Long-time asymptotics for the good Boussinesq equation via a Riemann-Hilbert approach

Christophe Charlier

Integrable systems and applications,
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Joint work with J. Lenells (KTH) and D. Wang (Beijing).

The good Boussinesq equation

The mathematician Boussinesq derived (back in 1872) the following equation for shallow water waves propagating in a rectangular channel

$$u_{tt} - u_{xx} - (u^2)_{xx} - u_{xxxx} = 0.$$

The good Boussinesq equation takes the form

$$u_{tt} + (u^2)_{xx} + u_{xxxx} = 0.$$

The initial data are $u_0(x) = u(x,0)$ and $u_1(x) = u_t(x,0)$.
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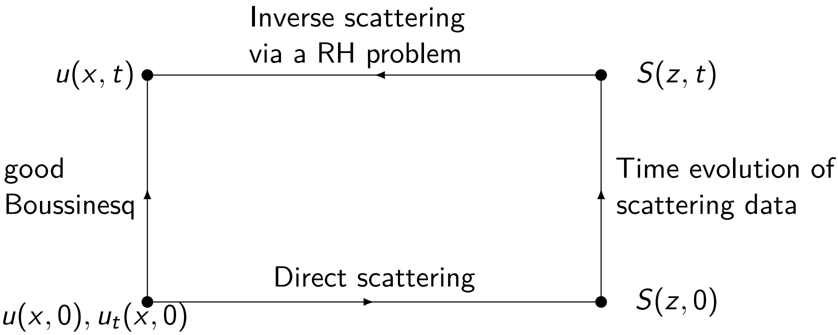
Earlier works on long-time asymptotics: Linares & Scialom (1995), Liu (1997), Farah (2008), Wang (2009). Functional analytic approaches.

Scattering and inverse scattering method via a Riemann-Hilbert (RH) approach



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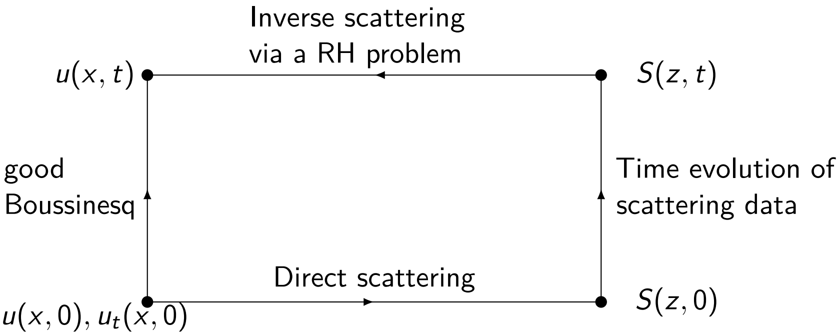

Deift & Zhou (1993): long-time asymptotics for the mKdV equation

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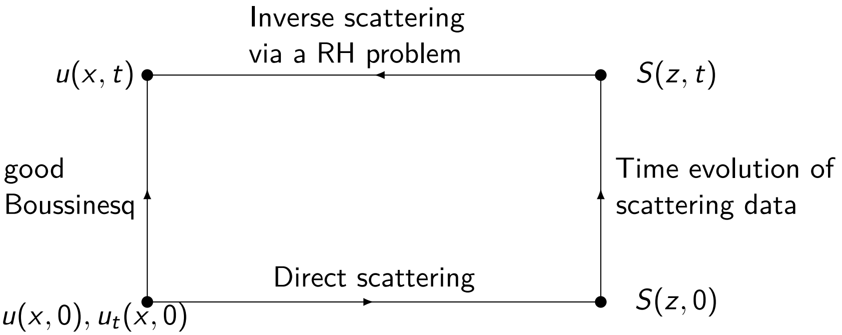

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Deift, Tomei & Trubowitz (1982): An inverse scattering for the Boussinesq equation was outlined.

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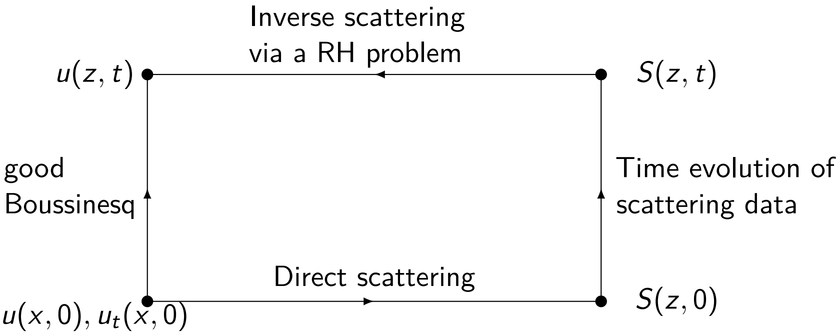

At his 60th birthday conference in 2005, P. Deift presented a list of sixteen open problems, among which he pointed out that “The long-time behavior of the solutions of the Boussinesq equation with general initial data is a very interesting problem with many challenges.”

Direct scattering

The good Boussinesq equation

$$u_{tt} + (u^2)_{xx} + u_{xxxx} = 0.$$

is equivalent to the compatibility condition

$$M_{xt}(x,t,z) = M_{tx}(x,t,z), \quad \text{where } z \in \mathbb{C} \text{ is a new parameter}$$

and M is a 3×3 matrix that satisfies the following Lax pair (Zakharov, 1974):

$$\begin{cases} M_x - [\mathcal{L}, M] = \mathsf{U}M, \\ M_t - [\mathcal{Z}, M] = \mathsf{V}M. \end{cases}$$

$$\mathcal{L} = \operatorname{diag}(l_1, l_2, l_3), \quad \mathcal{Z} = \operatorname{diag}(z_1, z_2, z_3), \quad \lim_{x \to \pm\infty} \mathsf{U} = 0, \quad \lim_{x \to \pm\infty} \mathsf{V} = 0.$$

Direct scattering

Fix $t = 0$ and consider the x-part of the Lax pair:

$$M_x - [\mathcal{L}, M] = \mathrm{U}M.$$

where

$$\mathcal{L}(z) = \begin{pmatrix} l_1(z) & 0 & 0 \\ 0 & l_2(z) & 0 \\ 0 & 0 & l_3(z) \end{pmatrix} = \begin{pmatrix} \omega z & 0 & 0 \\ 0 & \omega^2 z & 0 \\ 0 & 0 & z \end{pmatrix}, \quad \omega = e^{\frac{2\pi i}{3}},$$

$$\mathrm{U}(x,0,z) = -\frac{2u(x,0)}{3z} \begin{pmatrix} \omega^2 & \omega & 1 \\ \omega^2 & \omega & 1 \\ \omega^2 & \omega & 1 \end{pmatrix} - \frac{v(x,0) + u_x(x,0)}{3z^2} \begin{pmatrix} \omega & \omega^2 & 1 \\ \omega & \omega^2 & 1 \\ \omega & \omega^2 & 1 \end{pmatrix}$$

What is M for $t = 0$?

Direct scattering

Consider the solutions to the following linear Volterra integral equations

$$X(x,z) = I - \int_x^{\infty} e^{(x-x')\mathcal{L}(z)} (\mathsf{U}X)(x',z) e^{-(x-x')\mathcal{L}(z)} dx',$$
$$Y(x,z) = I + \int_{-\infty}^{x} e^{(x-x')\mathcal{L}(z)} (\mathsf{U}Y)(x',z) e^{-(x-x')\mathcal{L}(z)} dx'.$$

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The columns of X and Y do not exist for every value of z !

The integrand is of the form

$$\begin{pmatrix} \star & \star\, e^{(x-x')(l_1(z)-l_2(z))} & \star\, e^{(x-x')(l_1(z)-l_3(z))} \\ \star\, e^{(x-x')(l_2(z)-l_1(z))} & \star & \star\, e^{(x-x')(l_2(z)-l_3(z))} \\ \star\, e^{(x-x')(l_3(z)-l_1(z))} & \star\, e^{(x-x')(l_3(z)-l_2(z))} & \star \end{pmatrix}$$

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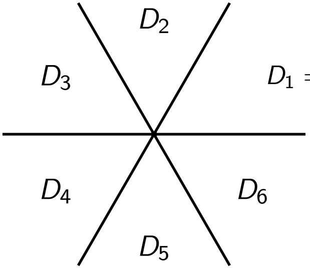

$$D_1 = \{z : \text{Re}\, l_1 < \text{Re}\, l_2 < \text{Re}\, l_3\}$$

Direct scattering

Let X_j (resp. Y_j) be the j-th column of X (resp. Y) $j = 1, 2, 3$.

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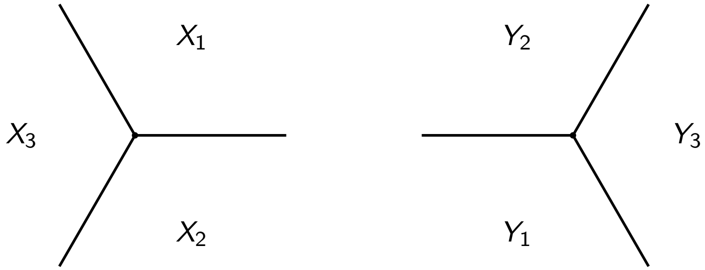

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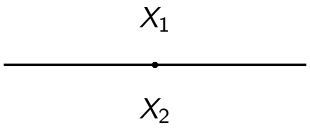

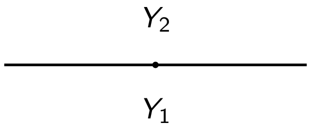

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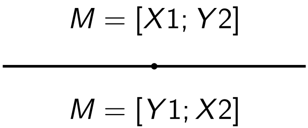

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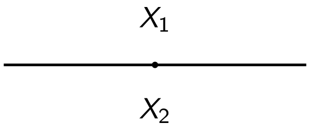

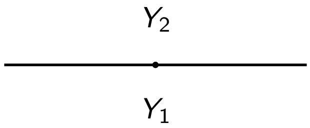

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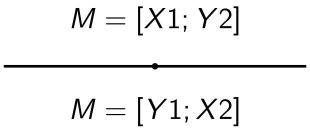

(In fact this is not as simple as that, because we want $\det M(x, t, k) = 1$...)

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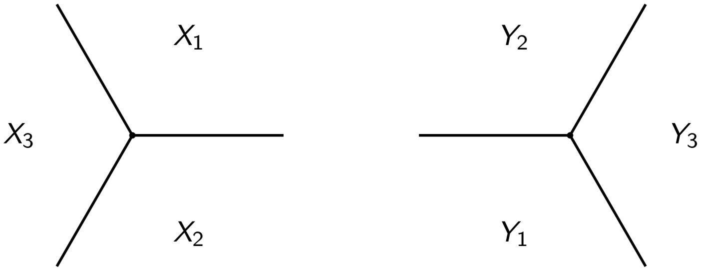

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Assume M satisfies

$$M_x - [\mathcal{L}, M] = \mathsf{U}M,$$

and consider $M^A = (M^{-1})^T$. It satisfies the following x-part:

$$(M^A)_x + [\mathcal{L}, M^A] = -\mathsf{U}^T M^A.$$

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We consider the associated Volterra equations:

$$X^A(x,z) = I + \int_x^{\infty} e^{-(x-x')\mathcal{L}(z)}(\mathrm{U}^T X^A)(x',z)e^{(x-x')\mathcal{L}(z)}dx',$$
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Direct scattering

Summary of the ingredients to build M: X, Y, X^A, Y^A.

How to construct M ? It is still not obvious.

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The solution to

$$M_x - [\mathcal{L}, M] = \mathrm{U}M$$

can also be analyzed via a Fredholm integral equation:

$$M(x,z) = I + \int_\gamma e^{(x-x')\mathcal{L}(z)}(\mathrm{U}M)(x',z)e^{-(x-x')\mathcal{L}(z)}dx',$$

where the contours $\gamma = \gamma_{ij}(x,z)$, $i,j = 1,2,3$, are defined by

$$\gamma_{ij}(x,z) = \begin{cases} (-\infty, x), & \text{Re}\, l_i(z) < \text{Re}\, l_j(z), \\ (+\infty, x), & \text{Re}\, l_i(z) \geq \text{Re}\, l_j(z), \end{cases}.$$

Direct scattering

Advantage of the Fredholm equation:

- All the columns of M exist simultaneously
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Disadvantage of the Fredholm equation:

- It is considerably harder to analyze than the Volterra equations.

  The solution does not exist for $z \in \mathcal{Z}$, where $\mathcal{Z}$ is the zero set of the associated Fredholm determinant (the kernel is not scalar but 3×3 matrix-valued).

Direct scattering

We try to analyze a solution of $M_x - [\mathcal{L}, M] = \mathsf{U}M$.

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The good news is that it is possible to relate M with X, Y, X^A, Y^A

$$M = \begin{pmatrix} X_{11} & \frac{Y_{31}^A X_{23}^A - Y_{21}^A X_{33}^A}{s_{11}} & \frac{Y_{13}}{s_{33}^A} \\ X_{21} & \frac{Y_{11}^A X_{33}^A - Y_{31}^A X_{13}^A}{s_{11}} & \frac{Y_{23}}{s_{33}^A} \\ X_{31} & \frac{Y_{21}^A X_{13}^A - Y_{11}^A X_{23}^A}{s_{11}} & \frac{Y_{33}}{s_{33}^A} \end{pmatrix}, \qquad z \in D_1$$

where s is given by $s(z) = I - \int_{\mathbb{R}} e^{-x\mathcal{L}(z)}(\mathrm{U}X)(x,z)e^{x\mathcal{L}(z)}dx$.

Direct scattering

It is possible to relate $M(x, 0, z)$ with X, Y, X^A, Y^A

This allows to show that $\mathcal{Z} = \emptyset$.

We can transfer other properties of X, Y, X^A, Y^A to $M(x, 0, z)$.

The evolution in time $M(x, 0, z) \to M(x, t, z)$ is simpler to analyze.

Properties of M

(a) $M(x,t,\cdot) : \mathbb{C} \setminus \Gamma \to \mathbb{C}^{3\times 3}$ is analytic, where $\Gamma = \mathbb{R} \cup \omega\mathbb{R} \cup \omega^2\mathbb{R}$.

(b) For each $z \in \Gamma \setminus \{0\}$, the following limits exist

$$M_{\pm}(x,t,z) := \lim_{\epsilon \to 0_+} M(x,t,z \pm \epsilon\mathfrak{n}),$$

where $\mathfrak{n}$ goes in the normal direction to Γ at z (viewed as a complex number). The functions $z \mapsto M_{\pm}(x,t,z)$ are continuous and satisfy the jump

$$M_{+}(x,t,z) = M_{-}(x,t,z)v(x,t,z),$$

where the 3×3 function v is smooth and $v(x,t,z) \to I$ as $|z| \to +\infty$, $z \in \Gamma$.

(c) $M(x,t,z) = I + \mathcal{O}(z^{-1})$ as $z \to \infty$.

Properties of M

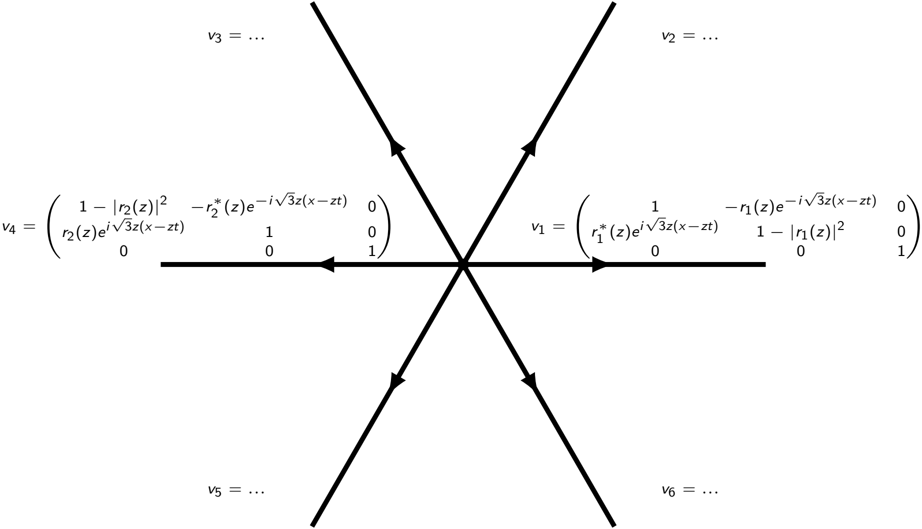
- M satisfies the following asymptotics

$$M(x,t,z) = I + M^{(1)}(x,t)z^{-1} + \mathcal{O}(z^{-2}), \qquad \text{as } z \to \infty,$$

- The solution u for the good Boussinesq equation can be recovered from M:

$$u(x,t) = -\frac{3}{2}\frac{d}{dx}M_{33}^{(1)}(x,t).$$

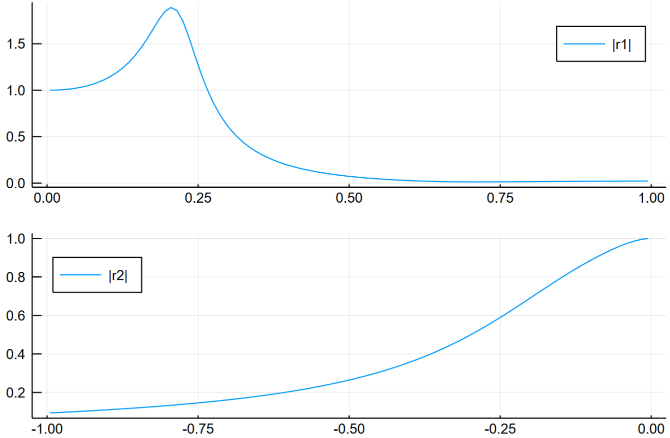
RH problem for M



r_1 and r_2

r_1 and r_2 are given explicitly in terms of the initial data u_0, u_1. For example,

$$r_1(z) = \frac{(s(z))_{12}}{(s(z))_{11}}, \qquad s(z) = I - \int_{\mathbb{R}} e^{-x\mathcal{L}(z)}(\mathrm{U}X)(x,z)e^{x\mathcal{L}(z)}dx.$$



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As $z \to 0$, $z \in D_1$, we have

$$M(x,t,z) = \frac{\alpha(x,t)}{z^2}\begin{pmatrix} \omega & 0 & 0 \\ \omega & 0 & 0 \\ \omega & 0 & 0 \end{pmatrix} + \frac{\beta(x,t)}{z}\begin{pmatrix} \omega^2 & 0 & 0 \\ \omega^2 & 0 & 0 \\ \omega^2 & 0 & 0 \end{pmatrix},$$

$$+ \frac{\gamma(x,t)}{z}\begin{pmatrix} \omega^2 & 0 & 0 \\ 1 & 0 & 0 \\ \omega & 0 & 0 \end{pmatrix} + \frac{\delta(x,t)}{z}\begin{pmatrix} 0 & 1-\omega & 0 \\ 0 & 1-\omega & 0 \\ 0 & 1-\omega & 0 \end{pmatrix} + \begin{pmatrix} \star & \star & \epsilon(x,t) \\ \star & \star & \epsilon(x,t) \\ \star & \star & \epsilon(x,t) \end{pmatrix}$$

$$+ \ldots$$

RH problems for M and n

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$$n(x,t,z) = \begin{pmatrix} \omega & \omega^2 & 1 \end{pmatrix} M(x,t,z), \qquad \omega = e^{\frac{2\pi i}{3}},$$

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- **If** the solution to the RH problem for n is unique, then solution u can be recovered from n via

$$u(x,t) = -\frac{3}{2}\frac{\partial}{\partial x}\lim_{z\to\infty} z(n_3(x,t,z) - 1).$$

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- The RH problem for n is of size 1×3. This is also not convenient.

3×3 RH problem for m

We define a 3×3 RH problem for m as follows:

- Same jumps as the RH problem for M
- Same behavior at ∞: $m(x,t,z) = I + \mathcal{O}(z^{-1})$.
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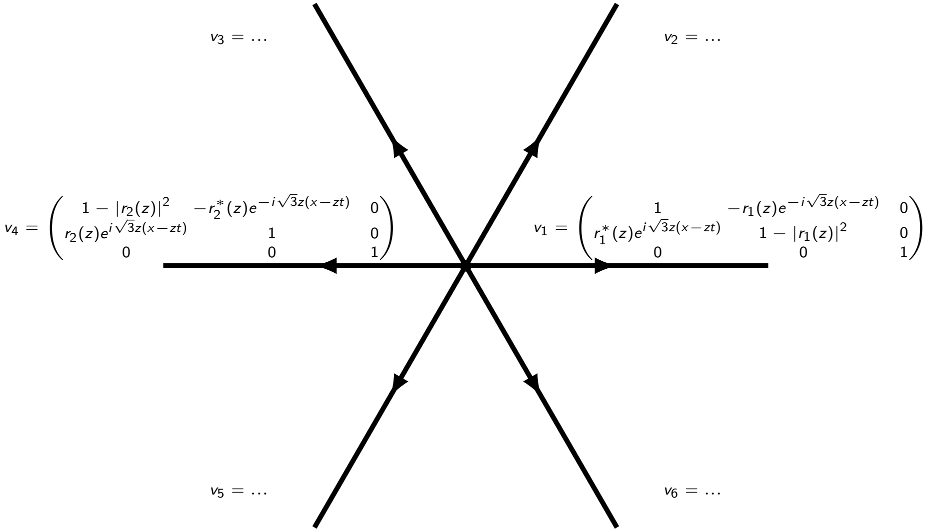
Existence of m will be guaranteed for large t from the steepest descent analysis. So we only need to assume existence of u for $t \in (0,T]$.

Steepest descent of m



Scalar additive RH problems

Seek for a complex-valued function such that

(a) $f : \mathbb{C} \setminus \mathbb{R}$ is analytic

(b) For each $x \in \mathbb{R}$, the following limits exist

$$f_{\pm}(x) := \lim_{\epsilon \to 0_+} f(x \pm i\epsilon).$$

The functions $x \mapsto f_{\pm}(x)$ are continuous and satisfy the jump

$$f_+(x) = f_-(x) + v(x),$$

where v is smooth enough and $v(x) \to 0$ as $|x| \to +\infty$.

(c) $f(z) = \mathcal{O}(z^{-1})$ as $z \to \infty$.

Solution ?

Scalar additive RH problems

Seek for a complex-valued function such that

(a) $f : \mathbb{C} \setminus \mathbb{R}$ is analytic

(b) For each $x \in \mathbb{R}$, the following limits exist

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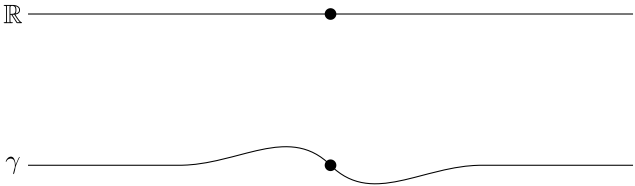
Solution ? Well-known Plemelj formula gives

$$f(z) = \int_{\mathbb{R}} \frac{v(x)}{x - z} \frac{dx}{2\pi i}.$$

Steepest descent method

Asymptotics for $\int_{\mathbb{R}} f(z)e^{-itz^2}dz$ as $t \to +\infty$:



Under suitable conditions on f, we have

$$\int_{\mathbb{R}} f(z)e^{-itz^2}dz = \int_{\gamma} f(z)e^{-itz^2}dz \approx \frac{\sqrt{\pi}f(0)}{\sqrt{t}}e^{-\frac{\pi i}{4}} + \ldots \qquad \text{as } t \to \infty.$$

The Deift–Zhou steepest descent method generalizes the classical steepest descent method for matrix RH problems.

Steepest descent of m

- The steepest descent is a method developed by Deift and Zhou (1993)
- In our case, we will apply several **invertible** transformation $m \mapsto m^{(1)} \mapsto m^{(2)} \mapsto m^{(3)} \mapsto \hat{m}$.
- Goal: obtain an RH problem for $\hat{m}$ such that its jumps are "close" to I.
- Such a RH problem is called "small norms" RH problem, and satisfies

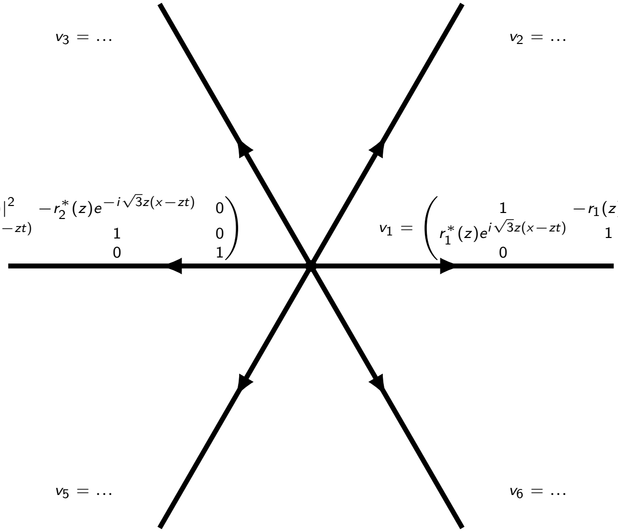
$$\hat{m}(x,t,z) = I + o(1), \qquad \text{as } t \to +\infty$$

uniformly for z in the complex plane.

- We start by computing the saddle point of the phase function

$$\frac{d}{dz}\Big(i\sqrt{3}z(x-zt)\Big) = 0, \qquad \Leftrightarrow \qquad z = z_0 := \frac{x}{2t}.$$

RH problem for m



$$v_4 = \begin{pmatrix} 1-|r_2(z)|^2 & -r_2^*(z)e^{-i\sqrt{3}z(x-zt)} & 0 \\ r_2(z)e^{i\sqrt{3}z(x-zt)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$v_1 = \begin{pmatrix} 1 & -r_1(z)e^{-i\sqrt{3}z(x-zt)} & 0 \\ r_1^*(z)e^{i\sqrt{3}z(x-zt)} & 1-|r_1(z)|^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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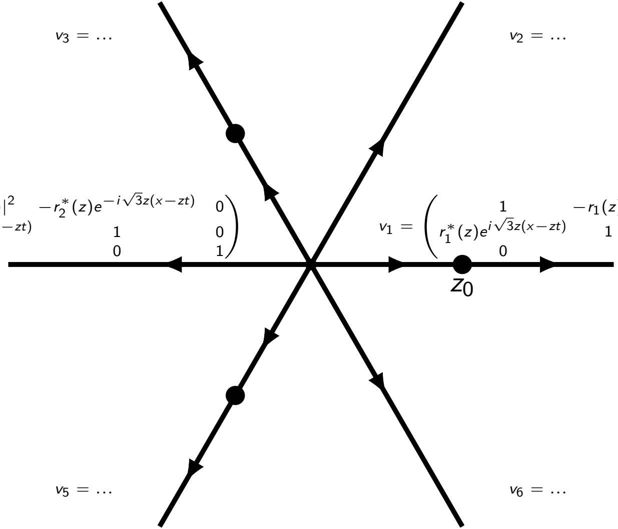

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Preparation to the steepest descent

The jumps are not analytic, so we cannot deform the contour at the moment. We use an idea from Deift and Zhou:

$$\begin{aligned}
r_2(z) &= r_{2,a}(x,t,z) + r_{2,r}(x,t,z), && z \in (-\infty, 0], \\
r_1(z) &= r_{1,a}(x,t,z) + r_{1,r}(x,t,z), && z \in [0, z_0], \\
\hat{r}_1(z) &= \hat{r}_{1,a}(x,t,z) + \hat{r}_{1,r}(x,t,z), && z \in [z_0, \infty),
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where

$$\hat{r}_1(z) = \frac{r_1(z)}{1 - |r_1(z)|^2}.$$

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Many estimates are needed. In particular, the $\frac{\partial}{\partial x}$ causes serious technicalities.

First transformation : $m \to m^{(1)}$

The jump matrix v_4 can be factorized as

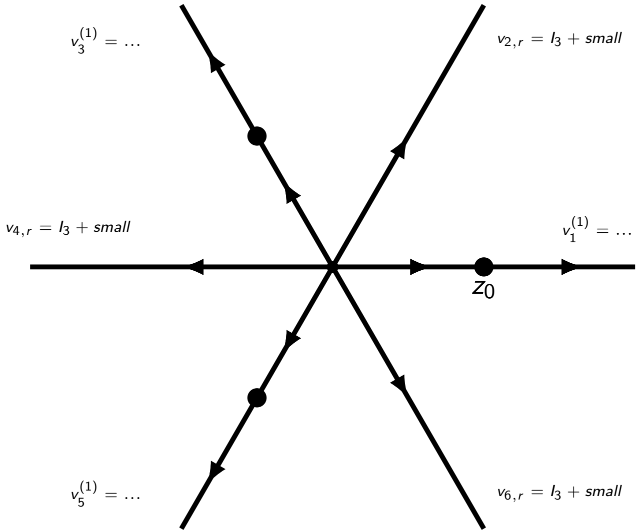
$$v_4 = v_{4,a}^U v_{4,r} v_{4,a}^L.$$

$$v_{4,a}^U = \begin{pmatrix} 1 & -r_{2,a}^*(z)e^{-i\sqrt{3}z(tz-x)} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad v_{4,a}^L = \begin{pmatrix} 1 & 0 & 0 \\ r_{2,a}(z)e^{i\sqrt{3}z(tz-x)} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\begin{aligned} v_{4,a}^U &= I + \text{small}, && \text{as } t \to +\infty, \; z \text{ above } \Gamma_4, \\ v_{4,a}^L &= I + \text{small}, && \text{as } t \to +\infty, \; z \text{ below } \Gamma_4, \\ v_{4,r} &= I + \text{small}, && \text{as } t \to +\infty, \; z \in \Gamma_4. \end{aligned}$$

There are similar factorizations for v_2 and v_6.

RH problem for $m^{(1)}$



Second and third transformations: $m^{(1)} \mapsto m^{(2)} \mapsto m^{(3)}$

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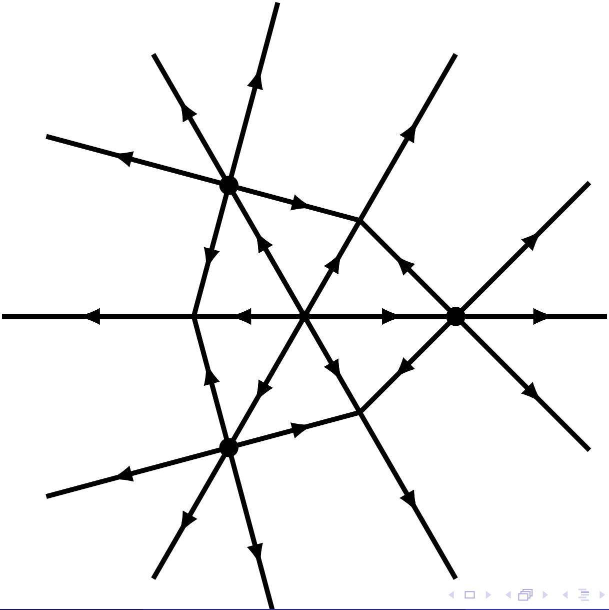
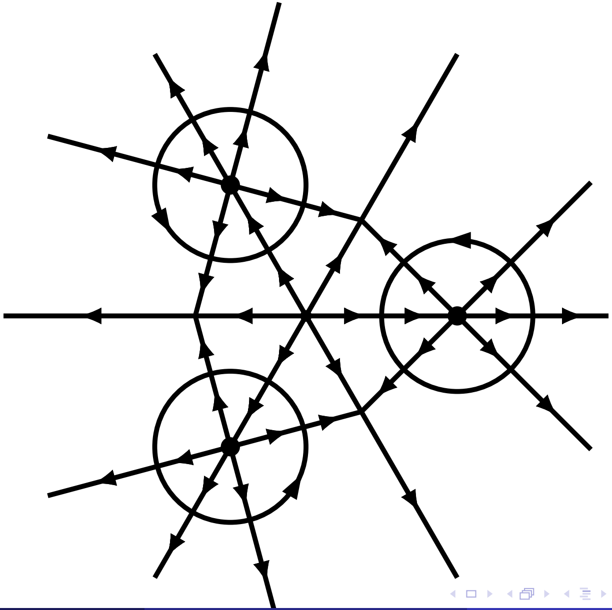
- Now, there are good factorizations for $v_1^{(2)}$ also on $(z_0, +\infty)$ and we can open lenses on both $(0, z_0)$ and $(z_0, +\infty)$ as in the first transformation; this is the $m^{(2)} \mapsto m^{(3)}$ transformation.

RH problem for $m^{(3)}$



RH problem for $\hat{m}$



Long-time asymptotics for the good Boussinesq

Theorem (C-Lenells-Wang 2020)

Assume that

- *The initial data are smooth with rapid decay,*
- $|r_1(z)| < 1$ *for all* $z \geq z_0 = \frac{x}{2t}$,
- *there exists a solution* $u : \mathbb{R} \times [0, +\infty)$ *to the good Boussinesq equation.*

Then, as $t \to +\infty$, *we have*

$$u(x,t) = -\frac{3^{5/4} z_0 \sqrt{\nu}}{\sqrt{2t}} \sin\bigg(\frac{19\pi}{12} + \nu \ln(6\sqrt{3} t z_0^2) - \sqrt{3} z_0^2 t - \arg q - \arg\Gamma(i\nu) - \frac{1}{\pi}\int_{z_0}^{\infty} \ln \frac{|s - z_0|}{|s - e^{\frac{2\pi i}{3}} z_0|} d\ln(1 - |r_1(s)|^2)\bigg) + O(t^{-1}\ln t),$$

where $\nu(z_0) = -\frac{1}{2\pi}\ln(1 - |r_1(z_0)|^2)$ *and* $q = r_1(z_0)$.

Thank you for your attention