Fredholm determinants, exact solutions to the Kardar-Parisi-Zhang equation and integro-differential Painlevé equations

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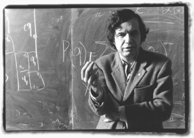
Outline

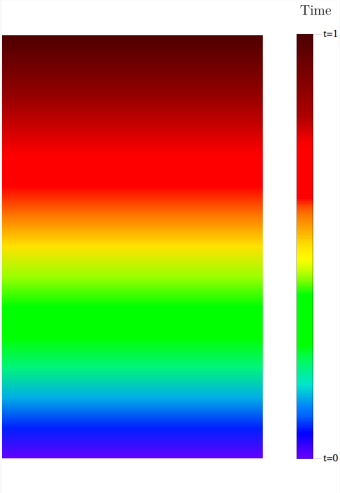


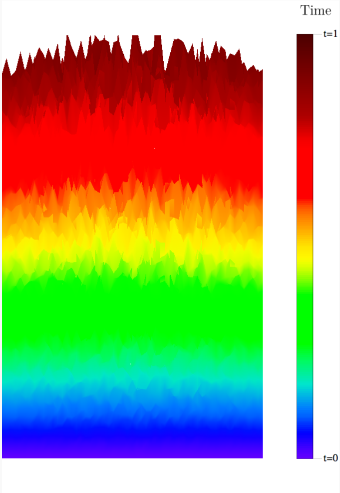
1. A short tale around Kardar, Parisi and Zhang.

2. Fredholm determinants, integrable systems and integro-differential hierarchies.

3. Towards an integro-differential Painlevé II hierarchy.

Stochastic growth of an interface





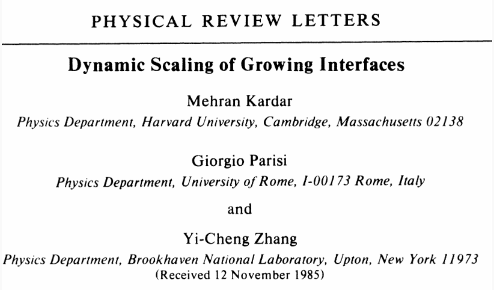
How to model such growth ?

Kardar-Parisi-Zhang equation

Consider a height field $h(x,t)$ obeying

$$\partial_t h(x,t) = \partial_x^2 h(x,t) + (\partial_x h(x,t))^2 + \sqrt{2}\,\xi(x,t) \ ,$$

where $\xi(x,t)$ is a standard white noise.


PHYSICAL REVIEW LETTERS

Dynamic Scaling of Growing Interfaces

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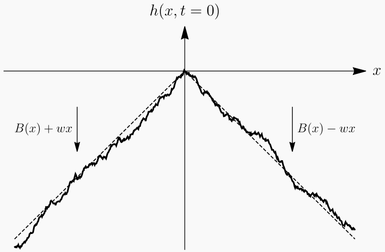
(Received 12 November 1985)

Geometries of interest

Full-space

$x \in \mathbb{R}$

- *Flat*
  $h(x, t = 0) = 0$
- *Droplet (wedge)*
  $h(x, t = 0) = -w|x| + \log(\frac{w}{2})$,
  with a slope $w \gg 1$
- *Brownian*
  $h(x, t = 0) = \mathcal{B}(x) - w|x|$

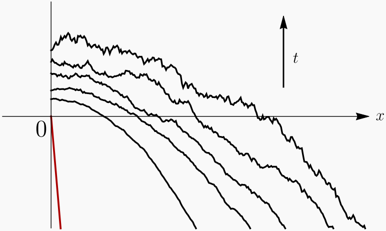


Half-space

$x \in \mathbb{R}_+$

with the b.c. $\partial_x h(x, t)\mid_{x=0} = A$, $\forall t > 0$.

It corresponds to the presence of a wall at the origin.



Exact solutions to the KPZ equation at all times

1. **Full-space**
   Droplet Sasamoto, Spohn ; Calabrese, Le Doussal, Rosso ; Dotsenko ; Amir, Corwin, Quastel ('10)
   Flat Calabrese, Le Doussal ('11)
   Brownian Imamura, Sasamoto ('12), Borodin, Corwin, Ferrari, Veto ('14)

2. **Half-space**
   Droplet
   - $A = \infty$ Gueudré, Le Doussal ('12)
   - $A = 0$ Borodin, Bufetov, Corwin ('15)
   - $A = -\frac{1}{2}$ Barraquand, Borodin, Corwin, Wheeler ('17)
   - $A \geqslant -\frac{1}{2}$ Krajenbrink, Le Doussal ('19)
   - $A \in \mathbb{R}$ De Nardis, Krajenbrink, Le Doussal, Thiery ('20)

   Brownian
   - $A = \infty$ Krajenbrink, Le Doussal ('19)
   - $A \geqslant -\frac{1}{2}$ Barraquand, Krajenbrink, Le Doussal ('20)

Exact solution to the KPZ equation with droplet data

Recall that the droplet data is $h(x,0) = -w|x| + \log(w/2)$, with $w \gg 1$. We will be interested in $H(t) = h(0,t) + \frac{t}{12}$, then

Result (Exact solution for droplet data)

$$\mathbb{E}_{\rm KPZ}\left[\exp\left(-ze^{H(t)}\right)\right] = \mathrm{Det}[I - \sigma_{z,t}K_{\rm Ai}]_{\mathbb{L}^2(\mathbb{R})} \, .$$

where $\mathbb{E}_{\rm KPZ} \equiv$ *average over the KPZ white noise.* $K_{\rm Ai}$ *is the Airy kernel,* $K_{\rm Ai}(u,u') = \int_0^\infty \mathrm{d}r \; \mathrm{Ai}(r+u)\mathrm{Ai}(r+u')$, *and the weight* $\sigma_{z,t}$ *is the Fermi factor*

$$\sigma_{z,t}(u) = \frac{z}{z + e^{-t^{1/3}u}}$$

At large time (take $z = e^{-st^{1/3}}$), the cumulative distribution of $H(t) = h(0,t) + \frac{t}{12}$ converges to the Tracy-Widom distribution for $\beta = 2$.

$$\lim_{t\to+\infty} \mathbb{P}\left(\frac{H(t)}{t^{1/3}} \leqslant s\right) = F_2(s) = \mathrm{Det}[I - K_{\rm Ai,s}]$$

Two surprising connections for the droplet initial data

- **Homogeneous case:** consider

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\log \mathrm{Det}[I - K_{\mathrm{Ai,s}}] = -q(s)^2$$

then you get the Painlevé II equation (Tracy-Widom)

$$q''(s) = 2q(s)^3 + sq(s)$$

- **Inhomogeneous case:** consider

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2}\log \mathrm{Det}[I - \sigma K_{\mathrm{Ai,s}}] = -\int_{\mathbb{R}} \mathrm{d}t\, q_t(s)^2 \sigma'(t)$$

then you get the integro-differential Painlevé II equation (Amir-Corwin-Quastel)

$$q_t''(s) = q_t(s)\big(s + t + 2\int_{\mathbb{R}} \mathrm{d}t'\, q_{t'}(s)^2\sigma'(t')\big)$$

Also some recent connections to the Kadomtsev–Petviashvili equation (Quastel, Remenik in maths, Le Doussal in physics)

Half-line problem: an intriguing symmetry

Take for initial data a Brownian motion with drift

$$h(x, t = 0) = \mathcal{B}(x) - (B + \frac{1}{2})x$$

and boundary condition

$$\partial_x h(x, t) \mid_{x=0} = A$$

Then there is a remarkable symmetry between the parameters A and B.

Result ($A \leftrightarrow B$ symmetry)

We have the equality in distribution

$$h_A^B(x = 0, t) = h_B^A(x = 0, t), \text{ for any } t > 0$$

Is the symmetry between initial and boundary condition specific to KPZ or is it more general in integrable non-linear systems ?

Phase diagram for the half-space KPZ problem

Consider the droplet initial data $h(x,t=0)=\mathcal{B}(x)-(B+\frac{1}{2})x$, $B\gg 1$ with boundary condition $\partial_x h(x,t)\mid_{x=0}=A$.

- For $A>-\frac{1}{2}$, the KPZ height has Tracy-Widom GSE fluctuations.

$$\lim_{t\to\infty}\frac{h(0,t)+\frac{t}{12}}{t^{1/3}}=\chi_4$$

- For $A=-\frac{1}{2}$, the KPZ height has Tracy-Widom GOE fluctuations.

$$\lim_{t\to\infty}\frac{h(0,t)+\frac{t}{12}}{t^{1/3}}=\chi_1$$

- For $A<-\frac{1}{2}$, the KPZ height has Gaussian fluctuations.

$$\lim_{t\to\infty}\frac{h(0,t)+t(\frac{1}{12}-(A+\frac{1}{2})^2)}{t^{1/2}\sqrt{|2A+1|}}=\mathcal{N}(0,1)$$

This resembles the Baik-Ben Arous-Péché phase transition for the largest eigenvalue of a rank-one spiked GSE matrix.

Some open problems around exact solutions to KPZ

- Is the generating function of the KPZ height determinantal for any initial condition in full-space ?
- Is the determinantal structure related to particular points in space (e.g. $x = 0$)
- What are the boundary conditions in half-space yielding a determinantal structure ?
- Is there a more general relation between random matrix theory and the exact solutions to the KPZ equation ?

From KPZ to Fredholm

Homogeneous Fredholm determinants

Take $\mathrm{Det}[I - K_s]_{\mathbb{L}^2(\mathbb{R}_+)}$, with K_s of the form of the square of a Hankel operator

$$K_s(x, y) = \int_0^\infty \mathrm{d}r\, A(x + r + s)A(y + r + s)$$

for some function A, it is related to various problems:

- Linear statistics of Ginibre, Elliptic, Gaussian random matrix spectra,
- Full-counting statistics and entropy of free fermions,
- Multi-critical fermions at the edge of interacting systems,
- Exact solutions of the Kardar-Parisi-Zhang equation,
- The Zakharov-Shabat system,
- The theory of solitons and τ-functions,
- Riemann-Hilbert and inverse scattering methods,
- Determinantal point processes,
- The Painlevé II hierarchy.

Attempt to a generalization

Let $s \in \mathbb{R}$ and a smooth function $A : \mathbb{R} \to \mathbb{R}$ exponentially decreasing towards $+\infty$ such that we define an operator A_s with kernel

$$A_s(x, y) = A(x + y + s).$$

We construct an operator K_s with kernel

$$K_s(x, y) = \int_0^\infty \mathrm{d}r \, A(x + r + s)A(y + r + s)$$

(or equivalently $K_s = A_s^2$) and assume that K_s is bounded by above by the identity so that its resolvent is well defined.

We are interested in two objects:

- $\mathrm{Det}[I - K_s]_{\mathbb{L}^2(\mathbb{R}_+)}$
- $\mathrm{Det}[I - \sigma K_s]_{\mathbb{L}^2(\mathbb{R})}$ with σ a smooth increasing function with asymptotics

$$\lim_{t \to -\infty} \sigma(t) = 0, \qquad \lim_{t \to +\infty} \sigma(t) \in (0, 1], \qquad \text{exponentially fast.}$$

A quasi-universal hierarchy (I)

Define two operators A_s with kernel $A_s(x,y) = A(x+y+s)$ and $K_s = A_s^2$, define $|\delta\rangle$ (resp. $\langle\delta|$) the right (resp. left) projector to 0.

Definition (Conjugated variables)

Let $p \in \mathbb{N}$, we define the quantities

$$q_p(s) = \langle\delta|\, A_s^{(p)} \frac{I}{I-K_s} \,|\delta\rangle$$

$$u_p(s) = \langle\delta|\, A_s^{(p)} \frac{I}{I-K_s} A_s \,|\delta\rangle$$

In integral representation

$$q_p(s) = \int_{\mathbb{R}_+} \mathrm{d}y\, A^{(p)}(y+s)(I-K_s)^{-1}(y,0)$$

$$u_p(s) = \int_{\mathbb{R}_+^2} \mathrm{d}y\mathrm{d}z\, A^{(p)}(y+s)(I-K_s)^{-1}(y,z)A(z+s)$$

The relation between the first functions and the Fredholm determinant is

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}s} \log \mathrm{Det}[I-K_s] & = u_0(s) \\ \frac{\mathrm{d}^2}{\mathrm{d}s^2} \log \mathrm{Det}[I-K_s] & = -q_0(s)^2 \end{cases}$$

A quasi-universal hierarchy (II)

The variables $\{q_p, u_p\}$ verify a universal hierarchy of equations.

Result (Hierarchy)

Let $p \in \mathbb{N}$, we have

$$\begin{cases} q_p' = q_{p+1} - q_0 u_p, \\ u_p' = -q_0 q_p. \end{cases}$$

This hierarchy is equivalent to the Zakharov-Shabat system, it is **independent** of the function A. The reflection coefficient of the related Riemann-Hilbert problem is the Fourier transform of the function A.

Result (Conservation laws)

Let $n \in \mathbb{N}$, the following quadratic quantity is constant

$$u_{2n+1} + \frac{1}{2}\sum_{k=0}^{2n}(-1)^{k+1}[u_k u_{2n-k} - q_k q_{2n-k}] = 0$$

Both results were known in the context of multi-critical fermions by Le Doussal, Majumdar and Schehr for general n.

A quasi-universal hierarchy (III)

The hierarchy of equations raises the index on $\{q_p\}$

$$\begin{cases} q_p' = q_{p+1} - q_0 u_p, \\ u_p' = -q_0 q_p. \end{cases}$$

so that the equations do not close. The only non-universal feature is the closure relation for some n

$$q_n = f(q_0, \ldots, q_{n-1}; u_0, \ldots, u_{n-1}) \quad \Rightarrow \quad \text{all the physics is here.}$$

this is where the explicit expression of A plays a role.

Viewing the hierarchy of differential equations as a member of the Lax pair for the considered system, this closure relation should correspond to a second member of the pair.

Inhomogeneous Fredholm determinants

Why also study $\mathrm{Det}[I - \sigma K_s]_{\mathbb{L}^2(\mathbb{R})}$?

- Finite-time solutions to the KPZ equation;
- Linear (multiplicative) statistics of determinantal / fermionic point processes.

To find back the homogeneous case, consider σ to be the projector onto $\mathbb{R}_+$.

We are still interested in the operator with kernel

$$K_s(x,y) = \int_0^\infty \mathrm{d}r\, A(x+r+s)A(y+r+s)$$

and we lift the operator A_s to $\mathbb{L}^2(\mathbb{R}) \to \mathbb{L}^2(\mathbb{R}_+)$ and denote its adjoint $A_s^\intercal$. By Sylvester's identity, we have

$$\mathrm{Det}[I - \sigma A_s A_s^\intercal]_{\mathbb{L}^2(\mathbb{R})} = \mathrm{Det}[I - A_s^\intercal \sigma A_s]_{\mathbb{L}^2(\mathbb{R}_+)}$$

.

We will be interested in the operator

$$K_2 = A_s^\intercal \sigma A_s$$

A quasi-universal integro-differential hierarchy (I)

Consider the operator $K_2 = A_s^{\intercal} \sigma A_s$,

Definition (Conjugated variables)

Let $p \in \mathbb{N}$, we define the quantities

$$q_p = A_s^{(p)} \frac{I}{I - K_2} |\delta\rangle$$

$$u_p = A_s^{(p)} \frac{I}{I - K_2} A_s^{\intercal}$$

In integral representation

$$\begin{cases} q_p(t) & = \int_{\mathbb{R}_+} \mathrm{d}y \, A^{(p)}(t+s+y)(I-K_{\mathbf{2}})^{-\mathbf{1}}(y,0) \\ u_p(t,t') & = \iint_{\mathbb{R}_+^{\mathbf{2}}} \mathrm{d}y \, \mathrm{d}z \, A^{(p)}(t+s+y)(I-K_{\mathbf{2}})^{-\mathbf{1}}(y,z) A(z+t'+s) \end{cases}$$

In the homogeneous case $\{q_p, u_p\}$ were scalar functions of the parameter s, and now

- q_p adopts a vector-like structure;
- u_p adopts a matrix-like structure.

A quasi-universal integro-differential hierarchy (II)

The relation between the first functions and the Fredholm determinant is

$$\begin{cases} \partial_s \log \mathrm{Det}(I - K_2) = \mathrm{Tr}(\sigma' u_0) \\ \partial_s^2 \log \mathrm{Det}(I - K_2) = -(q_0^{\intercal}\, \sigma' q_0) \end{cases}$$

with the canonical inner product $(a^{\intercal} \sigma' b) = \int_{\mathbb{R}} \mathrm{d}v\, a(v) \sigma'(v) b(v)$. The variables $\{q_p, u_p\}$ verify a universal hierarchy of equations.

Result (Hierarchy)

Let $p \in \mathbb{N}$, we have

$$\begin{cases} q_p' &= q_{p+1} - u_p \sigma' q_0, \\ u_p' &= -q_p q_0^{\intercal}. \end{cases}$$

where $' = \partial_s$

- The equation for q_p becomes integral due to the presence of σ';
- The derivative of u_p is a rank-one operator.

A quasi-universal integro-differential hierarchy (III)

Result (Conservation laws)

For all n in $\mathbb{N}$, the following quadratic quantities are invariant

$$u_{2n+1} + u_{2n+1}^{\intercal} + \sum_{k=0}^{2n} (-1)^{k+1} [u_{2n-k}\sigma' u_k^{\intercal} - q_{2n-k} q_k^{\intercal}] = 0$$

and

$$u_{2n} - u_{2n}^{\intercal} + \sum_{k=0}^{2n-1} (-1)^{k+1} [u_{2n-1-k}\sigma' u_k^{\intercal} - q_{2n-1-k} q_k^{\intercal}] = 0$$

The conservation law for even u_{2n} did not exist in the homogeneous case (when σ is the projector to $\mathbb{R}_+$, it yields $0 = 0$).

From Fredholm to Painlevé

Painlevé II hierarchy

The Painlevé II hierarchy is a sequence of ordinary non-linear differential equations obtained recursively upon the action of the Lenard operators. The first equations of the hierarchy read

$$q'' = sq + 2q^3$$
$$q'''' = sq + 10q(q')^2 + 10q^2q'' - 6q^5$$

The n-th member of the hierarchy can be solved upon a particular choice of asymptotic condition (Le Doussal, Majumdar, Schehr in physics, Cafasso, Claeys, Girotti in maths).

Let $A \equiv \mathrm{Ai}_{2n+1}$ be a higher-order Airy function $A^{(2n)}(x) = xA(x)$. The solution of the n-th member of the hierarchy is

$$q(s) = \langle\delta| \frac{A_s}{I - K_s} |\delta\rangle , \qquad q(s) \underset{s\to+\infty}{\sim} A(s)$$

There is a one-to-one correspondence between the PII hierarchy and the Fredholm determinant of the higher-order Airy functions. How about the inhomogeneous determinants ?

Closure relation for the Painlevé II hierarchy

Starting from the function $A^{(2n)}(x) = xA(x)$ and the kernel

$$K_2(x,y) = \int_{\mathbb{R}} \mathrm{d}r\, \sigma(r) A(x+r+s) A(y+r+s).$$

along with the $2n$-th variable of the hierarchy $q_{2n}(t) = A_s^{(2n)} \frac{I}{I-K_2} |\delta\rangle$ we obtain the closure relation

Result (Closure relation for the higher-order Airy function)

$$q_{2n} = (s+X)q_0 - \sum_{\ell=0}^{n-1} \left(u^{\intercal}_{2n-1-2\ell} \sigma' q_{2\ell} - u^{\intercal}_{2n-2-2\ell} \sigma' q_{2\ell+1} \right)$$

where X is the left multiplication, $\forall t \in \mathbb{R}$, $(Xq_0)(t) = t\, q_0(t)$.

Playing with the hierarchy of equations, the conservation laws and the closure relation allows to obtain closed integro-differential equations on q_0 akin to the Painlevé II hierarchy.

First member of the integro-differential Painlevé II hierarchy

Let $s \in \mathbb{R}$, consider the Airy function solution of $A''(x) = xA(x)$ (or $A \equiv \mathrm{Ai}$) and the Fredholm determinant with the kernel

$$K_2(x,y) = \int_{\mathbb{R}} \mathrm{d}r\, \sigma(r) A(x+r+s) A(y+r+s).$$

Consider $q_0(t) = A_s \frac{I}{I-K_2} |\delta\rangle$,

Result (First member of the integro-differential PII)

the function q_0 verifies the integro-differential extension of the Painlevé II equation (Amir-Corwin-Quastel)

$$q_0'' = (s+t) q_0 + 2 q_0 (q_0^{\intercal} \sigma' q_0)$$

subject $q_0(t) \sim \mathrm{Ai}(s+t)$ as $s \to +\infty$ for fixed $t \in \mathbb{R}$ and $' = \partial_s$.

When σ is the projector onto $\mathbb{R}_+$ this equation reduces to the standard Painlevé II equation.

Second member of the integro-differential Painlevé II hierarchy

Let $s \in \mathbb{R}$, consider the higher-order Airy function solution of $A^{(4)}(x) = xA(x)$ (or $A \equiv \mathrm{Ai}_5$) and the Fredholm determinant with the kernel

$$K_2(x,y) = \int_{\mathbb{R}} \mathrm{d}r\, \sigma(r) A(x+r+s) A(y+r+s).$$

Define $q_0(t) = A_s \frac{I}{I-K_2} |\delta\rangle$, then

Result (Second member of the integro-differential PII)

the function q_0 verifies the integro-differential extension of the second equation of the Painlevé II hierarchy

$$\begin{aligned} q_0'''' = (s+t)q_0 + 8q_0'(q_0^\intercal \sigma' q_0') + 6q_0(q_0^\intercal \sigma' q_0'') \\ - 6q_0(q_0^\intercal \sigma' q_0)^2 + 2q_0((q_0')^\intercal \sigma' q_0') + 4q_0''(q_0^\intercal \sigma' q_0) \end{aligned}$$

subject $q_0(t) \sim \mathrm{Ai}_5(s+t)$ as $s \to +\infty$ for fixed $t \in \mathbb{R}$ and $' = \partial_s$.

When σ is the projector onto $\mathbb{R}_+$ this equation reduces to the second member of the PII hierarchy.

Second member of the integro-differential Painlevé II hierarchy

Procedure.

1. Differentiate q_0 four times;
2. Replace the value of q_4 by the closure relation;
3. Use the conservation law for $u_3 + u_3^\intercal$;
4. Use the conservation law for $u_2 - u_2^\intercal$;
5. Replace q_2 by $q_1' + u_1\sigma' q_0$;
6. Use the symmetry of the inner-product $(v^\intercal u_1 v) = \frac{1}{2}(v^\intercal[u_1 + u_1^\intercal]v)$ and the conservation law for $u_1 + u_1^\intercal$;
7. Replace q_1 by $q_0' + u_0\sigma' q_0$;
8. Use that $u_0 = u_0^\intercal$;
9. Use the symmetry of the various inner-products $(v^\intercal w) = (w^\intercal v)$.

□

Direct outlooks and extensions of this work

- Extension of the Zakharov-Shabat system to an operator-valued one by considering $\mathrm{Det}[I - \sigma K_s]$;
- Extension from a symmetric to a Hermitian operator $K_s = A_s \bar{A}_s$;
- Extension to operators of type $K_s = A_s B_s$ for two Hankel operators A_s, B_s;
- Extension of the Beals-Coifman class of reflection coefficient ensuring unique solvability of RHP related to Zakharov-Shabat;
- Extension of the whole Painlevé II hierarchy to the integro-differential counterpart and definition of the related Lenard operators.

Thank you very much for listening!
Any questions ?