Painlevé II τ-function as a Fredholm determinant.

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Plan

Malgrange form and τ-function

Malgrange form: For a Riemann Hilbert problem on a contour Σ, depending on a parameter t,

$$\phi_+(z,t) = \phi_-(z,t)M(z,t); \quad \phi(\infty) = \mathbb{1} \tag{1}$$

Malgrange one-form is defined as

$$\omega_{\mathcal{M}} = \int_\Sigma \frac{dz}{2\pi i} \operatorname{Tr}\left[\phi_-^{-1}\phi_-' \delta M M^{-1}\right] \tag{2}$$

where $\delta \equiv \frac{\partial}{\partial t} dt$, $' \equiv \frac{\partial}{\partial z}$, $^{\cdot} \equiv \frac{\partial}{\partial t}$

τ-function:

$$\omega_{\mathcal{M}}(t) = \delta \log \tau(t) \tag{3}$$

For a Riemann Hilbert problem corresponding to an isomonodromic problem, the τ-function is related to the solution $u(t)$ of isomonodromic equation

$$u^2(t) \approx \frac{\partial^2}{\partial t^2} \log \tau[t]. \tag{4}$$

Zeros of the τ-function are the points where the Riemann Hilbert problem is not solvable.

A brief history

- Its, Izergin, Korepin, Slavnov '90 : Correlation function of Bose gas solves certain differential equation and the corresponding τ-function is Fredholm determinant of an integrable kernel.
- Tracy, Widom '93 : Fredholm determinants of integrable kernels solve integrable PDEs.
- Palmer '93: τ-functions can be interpreted as determinants of a singular Cauchy-Riemann operator acting on functions with prescribed monodromy.
- Cafasso '08: The SSW τ-function can be expressed as a Fredholm determinant of a particular combination of Toeplitz operators called the Widom constant.
- Cafasso, Lisovyy, Gavrylenko '17: The isomonodromic τ-function of certain Painlevé equations (VI, V, III) assume the form of Widom constant.

Painlevé equations

- Solutions of the Painlevé equations can be viewed as nonlinear analogues of special functions.

$$\begin{array}{ccccc} P_{\mathrm{VI}} & \longrightarrow & P_{\mathrm{V}} & \longrightarrow & P_{\mathrm{III}} & & \\ & & \downarrow & & \downarrow & & \\ & & P_{\mathrm{IV}} & \longrightarrow & P_{\mathrm{II}} & \longrightarrow & P_{\mathrm{I}}\,, \end{array}$$

Figure: Coalescence diagram for Painlevé equations

$$\begin{array}{ccccc} \text{Gauss} & \longrightarrow & \text{Kummer} & \longrightarrow & \text{Bessel} \\ & & \downarrow & & \downarrow \\ & & \text{Hermite-Weber} & \longrightarrow & \text{Airy}\,. \end{array}$$

- The Riemann-Hilbert problems of Painlevé equations are such that the local parametrices are described by special functions.

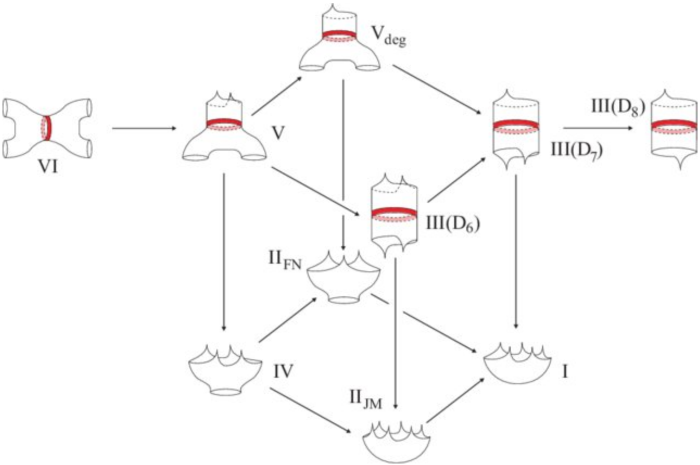
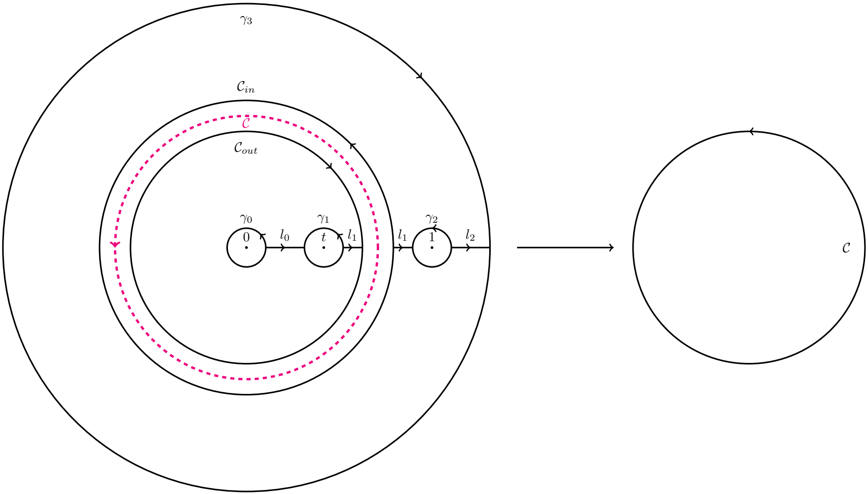


Figure: Confluence diagram for Painlevé equations[1]

[1] *Ref: Gavrylenko, P. and Lisovyy, O., 2018. Fredholm determinant and Nekrasov sum representations of isomonodromic tau functions. CMP, 363(1), pp.1-58.*

Painlevé VI



γ_3
$\mathcal{C}_{in}$
$\mathcal{C}$
$\mathcal{C}_{out}$
γ_0
γ_1
γ_2
0
t
1
l_0
l_1
l_1
l_2
$\mathcal{C}$

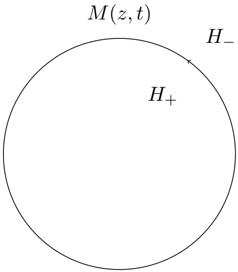
Widom constant

Consider a Riemann Hilbert problem defined on a unit circle.

$$\phi_+(z,t) = \phi_-(z,t)M(z,t); \quad \phi(\infty) = \mathbb{1} \tag{5}$$

$M(z,t)$ can be factorized in two different ways

$$M(z,t) = \phi_-^{-1}\phi_+ = \psi_+^{-1}\psi_- \tag{6}$$



- $L^2(S^1) = H_+ \oplus H_-$
- Define projection (Cauchy) operators $\Pi_\pm : L^2(S^1) \to H_\pm$
- Toeplitz operator is defined as $T_M = \Pi_+ M$

$M(z,t)$ is a matrix valued 'symbol' and the Widom constant is

$$\tau_W[t] = \det_{H_+}[T_M \circ T_{M^{-1}}] \tag{7}$$

- The zeros of T_M correspond to unsolvability of the RHP and the zeros of $T_{M^{-1}}$ correspond to the unsolvability of the dual RHP.
- Logarithmic derivatives of $\tau_W(t)$, Malgrange $\tau(t)$ coincide up to explicit terms

$$\partial_t \log \tau_W[t] = \partial_t \log \tau[t] + \text{explicit terms} \tag{8}$$

Can a generic τ-function of Painlevé II equation be expressed as a Fredholm determinant?

Plan

What is known?

$$\text{Painlevé II:} \quad u_{xx} = 2u^3 + xu \tag{9}$$

- Ablowitz-Segur family of solutions:

$$u(x) \approx \kappa Ai(x); \quad x \to +\infty, \quad \kappa \in \mathbb{C} \tag{10}$$

- Tracy, Widom '99: For the Ablowitz-Segur solutions,

$$u^2(x) = -\frac{\partial^2}{\partial x^2} \log \underbrace{\det\left[1 - \kappa K_{Ai}|_{[x,\infty)}\right]}_{\tau(x)} \tag{11}$$

Relation to the Widom constant?

⇝ The Ablowitz-Segur τ-function can be expressed as Widom constant. Further, we can obtain a minor expansion of the Airy kernel.

RHP

$\Psi(\lambda)$ is piecewise holomorphic 2×2 matrix valued function such that

- $\Psi(\lambda)$ is holomorphic for $\lambda \in \mathbb{C} \ \cup \{\gamma_k\}$
- Boundary conditions on each Stokes' ray are

$$\Psi_+(\lambda) = \Psi_-(\lambda) S_k, \ \lambda \in \gamma_k \tag{12}$$

  Stokes' data satisfies the constraint
  $s_{k+3} = -s_k \ , \ s_1 - s_2 + s_3 + s_1 s_2 s_3 = 0$

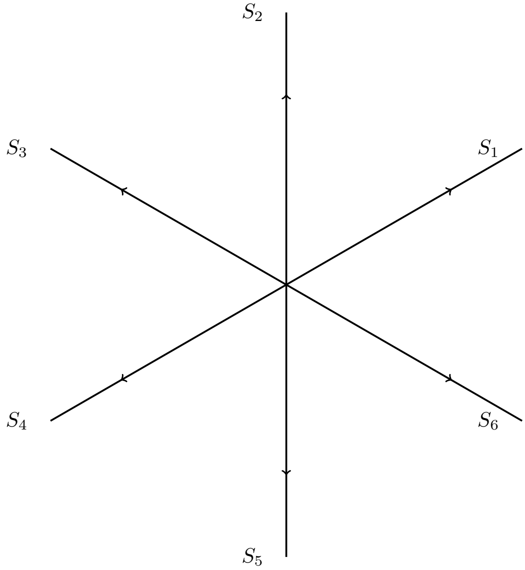
- Asymptotic behaviour is specified by

$$\Psi(\lambda) e^{\theta(\lambda,x)\sigma_3} \to I \ ; \ \theta(\lambda, x) = i\left(\frac{4}{3}\lambda^3 + x\lambda\right), \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{13}$$

  Changing the coordinates $\lambda = (-x)^{1/2} z, \quad t = (-x)^{3/2}$ and defining the parameter $\nu = \frac{1}{2\pi} \log(1 - s_1 s_3)$

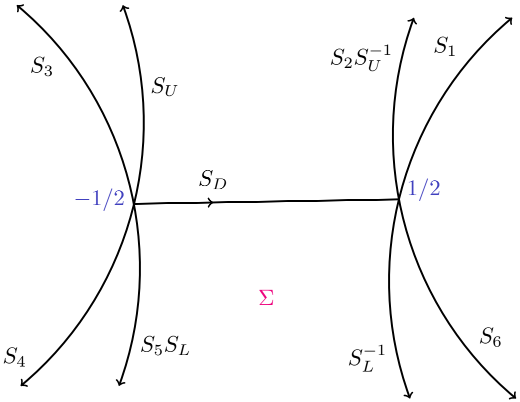
Disclaimer: All functions depend on z, t unless mentioned otherwise and all the solutions of RHPs are normalised at ∞.

Contour of the RHP



[1] Painlevé transcendents: the Riemann-Hilbert approach (No. 128). AMS

Contour of the RHP



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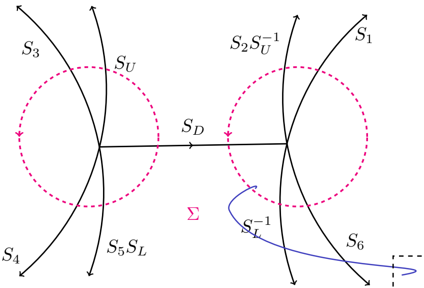
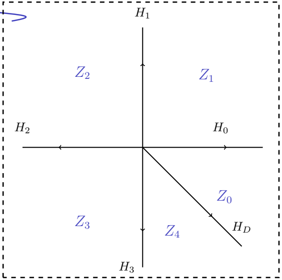
Contour of the RHP



$$\zeta(z) = 2t^{1/2}\sqrt{\frac{-4i}{3}z^3 + iz - \frac{i}{3}}$$



RHP of Parabolic Cylinder function

[1] Painlevé transcendents: the Riemann-Hilbert approach (No. 128). AMS

Parabolic cylinder

$Z^{RH}(\zeta) = Z_i(\zeta);\ i = 0, ..., 4$ solve the following Riemann Hilbert problem.

- The following jump conditions are valid

$$Z_+^{RH}(\zeta) = Z_-^{RH}(\zeta) H_k,\ \arg\zeta = \frac{\pi}{2}k,\ \ k = 0, 1, 2, 3 \tag{14}$$

$$Z_+^{RH}(\zeta) = Z_-^{RH}(\zeta) e^{2\pi i \nu \sigma_3},\ \arg\zeta = -\frac{\pi}{4} \tag{15}$$

Wronskian of the parabolic cylinder functions $D_\nu(\zeta)$ and $D_{-\nu-1}(i\zeta)$ is the solution of the RHP in one sector

$$Z_0(\zeta) = 2^{-\sigma_3/2} \begin{pmatrix} D_{-\nu-1}(i\zeta) & D_\nu(\zeta) \\ \frac{d}{d\zeta} D_{-\nu-1}(i\zeta) & \frac{d}{d\zeta} D_\nu(\zeta) \end{pmatrix} \begin{pmatrix} e^{i\frac{\pi}{2}(\nu+1)} & 0 \\ 0 & 1 \end{pmatrix} \tag{16}$$

explicit form of the jump functions H_k

$$H_{k+2} = e^{i\pi(\nu+\frac{1}{2})\sigma_3} H_k e^{-i\pi(\nu+\frac{1}{2})\sigma_3},\ H_0 = \begin{pmatrix} 1 & 0 \\ h_0 & 1 \end{pmatrix},\ H_1 = \begin{pmatrix} 1 & h_1 \\ 0 & 1 \end{pmatrix} \tag{17}$$

and the parameters h_0 and h_1 are dependent on ν

$$h_0 = -i\frac{\sqrt{2\pi}}{\Gamma(\nu+1)},\ h_1 = \frac{\sqrt{2\pi}}{\Gamma(-\nu)} e^{i\pi\nu},\ 1 + h_0 h_1 = e^{2\pi i\nu} \tag{18}$$

In terms of Z^{RH}, the local solution of the right parametrix is given by

$$\Phi_R = e^{t\theta(z)\sigma_3}\left(\zeta(z)\frac{z-1/2}{z+1/2}\right)^{\nu\sigma_3}\left(\frac{-h_1}{s_3}\right)^{-\sigma_3/2} e^{\frac{it}{3}\sigma_3}2^{-\sigma_3/2}\begin{pmatrix}\zeta(z) & 1\\ 1 & 0\end{pmatrix} \times Z^{RH}(\zeta(z))\left(\frac{-h_1}{s_3}\right)^{\sigma_3/2} \quad (19)$$

$\Psi(z)$ is the global solution of Painlevé II RHP on Σ.

$$\Psi_+ = \Psi_- G \quad (20)$$

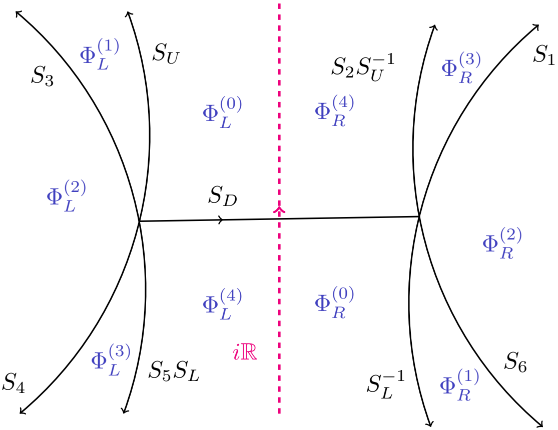
Define the functions $\mathcal{R}$, $\mathcal{L}$ in terms of the solutions of the local parametrices and Ψ.

$$\mathcal{R}(z,t) = \Psi(z,t){\Phi_R^{(0)}}^{-1}; \quad \mathcal{L}(z,t) = \Psi(z,t){\Phi_L^{(4)}}^{-1} \quad (21)$$

$\mathcal{R}$,$\mathcal{L}$ have a jump only on $i\mathbb{R}$

$$\mathcal{R} = \mathcal{L}\, J \quad (22)$$

The transformation $\zeta(z) = 2t^{1/2}\sqrt{\frac{-4i}{3}z^3 + iz - \frac{i}{3}}$ induces additional stationary points at ± 1. So, defining the dual RHP on $i\mathbb{R}$ is not straightforward. However, we do have a way to construct an integrable kernel on the line contour!



$\Phi_L^{(1)}$
S_U
S_3
$S_2 S_U^{-1}$
$\Phi_R^{(3)}$
S_1
$\Phi_L^{(0)}$
$\Phi_R^{(4)}$
$\Phi_L^{(2)}$
S_D
$\Phi_R^{(2)}$
$\Phi_L^{(4)}$
$\Phi_R^{(0)}$
$i\mathbb{R}$
$\Phi_L^{(3)}$
$S_5 S_L$
S_4
S_L^{-1}
$\Phi_R^{(1)}$
S_6

Theorem 1 (H.D, 2020)

The τ-function of Painlevé II equation can be expressed in terms of a Fredholm determinant of an integrable operator $\mathcal{K}$

$$\partial_t \log \tau_{PII} = \partial_t \log \det \left[\mathbb{1}_{L^2(l_1 \cup l_3)} - \mathcal{K} \right] - \left[\frac{4i\nu}{3} + \frac{2\nu^2}{t} \right] + \mathcal{F}(t, \nu, h), \qquad (23)$$

Its-Izergin-Korepin-Slavnov (IIKS) kernel

Theorem 2 (IIKS)

Given a RHP of the form

$$Y_+ = Y_- J \tag{24}$$

where the jump assumes the form $J = 1 - 2\pi i f(z) g^T(z)$; a Kernel

$$K(z, w) = \frac{f^T(z) g(w)}{z - w} \tag{25}$$

can be constructed such that the RHP is solvable iff $(1 - K)$ is invertible.

The jump on $i\mathbb{R}$ is

$$J = \Phi_L^{(4)} \Phi_R^{(0)^{-1}}. \tag{26}$$

Now the task reduces to constructing the integrable kernel.

Jump on $i\mathbb{R}$

The jump

$$J = \Phi_L^{(4)} {\Phi_R^{(0)}}^{-1} = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix}, \tag{27}$$

where

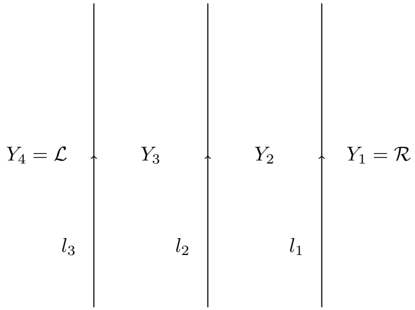
$$\begin{aligned}
\mathcal{A}(z,t) &= \zeta^{\nu}\xi^{\nu} e^{\frac{2i}{3}t} \left(e^{-\pi i\nu} D_{-\nu}(i\zeta) D_{-\nu}(i\xi) + \nu^2 h^{-4} e^{2\pi i\nu} D_{\nu-1}(\zeta) D_{\nu-1}(\xi)\right) \\
\mathcal{B}(z,t) &= \left(\frac{z-z_-}{z-z_+}\right)^{2\nu} \zeta^{\nu}\xi^{-\nu} \left(ih^2 e^{-i\pi\nu} D_{-\nu}(i\zeta) D_{-\nu-1}(i\xi) + \nu h^{-2} e^{2\pi i\nu} D_{\nu-1}(\zeta) D_{\nu}(\xi)\right) \\
\mathcal{C}(z,t) &= \left(\frac{z-z_-}{z-z_+}\right)^{-2\nu} \zeta^{-\nu}\xi^{\nu} \left(ih^2 e^{-i\pi\nu} D_{-\nu-1}(i\zeta) D_{-\nu}(i\xi) + \nu h^{-2} e^{2\pi i\nu} D_{\nu}(\zeta) D_{\nu-1}(\xi)\right) \\
\mathcal{D}(z,t) &= \zeta^{-\nu}\xi^{-\nu} e^{-\frac{2i}{3}t} \left(-e^{-\pi i\nu} h^4 D_{-\nu-1}(i\zeta) D_{-\nu-1}(i\xi) + e^{2\pi i\nu} D_{\nu}(\zeta) D_{\nu}(\xi)\right)
\end{aligned} . \tag{28}$$

LDU decomposition

- One can decompose the jump function as LDU

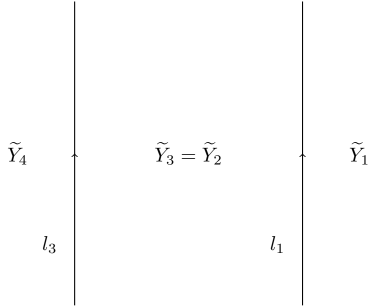
$$J = \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{\mathcal{C}}{\mathcal{A}} & 1 \end{bmatrix} \begin{bmatrix} \mathcal{A} & 0 \\ 0 & \frac{1}{\mathcal{A}} \end{bmatrix} \begin{bmatrix} 1 & \frac{\mathcal{B}}{\mathcal{A}} \\ 0 & 1 \end{bmatrix} = \prod_{i=1}^{3} F_i. \quad (29)$$

- Similar to the previous case, a RHP can be defined on a set of three parallel lines.



LDU decomposition

- Noticing that the RHP with the diagonal jump on l_2 can be solved locally, the RHP on LDU can be transformed on to two parallel lines with lower and upper triangular jumps.
- Let $\varphi(z,t)^{\sigma_3}$ solve the RHP on l_2. Then, $\tilde{Y} = Y\varphi^{-1}$ has jumps on $l_1 \cup l_3$.



RHP on $l_1 \cup l_3$ reads

$$\widetilde{Y}_+ = \widetilde{Y}_- \widetilde{F} \tag{30}$$

with the jumps

$$\widetilde{F} = \begin{cases} \widetilde{F}_1 = \begin{pmatrix} 1 & 0 \\ \frac{\mathcal{C}}{\mathcal{A}}\varphi^2 & 1 \end{pmatrix} ; \text{on } l_1 \\ \widetilde{F}_3 = \begin{pmatrix} 1 & \frac{\mathcal{B}}{\mathcal{A}}\varphi^{-2} \\ 0 & 1 \end{pmatrix} ; \text{on } l_3 \end{cases} . \tag{31}$$

and

$$\varphi(z) = \exp\left[\int_{i\mathbb{R}} \frac{\log \mathcal{A}(z', t)}{z' - z} \frac{dz'}{2\pi i}\right] \tag{32}$$

Defining the chacterstic functions $\chi_1(z)$, $\chi_3(z)$ on the contours l_1, l_3 respectively, the jumps can be expressed in integrable form. Define the functions

$$f(z,t) = \frac{1}{2\pi i}\begin{pmatrix} \chi_3(z) \\ \\ \chi_1(z) \end{pmatrix} \quad ; \quad g(z,t) = \begin{pmatrix} \frac{\mathcal{C}}{\mathcal{A}}\varphi^2\chi_1(z) \\ \\ \frac{\mathcal{B}}{\mathcal{A}}\varphi^{-2}\chi_3(z) \end{pmatrix}. \tag{33}$$

$\widetilde{F}$ can be written as

$$\widetilde{F} = \mathbb{1} - 2\pi i f(z) g^T(z). \tag{34}$$

and one can verify that $f^T(z)g(z) = 0$. The integrable kernel is then

$$K(z,w) = \frac{f^T(z)g(w)}{z-w}$$

$$= \frac{1}{z-w}\begin{pmatrix} \chi_1(z) & \chi_3(z) \end{pmatrix}\begin{pmatrix} 0 & \frac{\mathcal{B}(w,t)}{\mathcal{A}(w,t)}\varphi^{-2}(w,t) \\ \frac{\mathcal{C}(w,t)}{\mathcal{A}(w,t)}\varphi^2(w,t) & 0 \end{pmatrix}\begin{pmatrix} \chi_1(w) \\ \chi_3(w) \end{pmatrix}$$

and the corresponding τ-function is

$$\tau_{LU} = \det\left[I - \mathcal{K}\right]. \tag{35}$$

Relating the Malgrange forms

$$\partial_t \log \tau_{PII} = \partial_t \log \tau_{i\mathbb{R}} - \int_{i\mathbb{R}} \mathrm{Tr}\left[\Phi_R' \Phi_R^{-1} \Delta\left(\dot{\Phi}\Phi^{-1}\right)\right] - \left[\frac{4i\nu}{3} + \frac{2\nu^2}{t}\right]$$

$$= \partial_t \log \tau_{LDU} - \int_{i\mathbb{R}} \frac{dz}{2\pi i}\left(\frac{\dot{\mathcal{B}}}{\mathcal{A}}\right)\left(\mathcal{A}\mathcal{C}' - \mathcal{A}'\mathcal{C}\right) - \int_{i\mathbb{R}} \mathrm{Tr}\left[\Phi_R' \Phi_R^{-1} \Delta\left(\dot{\Phi}\Phi^{-1}\right)\right] - \left[\frac{4i\nu}{3} + \frac{2\nu^2}{t}\right]$$

$$= \partial_t \log \tau_{LU} + 2\int_{i\mathbb{R}} \frac{dz}{2\pi i}\frac{\dot{\mathcal{A}}(z,t)}{\mathcal{A}(z,t)}\int_{i\mathbb{R}_-}\frac{dw}{2\pi i}\frac{\mathcal{A}'(w,t)}{\mathcal{A}(w,t)(z-w)} + \int_{i\mathbb{R}} \frac{dz}{2\pi i}\left(\frac{\dot{\mathcal{B}}}{\mathcal{A}}\right)\left(\mathcal{A}\mathcal{C}' - \mathcal{A}'\mathcal{C}\right)$$

$$- \int_{i\mathbb{R}} \mathrm{Tr}\left[\Phi_R' \Phi_R^{-1} \Delta\left(\dot{\Phi}\Phi^{-1}\right)\right] - \left[\frac{4i\nu}{3} + \frac{2\nu^2}{t}\right]$$

$$= \partial_t \log \det\left[1 - \mathcal{K}\right] + 2\int_{i\mathbb{R}} \frac{dz}{2\pi i}\frac{\dot{\mathcal{A}}(z,t)}{\mathcal{A}(z,t)}\int_{i\mathbb{R}_-}\frac{dw}{2\pi i}\frac{\mathcal{A}'(w,t)}{\mathcal{A}(w,t)(z-w)}$$

$$+ \int_{i\mathbb{R}} \frac{dz}{2\pi i}\left(\frac{\dot{\mathcal{B}}}{\mathcal{A}}\right)\left(\mathcal{A}\mathcal{C}' - \mathcal{A}'\mathcal{C}\right)$$

$$- \int_{i\mathbb{R}} \mathrm{Tr}\left[\Phi_R' \Phi_R^{-1} \Delta\left(\dot{\Phi}\Phi^{-1}\right)\right] - \left[\frac{4i\nu}{3} + \frac{2\nu^2}{t}\right]$$

$$= \partial_t \log \det\left[\mathbb{1}_{L^2(l_1 \cup l_3)} - \mathcal{K}\right] + \mathcal{F}(t,\nu,h) - \left[\frac{4i\nu}{3} + \frac{2\nu^2}{t}\right] \tag{36}$$

Plan

Introduction

Painlevé II

Outlook

What's next?

- **Minor expansion:** After appropriate transformations, we expect to transform the kernel on to the imaginary axis, thereby obtaining a minor expansion as in the case of the Airy kernel.
- **Painlevé I and IV:** Painlevé I should have a similar structure with the local parametrices defined by Airy function and Parabolic cylinder functions.
- **Other integrable equations:** The τ-function of modified Korteweg de Vries (mKdV) equation can be written as a Fredholm determinant.

References

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