Matrix generalization of the PII hierarchy: Lax pair and solutions in terms of Fredholm determinants

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Integrable systems around the world

SISSA, 14-16 September 2020

Plan

Outline

The Airy kernel

Here $Ai(x)$ is the Airy function.

Definition

Consider the Airy convolution operator $\mathcal{A}$ acting on any $f \in L^2(\mathbb{R}_+)$ as

$$(\mathcal{A}f)(x) := \int_{\mathbb{R}_+} Ai(x+y)f(y)dy.$$

The Airy kernel can be obtained as the square of the Airy convolution operator

$$K_{Ai}(x,y) := \mathcal{A}^2(x,y) = \int_{\mathbb{R}_+} Ai(x+z)Ai(y+z)dz$$

The Fredholm determinant $F(s) := \det(Id - K_{Ai}\chi_{[s,\infty)})$ is connected to Painlevé trascendents.

Tracy-Widom results about the Airy kernel

(1999, C.A. Tracy - H. Widom) The Fredholm determinant $F(s)$ and the Hastings-McLeod solution of the Painlevé II equation are related through the formula

$$\frac{d^2}{ds^2}\log F(s) = -q^2(s),$$

where $q(s)$ solves the equation

$$q''(s) = 2q^3 + sq$$

with boundary condition $q(s) \sim Ai(s)$ at $s \to +\infty$.

Outline

1. Introduction
   - Around the Airy kernel
   - Generalizations and motivations
2. Our setting and result
3. Proof of result
   - Property of the Fredholm determinants
   - The isomonodromic Lax pair for the matrix PII hierarchy

Generalization: higher order Airy kernels

Consider the Fredholm determinant $F_{2n+1}(s) := \det(Id - K_{2n+1}\chi_{[s,\infty)})$, where the kernel operator K_{2n+1} is defined as before

$$K_{2n+1}(x,y) := \mathcal{A}^2_{2n+1}(x,y) = \int_{\mathbb{R}_+} Ai_{2n+1}(x+z)Ai_{2n+1}(y+z)dz.$$

Here Ai_{2n+1} is the n-th Airy function.

- (2018, P. Le Doussal - S.N. Majumdar - G. Schehr) proved their connection with statistichal mechanic model.
- (2019, M. Cafasso - T. Claeys - M. Girotti) showed that

$$\frac{d^2}{ds^2}\log F_{2n+1}(s) = -q^2((-1)^{n+1}s)$$

with $q(s)$ now solving the n-th equation of the PII hierarchy.

Matrix valued n-th Airy convolution operators

Consider the n-th matrix Airy convolution operator acting on $\vec{f} \in L^2(\mathbb{R}_+, \mathbb{C}^r)$ as

$$\left(\mathcal{A}_{2n+1}\vec{f}\right)(x) := \int_{\mathbb{R}_+} \mathbf{Ai}_{2n+1}(x+y;\vec{s})\vec{f}(y)dy,$$

where now $\mathbf{Ai}_{2n+1}(x)$ is the n-th matrix Airy function defined as

$$\mathbf{Ai}_{2n+1}(x) := \left\{c_{jk}Ai_{2n+1}(x+s_j+s_k)\right\}_{j,k=1}^{r},$$

with $C := (c_{jk})_{j,k=1}^{r} = C^\dagger$ and $s_k \in \mathbb{R}$ for any k.

Remark Provided that C has eigenvalues inside $[-1,1]$, the kernel operator obtained as the square of the matrix n-th Airy convolution operator $\mathcal{A}_{2n+1}^2$ defines a determinantal point process on $\{1,\ldots,r\} \times \mathbb{R}$.

Result

Theorem (S.T.)

The Fredholm determinants $F_{2n+1}(\vec{s}) := \det(Id - \mathcal{A}^2_{2n+1})$ *satisfies*

$$-\frac{d^2}{dS^2}\log F_{2n+1}(\vec{s}) = \mathrm{Tr}(Q^2(\vec{s})),$$

where the matrix Q solves the n-th member of a certain matrix Painlevé II hierarchy.

Here the differential operator $\frac{d}{dS}$ is defined as

$$\frac{d}{dS} := \sum_{k=1}^{r}\frac{\partial}{\partial s_k}.$$

Matrix Lenard operators

The matrix Lenard operators are constructed through the following recursion

$$\begin{cases} \mathcal{L}_0 = \frac{1}{2} I_r \\ \frac{\mathrm{d}}{\mathrm{d}S} \mathcal{L}_n = \left(\frac{\mathrm{d}^3}{\mathrm{d}S^3} + [Q, \cdot]_+ \frac{\mathrm{d}}{\mathrm{d}S} + \frac{\mathrm{d}}{\mathrm{d}S} [Q, \cdot]_+ + [Q, \cdot] \frac{\mathrm{d}}{\mathrm{d}S}^{-1} [Q, \cdot] \right) \mathcal{L}_{n-1}, \ n \geq 1 \end{cases}$$

- I_r is the identity matrix of dimension r;
- $[Q, \cdot]$ and $[Q, \cdot]_+$ are the standard matrix commutator and anticommutator;
- $\frac{\mathrm{d}}{\mathrm{d}S}^{-1}$ is the formal integration.

EXAMPLES

1. $\mathcal{L}_1[Q] = Q$
2. $\mathcal{L}_2[Q] = Q_{2S} + 3Q^2$
3. $\mathcal{L}_3[Q] = Q_{4S} + 5\,[Q, Q_{2S}]_+ + 5Q_S^2 + 10Q^3$

The matrix PII hierarchy

The matrix PII hierarchy related to the Fredholm determinants $F_{2n+1}(\vec{s})$ is obtained as

$$\mathrm{PII}_{\mathrm{NC}}^{(n)}: \ \left(\frac{\mathrm{d}}{\mathrm{d}S} + [Q, \cdot]_+\right) \mathcal{L}_n \left[\frac{\mathrm{d}}{\mathrm{d}S} Q - Q^2\right] = (-1)^{n+1} 4^n [S, Q]_+ \,,$$

with $S := \mathrm{diag}(s_1, \ldots, s_r)$.

EXAMPLES (here $' = \frac{\mathrm{d}}{\mathrm{d}S}$)

- For $n = 1$ the 2nd order equation: $Q'' = 2Q^3 + 4\,[S, Q]_+$.
  See also (2011, M. Bertola - M. Cafasso).
- For $n = 2$ the 4th order equation:

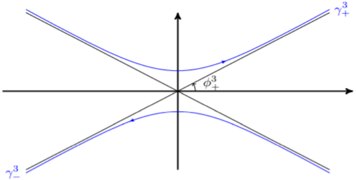
$$Q'''' = 6Q^5 + 4\left[Q^2, Q''\right]_+ + 2QQ''Q + 2\left[(Q')^2, Q\right]_+ + 6Q'QQ' - 4^2\,[S, Q]_+ \,.$$

See also (2016, P.R. Gordoa - A. Pickering - Z.N. Zhu) for another matrix PII hierarchy.

Sketch of the proof

$$\boxed{F_{2n+1}(\vec{s}) = \det\left(Id - \mathcal{M}_n\right)}$$

with $\mathcal{M}_n$ an *integrable* kernel operator acting on $L^2(\gamma_+, \mathbb{C}^r)$ with matrix-valued kernel.



$\downarrow$

R-H problems for block matrices $\Xi(\lambda)$ with jump $J(\lambda, \vec{s})$ for $\lambda \in \gamma_\pm$.

$\swarrow$ $\searrow$

$\Psi := \Xi \exp\left(\theta \otimes \sigma_3\right)$ solves

$$\begin{cases} \partial_\lambda \Psi = M^{(n)} \Psi \\ \frac{\mathrm{d}}{\mathrm{d}S} \Psi = L \Psi. \end{cases}$$

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}S} \log F_{2n+1}(\vec{s}) = -2i \operatorname{Tr}(\alpha_1)}$$

with $\alpha_1 := \lim_{|\lambda| \to \infty} \lambda(I_r - \Xi_{1,1})$.

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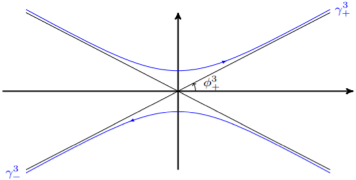

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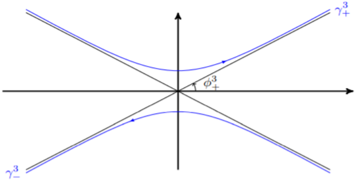

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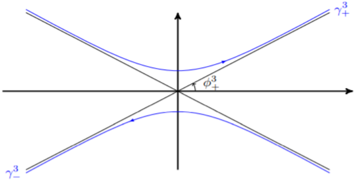

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Outline

R-H problem

Fix $n \geq 1$. Find analytic block matrix valued function $\Xi(\lambda) : \mathbb{C} \setminus (\gamma_+ \cup \gamma_-) \to GL(2r, \mathbb{C})$ satisfying the jump condition

$$\Xi_+(\lambda) = \Xi_-(\lambda) \underbrace{\left(\begin{array}{c|c} I_r & -r(\lambda)\chi_{\gamma_+}(\lambda) \\ \hline -r(-\lambda)\chi_{\gamma_-}(\lambda) & I_r \end{array}\right)}_{:=J(\lambda,\vec{s})}, \ \lambda \in \gamma_\pm$$

where

$$r(\lambda) := \exp(\theta(\lambda, \vec{s}))C\exp(\theta(\lambda, \vec{s}))$$

and $\theta(\lambda, \vec{s}) := \frac{i(-1)^{n+1}\lambda^{2n+1}}{2(2n+1)} I_r + i\lambda S$.

And the asymptotic condition for $|\lambda| \to \infty$

$$\Xi(\lambda) \longrightarrow I_{2r} + \sum_{j \geq 1} \frac{\Xi_j}{\lambda^j}; \quad \text{with } \ \Xi_1 = \alpha_1 \otimes \sigma_3 + \beta_1 \otimes \sigma_2.$$

Here σ_j are the Pauli's matrices.

Proof of the formula for of $\frac{d}{dS} \log F_{2n+1}$

1. It is known that $F_{2n+1} = \det\left(Id - \mathcal{M}_n\right)$. This last Fredholm determinant can be interpreted as a Tau function associated to the space of deformation of the R-H problem for Ξ (see (2010, M. Bertola)). This means that

$$\frac{d}{dS} \log F_{2n+1}(\vec{s}) = \int_{\gamma_\pm} \mathrm{Tr}\left((\Xi_-)^{-1} \partial_\lambda (\Xi_-) \frac{d}{dS} J J^{-1} \right) \frac{d\lambda}{2\pi i}$$

2. Due to the special form of the deformed jump matrix

$$J(\lambda, \vec{s}) = \exp\left(\theta(\lambda, \vec{s}) \otimes \sigma_3\right) J_0 \exp\left(-\theta(\lambda, \vec{s}) \otimes \sigma_3\right)$$

the integral above can be explicitely computed as the formal residue at ∞ of the function $\mathrm{Tr}\left(\Xi^{-1}\partial_\lambda \Xi i\lambda I_r \otimes \sigma_3\right)$.

3. Direct computation of that finally implies

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Outline

Conctruction of Ψ

Theorem

The R-H problem admits solution Ξ if and only if the matrix C has eigenvalues in the interval $[-1, 1]$.

In this case we can take the function
$\Psi(\lambda, \vec{s}) := \Xi(\lambda, \vec{s}) \exp\left(\theta(\lambda, \vec{s}) \otimes \sigma_3\right)$.
For $\lambda \in \gamma_+ \cup \gamma_-$, it has a constant jump condition

$$\Psi_+(\lambda) = \Psi_-(\lambda) J_0$$

and for $|\lambda| \to \infty$ it has the asymptotic condition

$$\Psi(\lambda) \to \left(I_{2r} + \sum_{j \geq 1} \frac{\Xi_j}{\lambda^j} \right) \exp\left(\theta(\lambda, \vec{s}) \otimes \sigma_3\right).$$

Block Matrix Lax pair for the matrix PII hierarchy

Proposition

There exist $L^{(n)} = L$ and $M^{(n)}$ polynomial matrices in λ of degree respectively 1 and $2n$, such that Ψ solves the system

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}S}\Psi(\lambda,\vec{s}) = L(\lambda,\vec{s})\Psi\left(\lambda,\vec{s}\right) \\ \partial_\lambda \Psi(\lambda,\vec{s}) = M^{(n)}(\lambda,\vec{s})\Psi\left(\lambda,\vec{s}\right) \end{cases},$$

where

$$L(\lambda,\vec{s}) = \left(\begin{array}{c|c} i\lambda I_r & Q(\vec{s}) \\ \hline Q(\vec{s}) & -i\lambda I_r \end{array}\right) \text{ with } Q(\vec{s}) = 2\beta_1(\vec{s})$$

$$M^{(n)}(\lambda,\vec{s}) = \sum_{k=0}^{2n} \lambda^{2n-k} M_{2n-k}(\vec{s}, Q(\vec{s})),$$

with M_{2n-k} block matrices of dimension $2r$.

The zero curvature equation

Proposition

For each fixed n, the zero curvature equation $\partial_\lambda L - \frac{d}{dS} M^{(n)} + \left[L, M^{(n)}\right] = 0_{2r}$ *is equivalent to the n-th member of the matrix PII hierarchy.*

1. the zero curvature equation $\iff$ system of differential equations for the blocks composing the coefficients M_{2n-k} of the matrix $M^{(n)}$;
2. the blocks of all these coefficients M_{2n-k} can be written with formulas involving the matrix Lenard operators $\mathcal{L}_j(\frac{d}{dS} Q - Q^2)$ for $j = 1, \ldots, n$.
3. Finally there is only one last condition required from the zero curvature equation, that reads as

$$\left(\frac{d}{dS} + [Q, \cdot]_+\right) \mathcal{L}_n \left[\frac{d}{dS} Q - Q^2\right] = (-1)^{n+1} 4^n [S, Q]_+ .$$

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Conclusion

For any fixed $n \geq 1$, here is what we proved:

- The matrix Q defined as $Q := 2\beta_1 = -i \lim_{|\lambda|\to\infty} \lambda \Xi_{12}$, solves the n-th member of the matrix PII hierarchy.
- the logarithmic derivative of the Fredholm determinant $F_{2n+1}(\vec{s})$ is given by

$$-\frac{\mathrm{d}}{\mathrm{dS}} \log F_{2n+1} = 2i\,\mathrm{Tr}(\alpha_1).$$

By looking at the coefficient of λ^{-1} in the expansion at ∞ of the function $L := \frac{\mathrm{d}}{\mathrm{dS}} \Psi\Psi^{-1}$ we obtain that

$$\frac{\mathrm{d}}{\mathrm{dS}} \alpha_1 = -2i\beta_1^2, \text{ for any fixed } n.$$

$$\downarrow$$

The matrix Q solution of $\mathrm{PII}_{\mathrm{NC}}^{(n)}$, is related to the Fredholm determinant F_{2n+1} through

$$-\frac{\mathrm{d}^2}{\mathrm{dS}^2} \log F_{2n+1}(\vec{s}) = \mathrm{Tr}(Q^2(\vec{s})).$$

Thank you everybody!!!