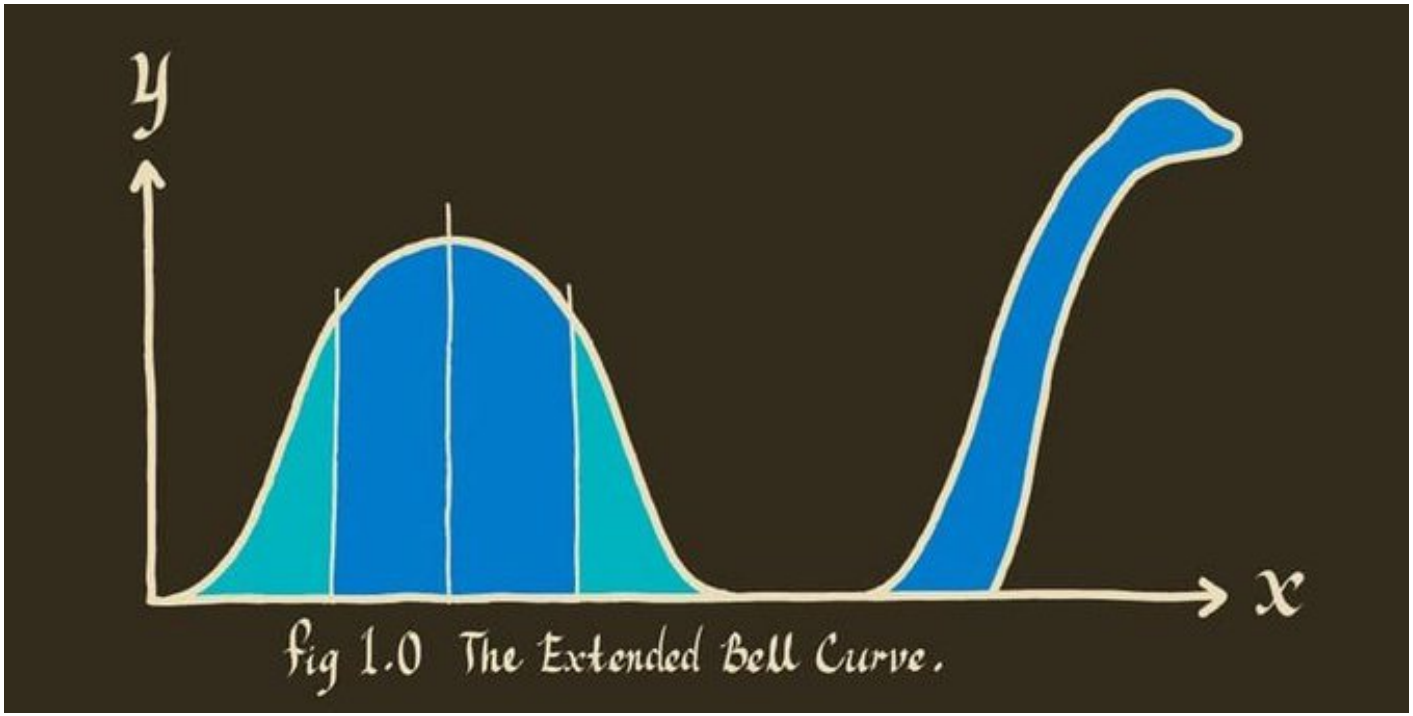
If $X_1, X_2, \ldots$ are iid random variables of mean μ, variance σ^2. Then

$$\text{Prob}\left(X_1 + \ldots + X_n \leq \mu n + \sigma\sqrt{n}\,x\right) \approx \int_{-\infty}^{x} \frac{\mathrm{e}^{-y^2/2}}{\sqrt{2\pi}}\,\mathrm{d}y,$$

for large n and $x \in \mathbb{R}$.

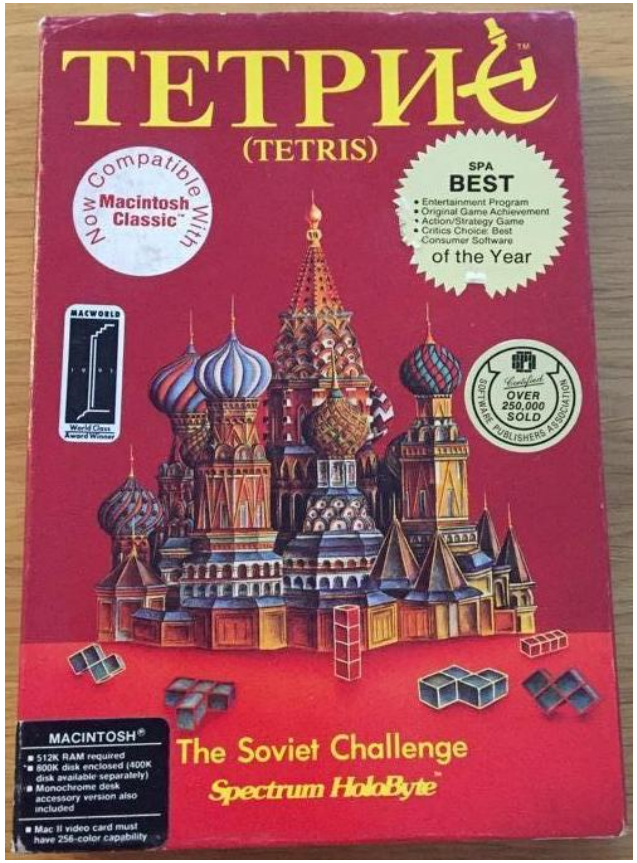
Extensions exist for the central limit theorem, though the bell curve is ubiquitous and forms the basis for much of classical statistics:

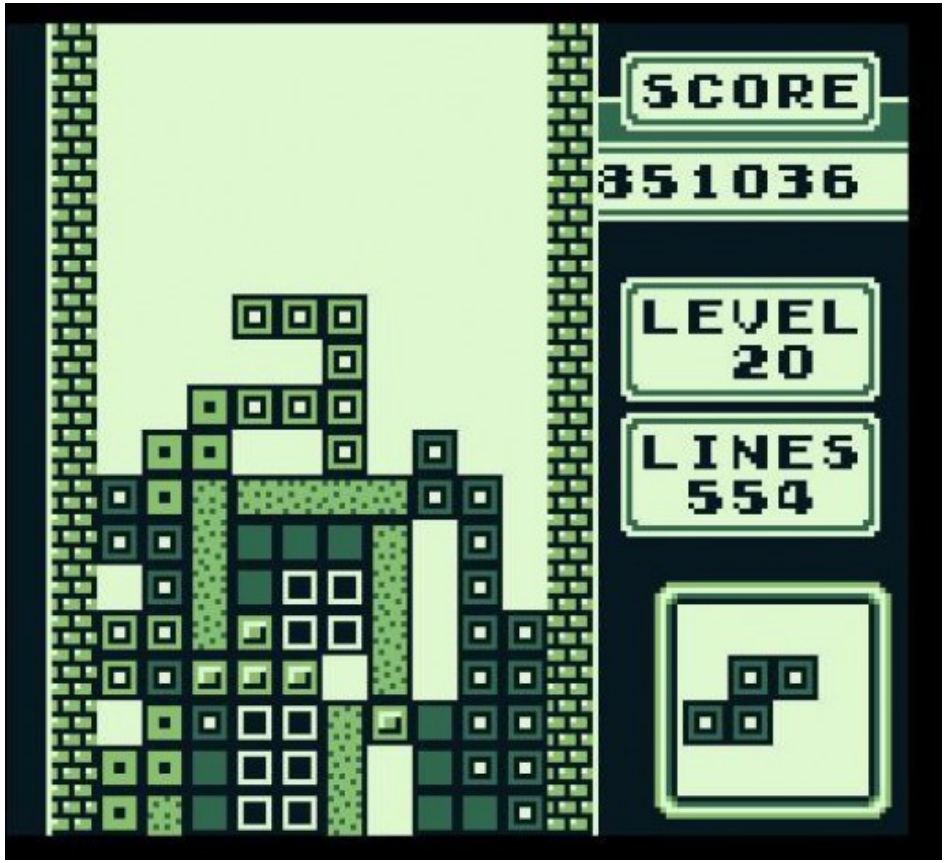
Gaussian universality class



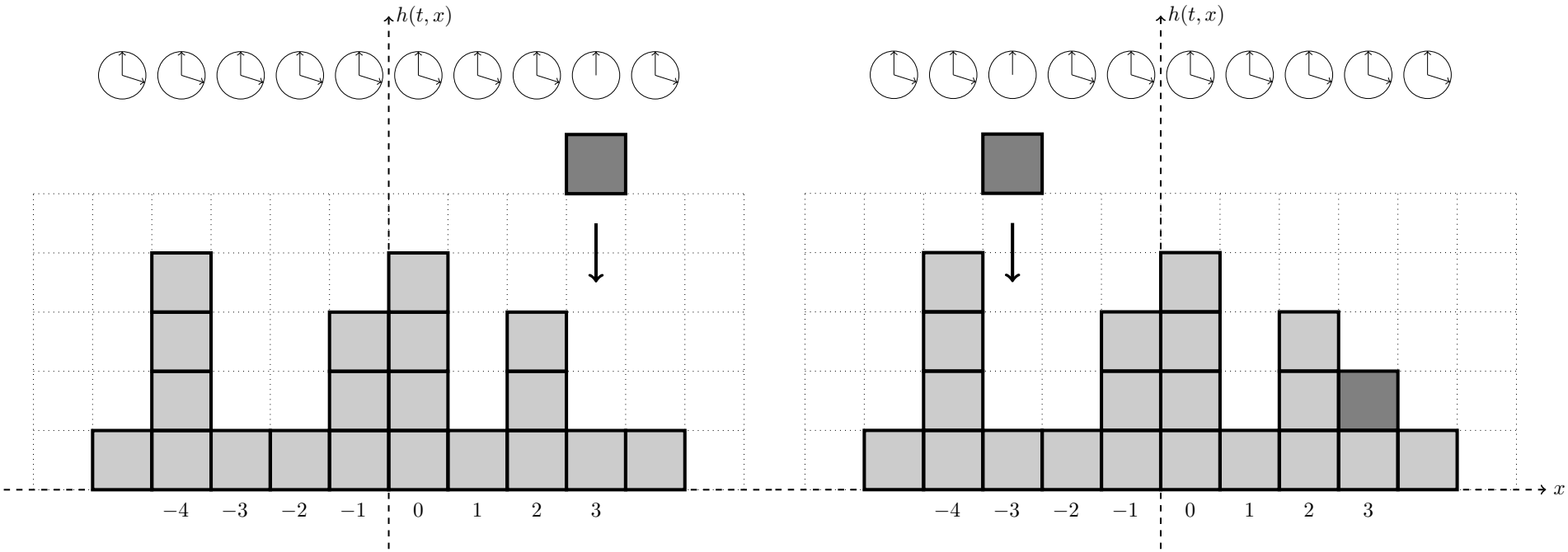
Towards random matrices via Tetris

Remember Tetris (designed by Alexey Pajitnov (1984))?





A simple (though not very realistic) Tetris model is the random deposition model: unit blocks fall independently and in parallel from the sky above each site of $\mathbb{Z}$, after some waiting time.



Interesting observable (with $h(0,x) := 0, x \in \mathbb{Z}$)

$$h(t,x) = \text{height above site } x \in \mathbb{Z} \text{ at time } t \geq 0$$

By the central limit theorem,

$$h(t,x) \approx \frac{t}{\lambda} + \frac{\sqrt{t}}{\lambda} Z, \qquad Z \sim N(0,1)$$

Linear growth with known speed, Gaussian fluctuations grow like $t^{\frac{1}{2}}$, no spatial correlations

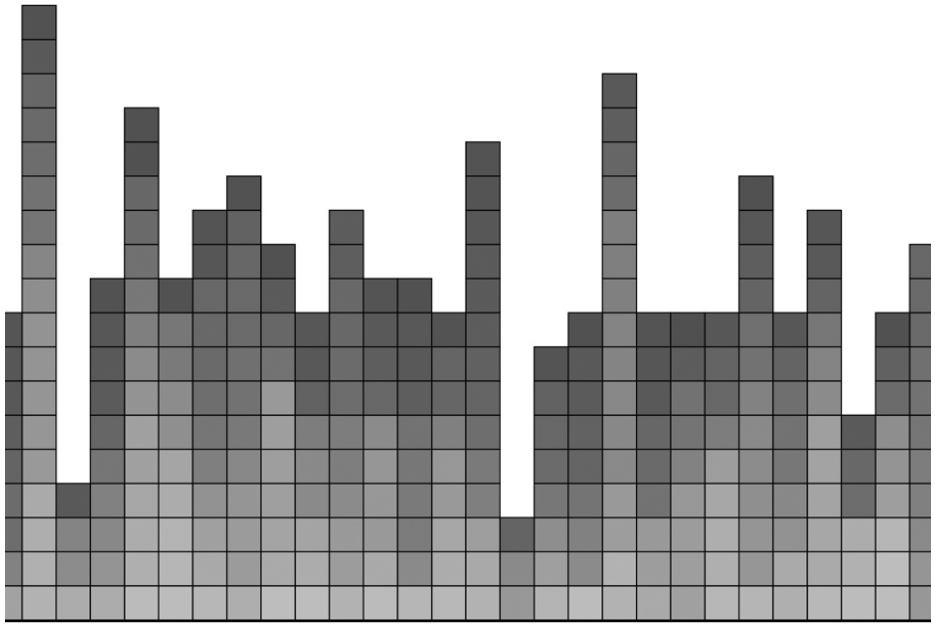
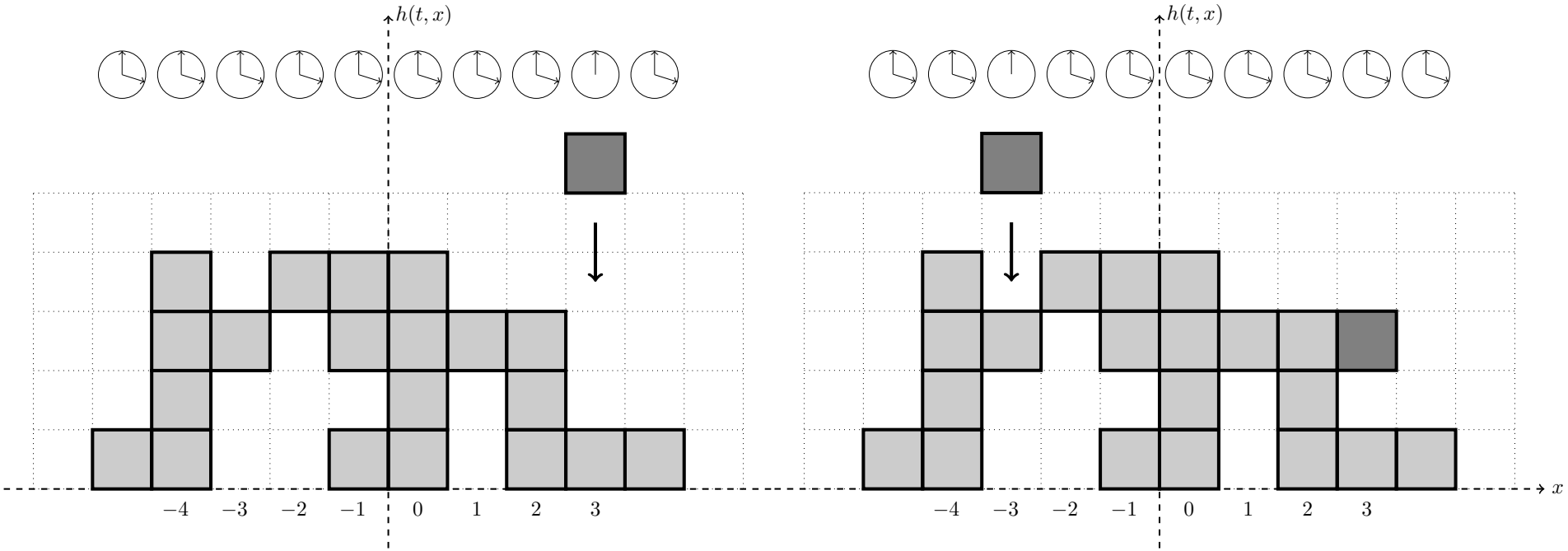


Figure 3: Random deposition. The model has a very rough top interface.

Consider the ballistic deposition model: unit blocks fall independently and in parallel from the sky as before. However, a block will now stick to the first edge against it becomes incident.



This time, overhangs are created and we are interested in

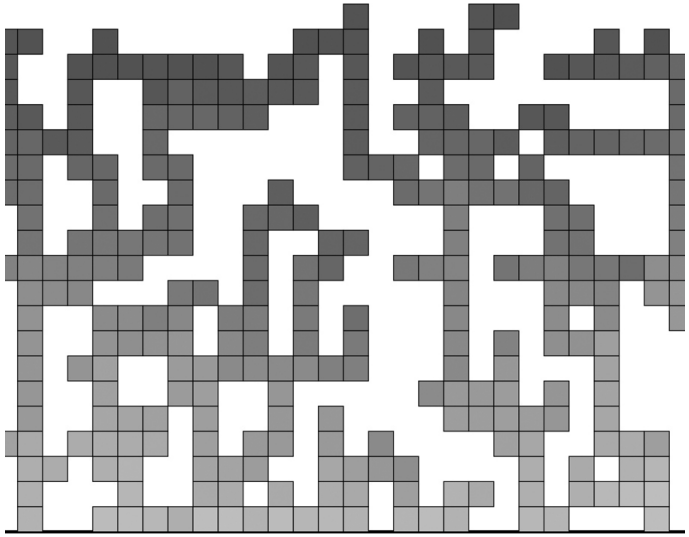
$$h(t,x) = \text{maximal height above site } x \in \mathbb{Z} \text{ at time } t \in \mathbb{R}_{\geq 0} \text{ which is occupied by a box}$$

It is believed that for some constants $c, d > 0$,

$$h(t,0) \approx ct + dt^{\frac{1}{3}} F_1. \tag{2}$$

Linear growth with unknown speed, Tracy-Widom fluctuations grow like $t^{\frac{1}{3}}$, spatial correlations

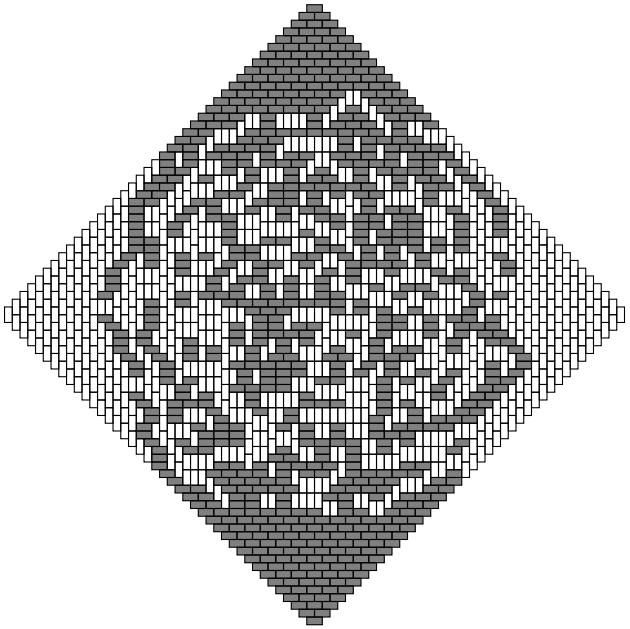
Ballistic deposition in "nature"

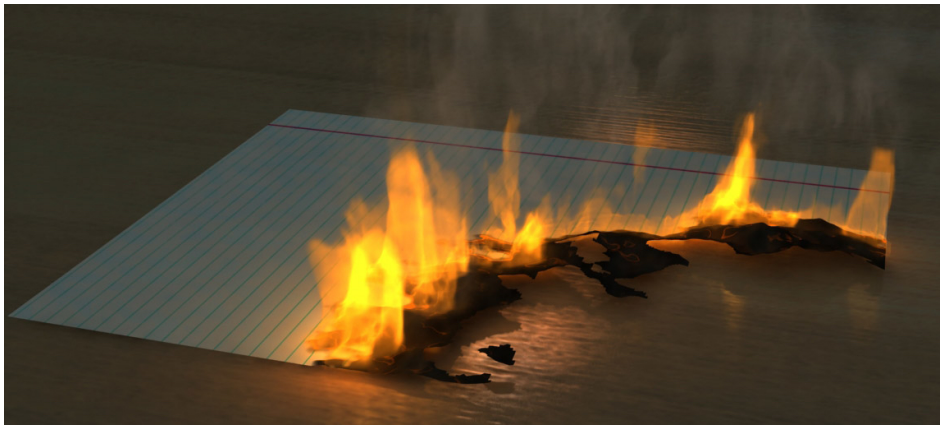


The behavior (2) is underlying a larger class of probabilistic models

Kardar-Parisi-Zhang universality class

KPZ class models





The origin of the Tracy-Widom distribution

The Gaussian Orthogonal Ensemble on $n \times n$ real matrices

$$\mathbf{X} = \frac{1}{2}(\mathbf{Y} + \mathbf{Y}^\top) : \quad Y_{ij} \overset{\text{iid}}{\sim} N(0,1).$$

Introduced by Wigner (1955) to model energy levels of atoms too complicated to solve analytically. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ denote the random real eigenvalues of $\mathbf{X}$, then

$$\lambda_1 \approx \sqrt{2n} + \frac{1}{\sqrt{2}n^{\frac{1}{6}}} F_1.$$

This scaling behavior is universal within the class of real Wigner matrices as shown by Soshnikov (1999).

The Tracy-Widom distribution is the modern-day bell curve. So what about exact formulæ? Instead of

$$\mathrm{Prob}(Z \leq x) = \int_{-\infty}^{x} \frac{\mathrm{e}^{-y^2/2}}{\sqrt{2\pi}} \, \mathrm{d}y$$

we have thanks to Tracy and Widom (1996)

$$\mathrm{Prob}(F_1 \leq s) = \exp\left[-\frac{1}{2}\int_s^{\infty}(x-s)q^2(x)\,\mathrm{d}x - \frac{1}{2}\int_s^{\infty} q(x)\,\mathrm{d}x\right],$$

where $q = q(x)$ solves a certain second order nonlinear ODE.

Shatter the symmetry

The Real Ginibre ensemble on $n \times n$ real matrices

$$\mathbf{X} = \mathbf{Y} : \quad Y_{ij} \overset{\text{iid}}{\sim} N(0,1).$$

Introduced by Ginibre (1965) and popularized by May (1972) in the stability analysis of ecosystems

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = \mathbf{F}(\mathbf{u}) \quad \rightarrow \quad \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = (-\mathbb{I}\mu + \mathbf{X})\mathbf{u}$$

In turn the spectral radius of $\mathbf{X}$ plays a crucial role ("stability versus diversity").

Some facts

The eigenvalues $\lambda_j = x_j + \mathrm{i}y_j \in \mathbb{C}$ of $\mathbf{X}/\sqrt{n}$ congregate for large n uniformly on the unit disk, a result known as the circular law

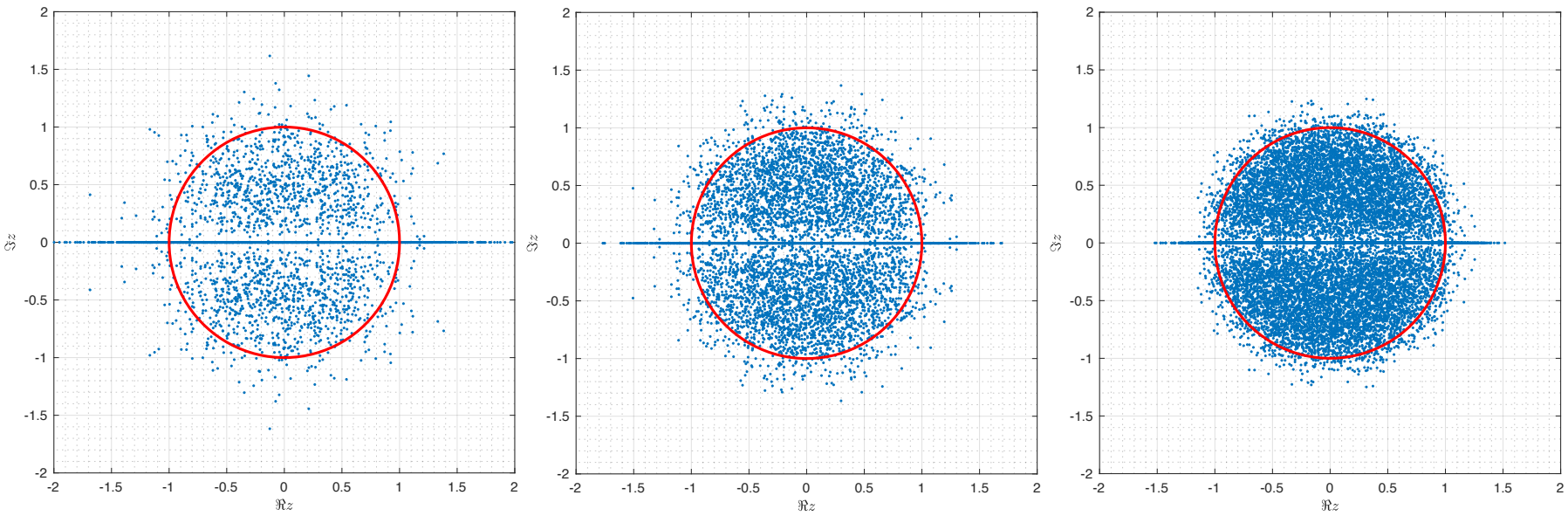


Figure 4: The circular law for 1000 real (rescaled) Ginibre matrices of varying dimensions $n \times n$ in comparison with the unit circle boundary. We plot $n = 4, 8, 16$ from left to right. A saturn effect is clearly visible on the real line.

The real Ginibre ensemble is more complicated than most other explicitly solvable matrix models, especially real eigenvalues in the Real Ginibre ensemble behave strangely

The saturn effect

$$\mathrm{Prob}(\lambda_1, \ldots, \lambda_n \in \mathbb{R}) = 2^{-\frac{n}{4}(n-1)} \qquad \text{Edelman (1997)}$$

So, what about the fluctuations of the spectral radius around $\sqrt{n}$?

$$\max_{j:\lambda_j \in \mathbb{C}\setminus\mathbb{R}} |\lambda_j| \Rightarrow \sqrt{n} + c_n + \frac{G}{d_n}, \qquad \text{Rider, Sinclair (2014)}$$

where G is Gumbel distributed and $c_n, d_n \approx \ln(n/(\ln n)^2)$

First result with Jinho

The largest real eigenvalue behaves differently,

$$\max_{j:\lambda_j \in \mathbb{R}} \lambda_j \Rightarrow \sqrt{n} + \chi, \qquad \text{Baik, Bothner (2018)}$$

where

$$\mathrm{Prob}(\chi \leq s) = \exp\left[-\frac{1}{2}\int_s^\infty (x-s)p^2(x)\,\mathrm{d}x - \frac{1}{2}\int_s^\infty p(x)\,\mathrm{d}x\right]$$

and $p = p(x)$ solves a certain inverse scattering problem for the Zakharov-Shabat system. This system is directly related to several of the most interesting nonlinear evolution PDEs in $1+1$ dimensions.

Proof ingredients - thinned real Ginibre ensemble

We will now discuss the threefold way in detail, in fact for the following more general model:

Consider the Pfaffian point process formed by the real eigenvalues $\lambda_j(\mathbf{X}) \in \mathbb{R}$ with $j = 1, \ldots, m_n \leq n$ and $\mathbf{X} \in \text{GinOE}$. Fix $\gamma \in [0, 1]$ and discard each $\lambda_j(\mathbf{X})$ independently with likelihood $1 - \gamma$.

What are the important statistical properties of the resulting particle system $\{\lambda_j^{\gamma}(\mathbf{X})\}_{j=1}^{m_{\gamma,n}}$, $m_{\gamma,n} \leq m_n \leq n$*?*

Exact formulæ (1) - Baik, Bothner 2020

We have for any $s \in \mathbb{R}$ and $\gamma \in [0,1]$,

$$\mathbb{P}\left(\max_{j=1,\dots,m_{\gamma,n}} \lambda_j^{\gamma}(\mathbf{X}) \leq s\right) = \sqrt{\det_2(1 - \gamma\chi_s \mathbf{K}_n \chi_s \restriction_{L^2(\mathbb{R})\oplus L^2(\mathbb{R})})}$$

where $\mathbf{K}_n$ is a certain Hilbert-Schmidt integral operator (parity dependent!). In order to prove the above, one makes crucial use of the observation that a thinned Pfaffian point process is again Pfaffian with an appropriately γ-modified kernel.

Exact formulæ (2) - Baik, Bothner 2020

Carefully manipulating the regularized operator determinant we then derive the simpler expression

$$F_{2n}(s,\gamma) = \sqrt{\det(1-\bar{\gamma}\chi_s T_{2n}\chi_s \restriction_{L^2(\mathbb{R})})\det\left(1-\sum_{k=1}^{3}\alpha_k\otimes\beta_k \restriction_{L^2(\mathbb{R})}\right)},$$

where $\bar{\gamma} := \gamma(2-\gamma)$ and T_n has kernel

$$T_n(x,y) = \frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}(x^2+y^2)}\mathfrak{e}_{n-2}(xy), \quad \mathfrak{e}_n(z) := \sum_{K=0}^{n}\frac{z^k}{k!}.$$

Scaling limits - Baik, Bothner 2020

At this point we rescale $s \mapsto s + \sqrt{n}$ and send $n \to \infty$,

$$\lim_{n\to\infty} \left(F_{2n}(s+\sqrt{2n},\gamma)\right)^2 = \det(1 - \bar{\gamma}\chi_s T \chi_s \restriction_{L^2(\mathbb{R})}) \times$$
$$\times \det \begin{bmatrix} 1-u & -v + \frac{\bar{\gamma}}{\sqrt{2}} & -v \\ -\frac{\gamma}{2\bar{\gamma}}(p+q-\frac{1}{\sqrt{2}}) & 1 - \frac{\gamma}{2\bar{\gamma}}(r+w) & -\frac{\gamma}{2\bar{\gamma}}(r+w) \\ \frac{\gamma}{\bar{\gamma}}p & \frac{\gamma}{\bar{\gamma}}r & 1+\frac{\gamma}{\bar{\gamma}}r \end{bmatrix}.$$

Here, $T : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ has kernel

$$T(x,y) := \frac{1}{\pi}\int_0^\infty \mathrm{e}^{-(x+u)^2}\mathrm{e}^{-(y+u)^2}\,\mathrm{d}u,$$

and the functions u, v, p, q, r, w involve the resolvent of T, for instance

$$u = \frac{\bar{\gamma}}{2} \int_s^\infty G(x)\big((1 - \bar{\gamma} T \chi_s \restriction_{L^2(\mathbb{R})})^{-1} g\big)(x)\,\mathrm{d}x,$$

where

$$g(x) := \frac{1}{\sqrt{\pi}}\,\mathrm{e}^{-x^2}, \qquad G(x) := \int_{-\infty}^x g(y)\,\mathrm{d}y.$$

In order to proceed, one first observes the following roadblock:

$$T(x,y) \neq \frac{\phi(x)\psi(y) - \psi(x)\phi(y)}{x-y},$$

thus the Tracy, Widom (1994) or Borodin, Deift (2002) methods are not directly applicable to our determinants.

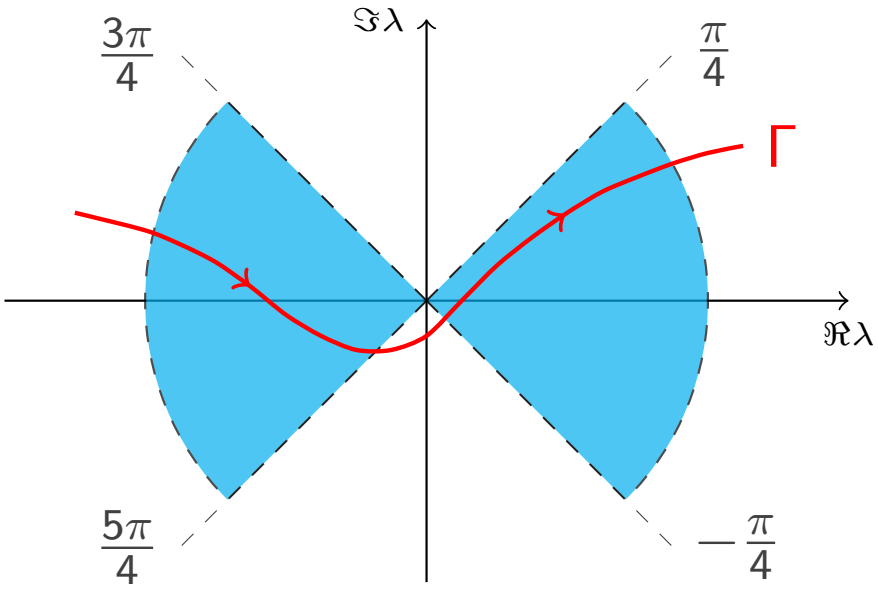
Fourier to the rescue

Recall that

$$T(x,y) = \int_0^\infty g(x+s)g(y+s)\,\mathrm{d}s; \qquad g(x) = \frac{1}{\sqrt{\pi}}\mathrm{e}^{-x^2} \tag{3}$$

and use

$$\mathrm{e}^{-x^2} = \frac{1}{2\sqrt{\pi}}\int_\Gamma \mathrm{e}^{-\frac{1}{4}\lambda^2 \pm \mathrm{i}x\lambda}\,\mathrm{d}\lambda, \quad x \in \mathbb{R}.$$



After substitution into (3),

$$T(x,y) = \frac{1}{(2\pi)^2} \int_{\Gamma_\lambda} \int_{\Gamma_w} \mathrm{e}^{-\frac{1}{4}(\lambda^2+w^2)} \mathrm{e}^{-\mathrm{i}(x\lambda - yw)} \left[\int_0^\infty \mathrm{e}^{-\mathrm{i}u(\lambda - w)} \, \mathrm{d}u \right] \mathrm{d}w \, \mathrm{d}\lambda$$

we choose $(\lambda, w) \in \Gamma_\lambda \times \Gamma_w$ such that $\Im w > \Im \lambda$ and obtain

$$T(x,y) = \frac{1}{(2\pi)^2} \int_{\Gamma_\lambda} \int_{\Gamma_w} \frac{\mathrm{e}^{-\frac{1}{4}(\lambda^2+w^2) - \mathrm{i}(x\lambda - yw)}}{\mathrm{i}(\lambda - w)} \, \mathrm{d}w \, \mathrm{d}\lambda.$$

Use residue theorem

Suppose $w \in \Gamma_w$ satisfies $\Im w > 0$. Then for any $y, s \in \mathbb{R} : y \neq s$,

$$\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\mu(y-s)} \frac{\mathrm{d}\mu}{\mu - w} = \mathrm{e}^{\mathrm{i}w(y-s)} \chi_{(s,\infty)}(y).$$

Now combine ($\Gamma_\lambda = \mathbb{R}, \Gamma_w \equiv \Gamma$),

$$T(x,y)\chi_{(s,\infty)}(y) = \iint_{\mathbb{R}^2} \frac{\mathrm{e}^{-\mathrm{i}x\lambda}}{\sqrt{2\pi}} \underbrace{\left[\frac{1}{(2\pi)^2}\int_\Gamma \frac{\mathrm{e}^{-\frac{1}{4}(\lambda^2+w^2)-\mathrm{i}s(\mu-w)}}{(\lambda-w)(w-\mu)}\,\mathrm{d}w\right]}_{=:E(\lambda,\mu)} \frac{\mathrm{e}^{\mathrm{i}y\mu}}{\sqrt{2\pi}}\,\mathrm{d}\mu\,\mathrm{d}\lambda$$

Thus, $T\chi_s$ on $L^2(\mathbb{R})$ is simply the operator composition $\mathcal{F}E\mathcal{F}^{-1}$. After properly modifying the function spaces, E is trace-class and can be massaged into the aforementioned Christoffel-Darboux structure.

Once the Christoffel-Darboux structure has been detected, the connection to the aforementioned Zakharov-Shabat system is rather straightforward. Some other tricks are needed for the remaining 3×3 determinant (curious integral identities).

Zakharov-Shabat formula

Baik, Bothner 2020

We have for any $(s, \gamma) \in \mathbb{R} \times [0, 1]$

$$\lim_{n\to\infty} \mathbb{P}\left(\max_{j=1,\dots,m_{\gamma,n}} \lambda_j^{\gamma}(\mathbf{X}) \leq \sqrt{n} + s \right) = \exp\left[-\frac{1}{2} \int_s^{\infty} (x-s) p^2(x;\bar{\gamma})\,\mathrm{d}x \right]$$
$$\times \sqrt{\frac{\gamma - 1 - \cosh\mu(s;\bar{\gamma}) + \sqrt{\bar{\gamma}}\sinh\mu(s;\bar{\gamma})}{\gamma - 2}},$$

using the abbreviations

$$\mu(s;\gamma) := \int_s^{\infty} p(x;\gamma)\,\mathrm{d}x,$$

and $p = p(x;\gamma)$ solves a γ-modified inverse scattering problem for the Zakharov-Shabat system.

Intermediate summary

1. Derive an exact formula for $\max_{j:\lambda_j\in\mathbb{R}} \lambda_j^\gamma(\mathbf{X})$ for finite n by exploiting the model's Pfaffian integrability.
2. Subject the formula to the scaling limit $n \to \infty$ and obtain limiting operator determinants for the distribution function. Then manipulate the operator determinants

3. Universality

Most importantly, Erdös et. al (2019) recently proved the universality of our scaling behavior within the class of real non-Hermitian random matrices.

$$\max_{j:\lambda_j\in\mathbb{R}} \lambda_j^\gamma \Rightarrow \sqrt{n} + \chi^\gamma, \qquad \text{Baik, Bothner (2020).}$$

Some corollaries - tail estimates for χ^γ

Baik, Bothner 2020

Let $\gamma \in [0,1]$, then as $s \to +\infty$,

$$F(s;\gamma) = 1 - \frac{\gamma}{4}\mathrm{erfc}(s) + \mathcal{O}\big(\gamma^{\frac{3}{2}} s^{-1} \mathrm{e}^{-2s^2}\big).$$

On the other hand, as $s \to -\infty$,

$$F(s;\gamma) = \mathrm{e}^{\eta_1(\bar{\gamma})s}\eta_0(\gamma)\big(1+o(1)\big), \quad \eta_1(\gamma) = \frac{1}{2\sqrt{2\pi}}\mathrm{Li}_{\frac{3}{2}}(\gamma)$$

in terms of the polylogarithm $\mathrm{Li}_s(z)$ and with the s-independent positive factor

$$\eta_0(\gamma) = \exp\left[\frac{1}{2}\ln\left(\frac{2}{2-\gamma}\right) + \frac{1}{4\pi}\int_0^{\bar{\gamma}}\left(\left(\mathrm{Li}_{\frac{1}{2}}(x)\right)^2 - \frac{\pi x}{1-x}\right)\frac{\mathrm{d}x}{x}\right].$$

Second, a Ferrari-Spohn formula for χ^γ:

Baik, Bothner 2020

We have

$$F(s;\gamma) = \sqrt{\frac{1-\sqrt{\bar{\gamma}}}{2(2-\gamma)}}\,\det(1+\sqrt{\bar{\gamma}}\chi_t S\chi_t \restriction_{L^2(\mathbb{R})})$$
$$+\sqrt{\frac{1+\sqrt{\bar{\gamma}}}{2(2-\gamma)}}\,\det(1-\sqrt{\bar{\gamma}}\chi_t S\chi_t \restriction_{L^2(\mathbb{R})})$$

where $S : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the trace class integral operator with kernel

$$S(x,y) = \frac{1}{2\sqrt{\pi}}\mathrm{e}^{-\frac{1}{4}(x+y)^2}.$$

Why so similar?

Given the striking formal similarities between

$$\mathrm{Prob}(F_1 \leq s) = \exp\left[-\frac{1}{2}\int_s^\infty (x-s)q^2(x)\,\mathrm{d}x - \frac{1}{2}\int_s^\infty q(x)\,\mathrm{d}x\right],$$

and

$$\mathrm{Prob}(\chi \leq s) = \exp\left[-\frac{1}{2}\int_s^\infty (x-s)p^2(x)\,\mathrm{d}x - \frac{1}{2}\int_s^\infty p(x)\,\mathrm{d}x\right]$$

it is only natural to identify an underlying universal foundation.

Following the steps of Alexandre Krajenbrink

In either ensemble, limiting GOE or GinOE, we are facing the following type of determinant

$$D_1(t,\gamma) = \sqrt{\det(1 - \bar{\gamma} K_t - \gamma \phi_t \otimes \varphi_t \restriction_{L^2(\mathbb{R}_+)})},$$

where the trace class $K_t : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$ is a composition of two Hankel operators with kernel

$$K_t(x,y) = \int_0^\infty \phi_t(x+z)\phi_t(z+y)\,\mathrm{d}z,$$

where $\phi_t(x) := \phi(x+t), \varphi_t(x) := 1 - \int_x^\infty \phi_t(y)\,\mathrm{d}y$.

Krajenbrink 2020

For $(t,\gamma) \in \mathbb{R} \times [0,1]$, provided $1-\gamma K_t$ is invertible on $L^2(\mathbb{R}_+)$,

$$D_1(t,\gamma) = \sqrt{D_2(t,\bar{\gamma})}\sqrt{\frac{\gamma - 1 - \cosh\mu(t,\bar{\gamma}) + \sqrt{\bar{\gamma}}\sinh\mu(t,\bar{\gamma})}{\gamma - 2}}$$

with

$$\mu(t,\gamma) := \int_t^\infty q(s,\gamma)\,\mathrm{d}s, \quad D_2(t,\gamma) = \exp\left[-\int_t^\infty (s-t)q^2(s,\gamma)\,\mathrm{d}s\right]$$

and where $q(t,\gamma) := \sqrt{\gamma}((1-\gamma K_t)^{-1}\phi_t)(0)$.

Most importantly, the above $q = q(t,\gamma)$ can be characterized through a naturally associated RHP!

Beyond the class of integrable operators

Consider trace class Hankel operators $M_t, N_t : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$,

$$(M_t f)(x) := \int_0^\infty \phi(x+y+t) f(y)\,\mathrm{d}y, \quad (N_t f)(x) := \int_0^\infty \psi(x+y+t) f(y)\,\mathrm{d}y,$$

where $\phi, \psi : \mathbb{R} \to \mathbb{C}$ are c-differentiable with $\phi(+\infty) = \psi(+\infty) = 0$.

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Let $K_t := M_t N_t : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$. If $\|K_t\| \leq 1$ in $L^2(\mathbb{R}_+)$ norm for every $t \in \mathbb{R}$, then for all $\lambda \in \mathbb{D}_1(0)$,

$$\det(1 - \lambda K_t \restriction_{L^2(\mathbb{R}_+)}) = \exp\left[-\int_t^\infty (s-t) q(s,\lambda) q^*(s,\lambda)\,\mathrm{d}s\right],$$

with
$q(t,\lambda) = \lambda^{\frac{1}{2}}((1-\lambda K_t)^{-1}\phi_t)(0), q^*(t,\lambda) = \lambda^{\frac{1}{2}}((1-\lambda K_t^*)^{-1}\psi_t)(0).$

Bothner 2020

If in addition $\|K_t\| \leq 1$ in $L^1(\mathbb{R}_+) \cap L^2(\mathbb{R}_+)$ norm for every $t \in \mathbb{R}$ and

$$\phi, \psi \in \left\{ f \in L^1(\mathbb{R}) : \ f' \in L^1(\mathbb{R}), \ xf(x) \in L^1(\mathbb{R}), \ f(-\infty) = 0 \right\},$$

then the following RHP is uniquely solvable for $(t, \lambda) \in \mathbb{R} \times \mathbb{D}_1^\circ(0)$,

(1) $\mathbf{X}(z)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$ and continuous on $\overline{\mathbb{H}}^{\pm}$.

(2) The continuous boundary values $\mathbf{X}_\pm(z) := \lim_{\epsilon \downarrow 0} \mathbf{X}(z \pm \mathrm{i}\epsilon), z \in \mathbb{R}$ obey

$$\mathbf{X}_+(z) = \mathbf{X}_-(z) \begin{bmatrix} 1 - r_1(z) r_2(z) & -r_2(z) \mathrm{e}^{-\mathrm{i}tz} \\ r_1(z) \mathrm{e}^{\mathrm{i}tz} & 1 \end{bmatrix}, \quad z \in \mathbb{R},$$

with $r_1(z) = -\mathrm{i}\lambda^{\frac{1}{2}} \int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}zy} \phi(y) \, \mathrm{d}y, r_2(z) = \mathrm{i}\lambda^{\frac{1}{2}} \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}zy} \psi(y) \, \mathrm{d}y$.

(3) As $z \to \infty$, we have $\mathbf{X}(z) = \mathbb{I} + \mathbf{X}_1 z^{-1} + o(z^{-1}), \ \mathbf{X}_1 = \mathbf{X}_1(t, \lambda)$.

and $q(t, \lambda) = \mathbf{X}_1^{21}(t, \lambda), q^*(t, \lambda) = \mathbf{X}_1^{12}(t, \lambda)$.

Concrete example

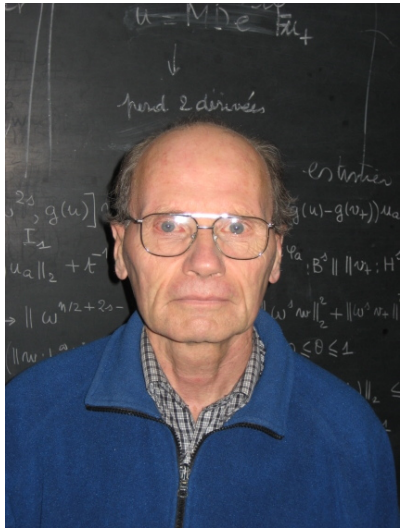
(1) In the GinOE, $\phi(x) = \psi(x) = \frac{1}{\sqrt{\pi}} \mathrm{e}^{-x^2}$ and so $r_1(z) = -\mathrm{i}\sqrt{\gamma}\, \mathrm{e}^{-\frac{1}{4}z^2} = \bar{r}_2(z)$ leading to

$$\mathbf{X}_+(z) = \mathbf{X}_-(z) \begin{bmatrix} 1 - \gamma \mathrm{e}^{-\frac{1}{2}z^2} & -\mathrm{i}\sqrt{\gamma}\, \mathrm{e}^{-\frac{1}{4}z^2 - \mathrm{i}tz} \\ -\mathrm{i}\sqrt{\gamma}\, \mathrm{e}^{-\frac{1}{4}z^2 + \mathrm{i}tz} & 1 \end{bmatrix}, \quad z \in \mathbb{R}.$$

(2) In the GOE, $\phi(x) = \psi(x) = \mathrm{Ai}(x)$ and so $r_1(z) = -\mathrm{i}\sqrt{\gamma}\, \mathrm{e}^{\frac{\mathrm{i}}{3}z^3} = \bar{r}_2(z)$ leading to

$$\mathbf{X}_+(z) = \mathbf{X}_-(z) \begin{bmatrix} 1 - \gamma & -\mathrm{i}\sqrt{\gamma}\, \mathrm{e}^{-\frac{\mathrm{i}}{3}z^3 - \mathrm{i}tz} \\ -\mathrm{i}\sqrt{\gamma}\, \mathrm{e}^{\frac{\mathrm{i}}{3}z^3 + \mathrm{i}tz} & 1 \end{bmatrix}, \quad z \in \mathbb{R}.$$

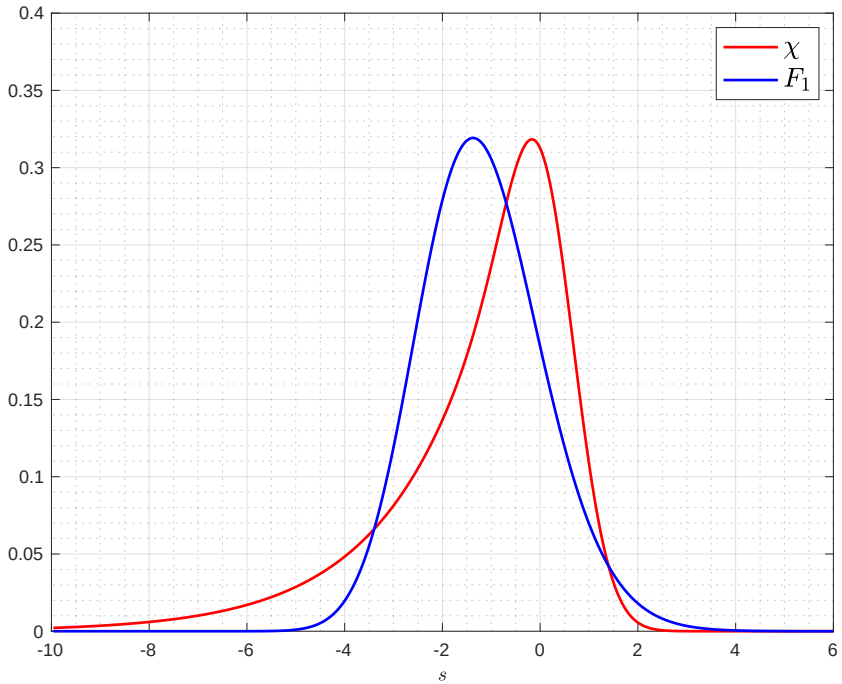
Thank you very much for your attention!!!

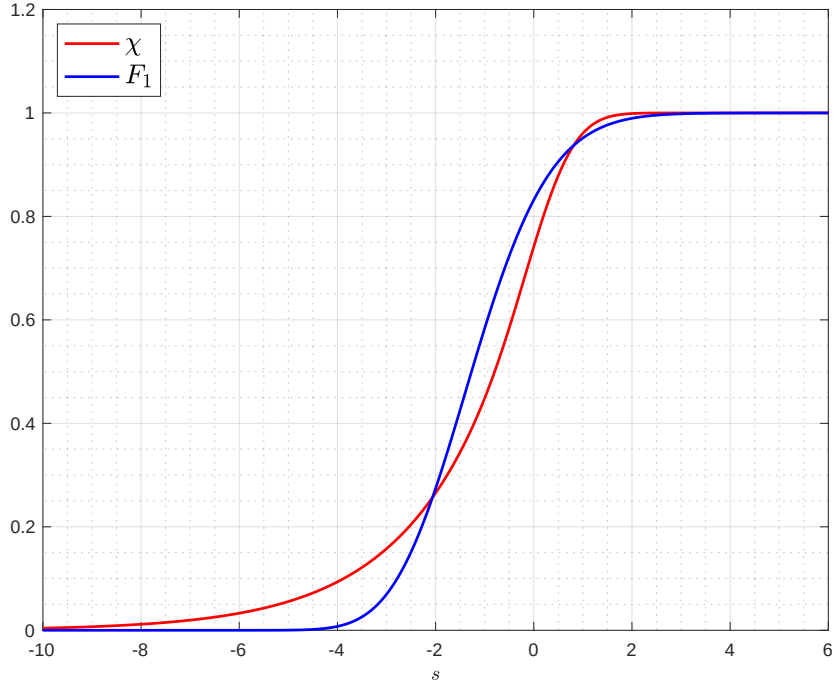


References I

- J. Baik, T. Bothner, *The largest real eigenvalue in the real Ginibre ensemble and its relation to the Zakharov-Shabat system*, Ann. Appl. Probab. **30** (2020), no. 1, 460-501.
- J. Baik, T. Bothner, *Edge distribution of thinned real eigenvalues in the real Ginibre ensemble*, arXiv:2008.01694.

Numerics





	mean	variance	skewness	kurtosis
χ	-1.30319	3.97536	-1.76969	5.14560
F_1	-1.20653	1.60778	0.29346	0.16524