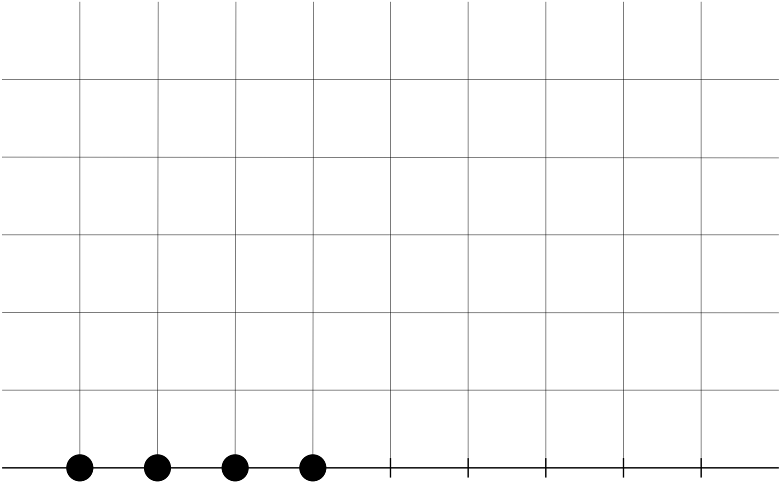
Integrable Systems in the Periodic TASEP

Guilherme Silva
(Universidade de São Paulo, Brazil)

Based on joint work with
Jinho Baik (University of Michigan) and Zhipeng Liu (University of Kansas)

Special thanks to FAPESP for the financial support

The TASEP model



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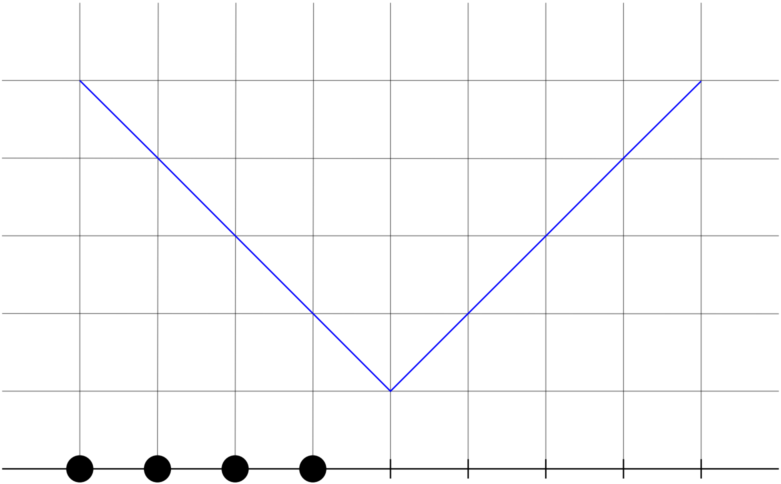

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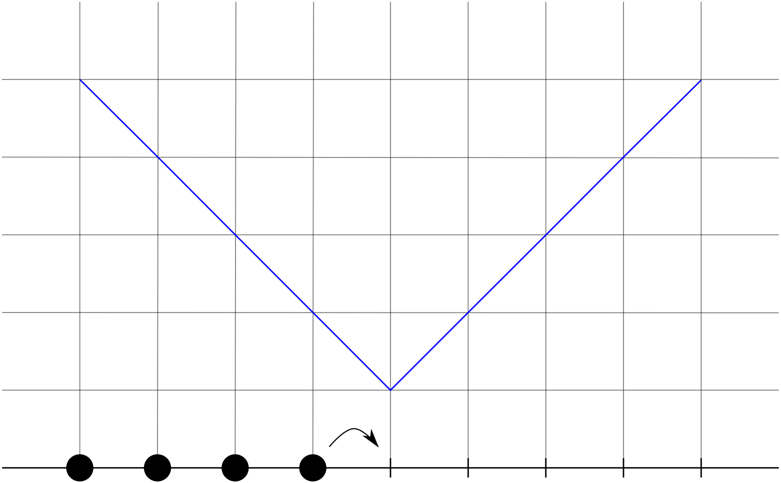

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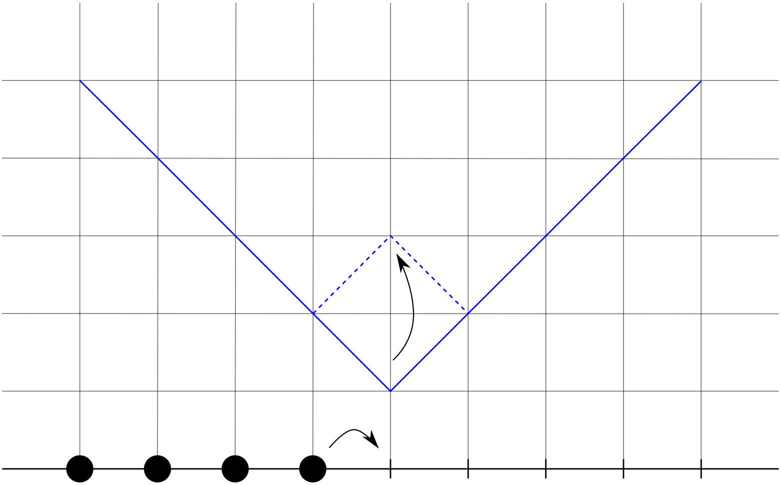

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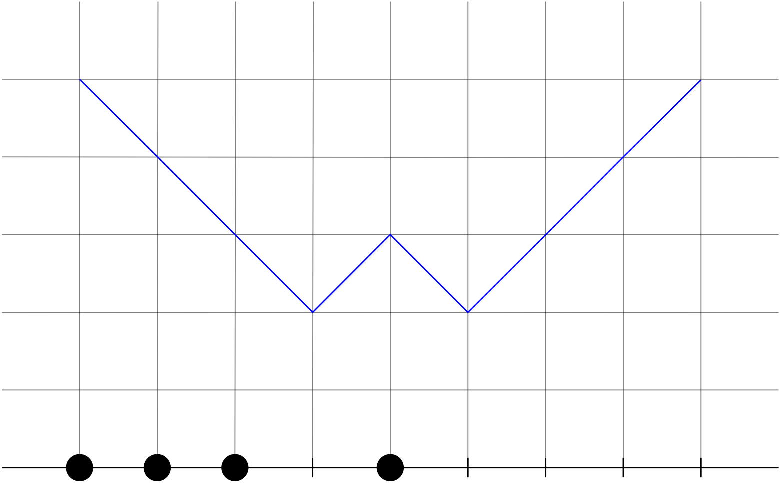

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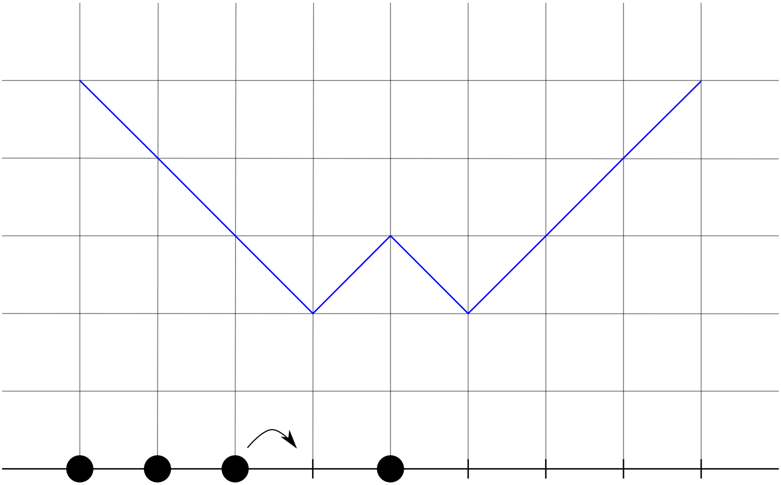

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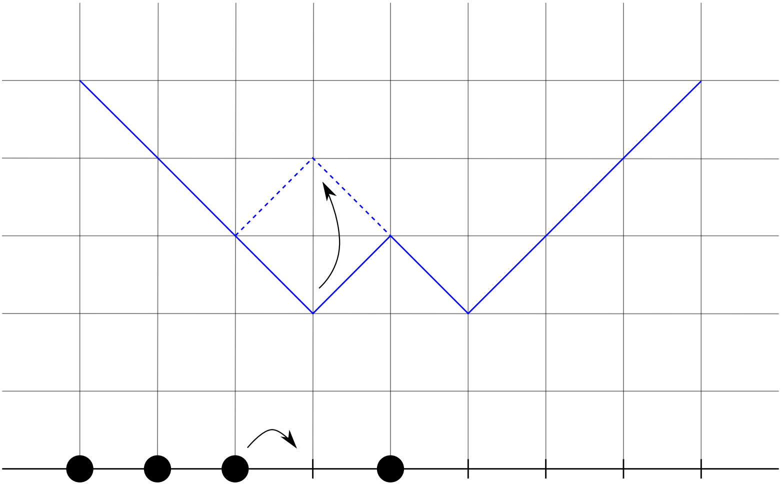

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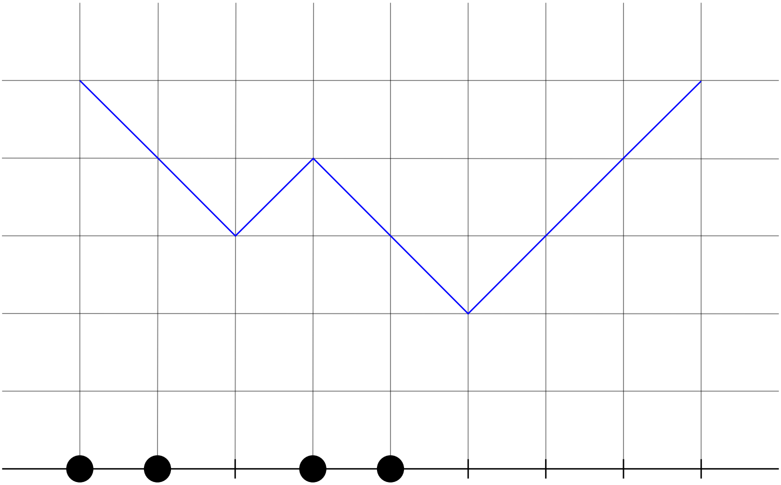

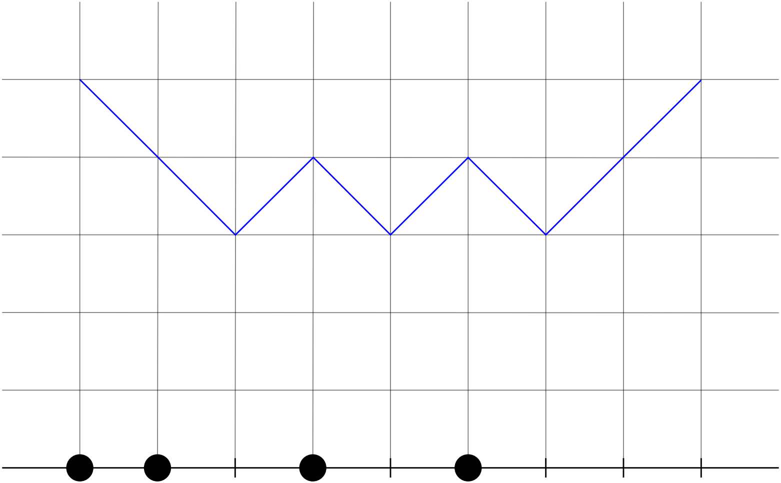

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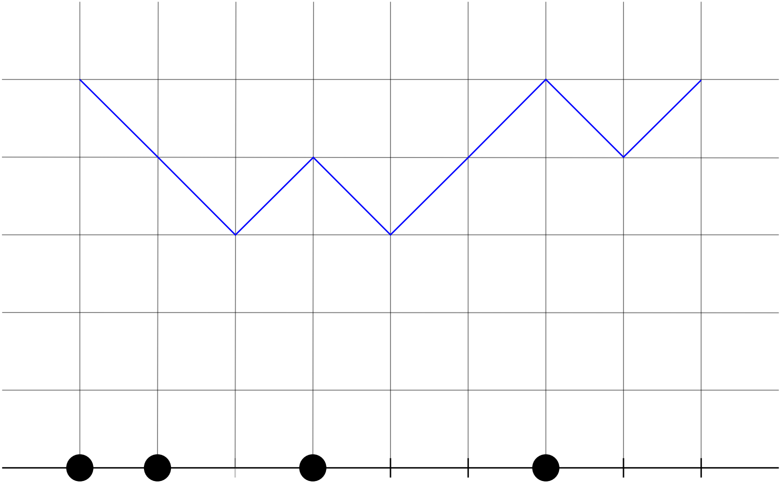

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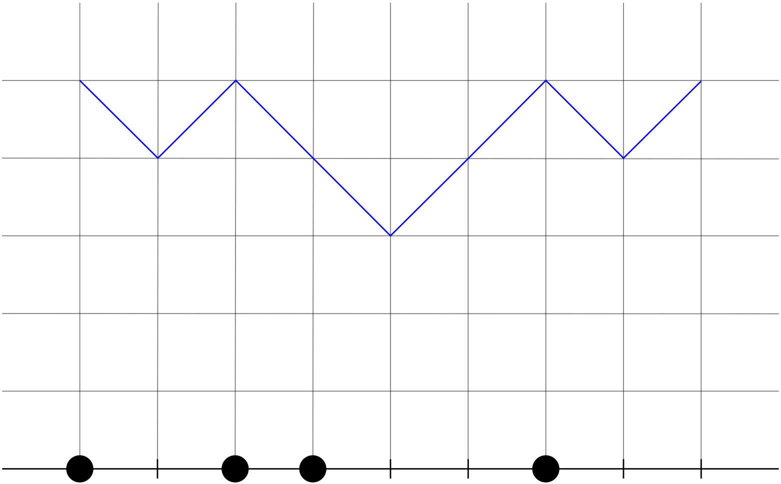

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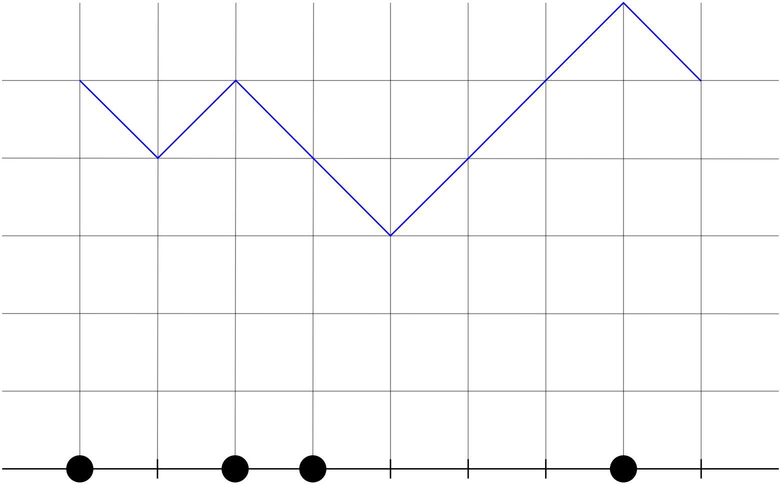
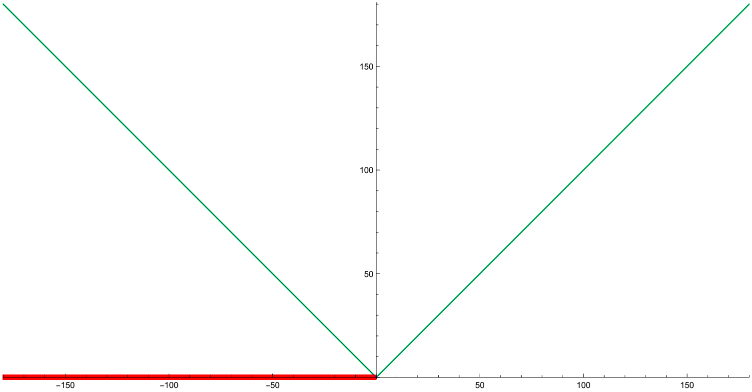


TASEP - 180 particles



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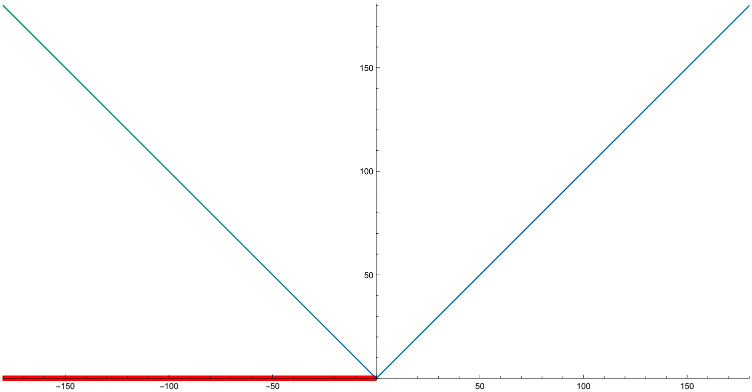
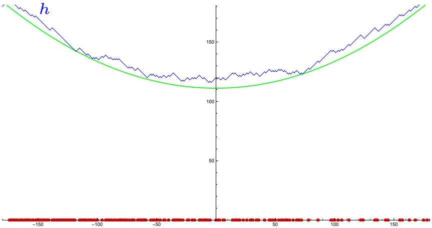
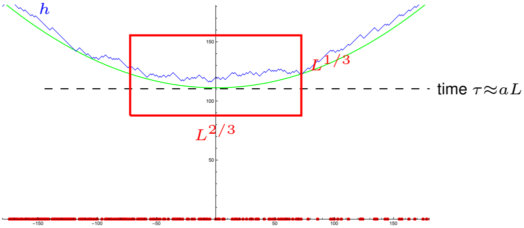
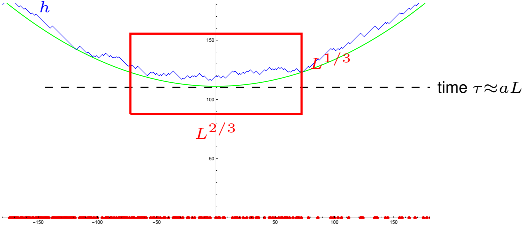


TASEP & Tracy-Widom





$h(\tau, \gamma) =$ height function at time τ and position γ.



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- [Johansson, 1999] For some constants c_1, c_2, c_3, c_4,

$$\lim_{L\to\infty} \mathbb{P}\left(\frac{h(L\tau, c_1 L^{2/3}\gamma) - (c_2 \tau L + c_3 (\tau L)^{2/3})}{c_4 (\tau L)^{1/3}} \leq x \right) = F_2\left(\frac{x}{\tau^{1/3}} + \frac{\gamma^2}{4\tau} \right)$$

where F_2 is the $\beta = 2$ Tracy-Widom distribution.

Tracy-Widom distribution, a different view

Limiting one-point distribution of TASEP with step i.c.:

$$F_{\mathrm{KPZ}}(x,\gamma,\tau) := F_2\left(\frac{x}{\tau^{1/3}} + \frac{\gamma^2}{4\tau^{4/3}}\right),$$

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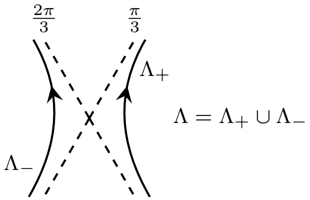

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- [Bertola & Cafasso, 2010's] $F_{\mathrm{KPZ}} = \det\left(\mathbb{I} - \mathbb{F}\right)$, where $\mathbb{F} : L^2(\Lambda) \to L^2(\Lambda)$ acts with kernel

$$\boldsymbol{F}(u,v) := \frac{1}{2\pi i}\frac{\vec{a}(u)^T\vec{b}(v)}{u-v}$$

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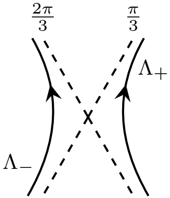

$$\Lambda = \Lambda_+ \cup \Lambda_-$$

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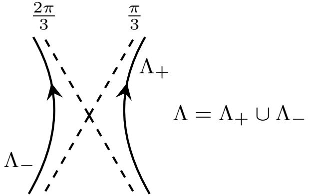
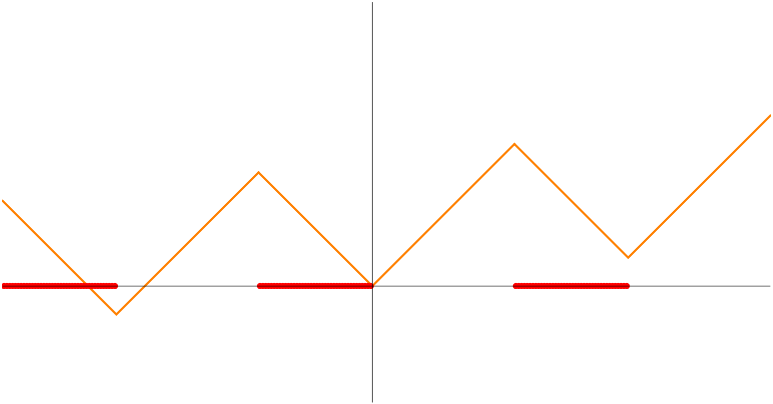


- [Tracy & Widom, 1993; Quastel & Remenik, 2019] $\frac{d^2}{dx^2}F_{\mathrm{KPZ}} := u(x,\gamma,\tau)$ solves the KP equation,

$$12u_{\gamma\gamma}+(12u_\tau+12uu_x+u_{xxx})_x=0.$$

The Periodic TASEP



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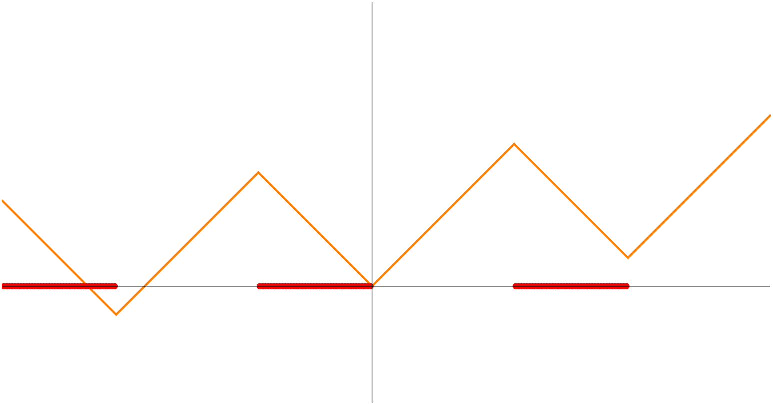

Periodic TASEP - three regimes

Three distinguished regimes when time and period go to ∞

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- For what follows, we assume periodic step IC.

The Periodic TASEP

- [Baik & Liu, 2018; also Prolhac, 2016] In relaxation time scale,

$$\lim_{L\to\infty} \mathbb{P}\left(\frac{h(c_1 s, c_2 t) - c_3 s - c_4 t}{c_5 L^{1/2}} \leq x \right) = F(\tau^{1/3} x; \gamma, \tau)$$

  with

$$F(x;\tau,\gamma) = \oint e^{xA_1(\xi)+\tau A_2(\xi)+B(\xi)} \det(\mathbb{I} - \mathbb{K}_\xi) \frac{d\xi}{2\pi i \xi}$$

  where $\mathbb{K}_\xi = \mathbb{K}_\xi(x,\gamma,\tau)$ is a discrete trace class operator and A_1, A_2, B are explicit parameter-free polylogs.

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- What can we say about $F(x;\gamma,\tau)$?

TASEP and Periodic TASEP (with Baik and Liu, on arxiv)

$$F_{\mathrm{KPZ}}(x;\gamma,\tau) = \det(\mathbb{I} - \mathbb{A}_{-\gamma}\mathbb{A}_{\gamma}), \quad \mathbb{A} : L^2(0,\infty) \to L^2(0,\infty),$$

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$$\mathcal{S}_-:=\{u=e^{-\xi^2/2}\}$$

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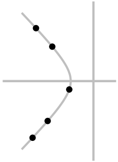

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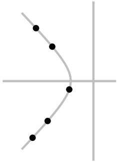

- Spacing between consecutive points $\approx u^{-1}$

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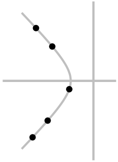

- Spacing between consecutive points $\approx u^{-1}$
- Q is a polylog function (independent of x,γ,τ)

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and

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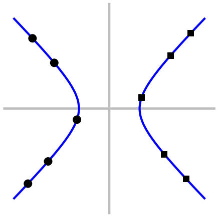

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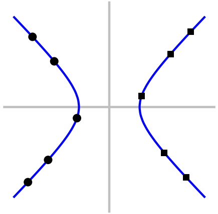

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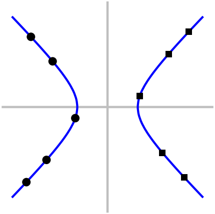

For pTASEP with periodic step i.c.,

$$F(x;\gamma,\tau) = \oint_{|\xi|=\varepsilon} E(\xi;x,\tau)\det(\mathbb{I}-\mathbb{H})\frac{d\xi}{2\pi i\xi},$$

where $\mathbb{H} : \ell^2(\mathcal{S}) \to \ell^2(\mathcal{S})$, with

$$\boldsymbol{H}(u,v) = \frac{\vec{f}(u)^T\vec{g}(v)}{u-v}, \quad u,v\in\mathcal{S},\ u\neq v,$$

and

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IIKS-integrable operators

$$\boldsymbol{H}(u,v)=\frac{\vec{f}(u)^T\vec{g}(v)}{u-v} \text{ on } \ell^2(\mathcal{S}), \quad \boldsymbol{F}(u,v)=\frac{1}{2\pi i}\frac{\vec{a}(u)^T\vec{b}(v)}{u-v} \text{ on } L^2(\Lambda),$$

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For pTASEP the RHP is discrete:

- $X : \mathbb{C} \setminus \mathcal{S} \to \mathbb{C}^{2\times 2}$ is analytic
- X has simple poles at each $w \in \mathbb{C}$, with $\operatorname*{Res}_{w\in\mathcal{S}} X = \lim_{z\to w} X(z) R_X(z)$,, where

$$R_X(z) = \begin{pmatrix} 0 & e^{-V(z)+Q(z)} \frac{1}{z}\chi_{\mathcal{S}_+} \\ e^{V(z)-Q(z)} \frac{1}{z}\chi_{\mathcal{S}_-} & 0 \end{pmatrix}, \quad V(z) := -\frac{\tau}{3}z^3 + \frac{\gamma}{2}z^2 + xz$$

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As usual, $\partial_x \log\det(\mathbb{I} - \mathbb{H}) = (X_1)_{11}$

RHP of interest

With $V(s) = -\frac{\tau}{3}u^3 + \frac{\gamma}{2}u^2 + xu$

- For pTASEP

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with R_0 independent of x, γ, τ

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Theorem (Baik, Liu & S., arxiv)

The functions

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take the form $u = pq$ *where* p, q *satisfy*

- *Coupled mKdV system*

$$\begin{cases} 3p_\tau + p_{xxx} + 6prp_x = 0 \\ 3r_\tau + r_{xxx} + 6rpr_x = 0 \end{cases}$$

- *Coupled heat equations*

$$\begin{cases} p_\gamma + p_{xx} + 2p^2 r = 0 \\ r_\gamma - r_{xx} - 2r^2 p = 0 \end{cases}$$

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- For pTASEP with flat i.c., we connect to KdV equation

Asymptotic results [Baik, Liu & S., arxiv]

The different representations

$$\det(\mathbb{I} - \mathbb{K}_\xi) = \det(\mathbb{I} - \mathbb{H}) = \det(\mathbb{I} - \mathbb{T}_{-\gamma}\mathbb{T}_{\gamma})$$

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- Large tail behavior:

$$\frac{1-F(x;\tau,\gamma)}{1-F_{\mathrm{KPZ}}(x;\tau,\gamma)}=1+\mathcal{O}(e^{-cx^{1/3}}),\quad x\to\infty$$

Time to wrap up

- We connected the limiting one-point distributions to integrable systems
- For multipoint distributions of both TASEP and pTASEP (with step i.c.), we recently generalized these results (work in progress with Baik and Prokhorov)
- Some asymptotic results are obtained, with distinct methods
- Other asymptotic questions may be within reach (in progress)

Thank you!