# Multiplicative statistics of the Airy process and the Korteweg-de Vries equation

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joint work with
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# Point processes in general

Let $R$ be a complete separable metric space.

Let $\mathrm{Conf}(R)$ be the set of locally finite configuration of points $\{r_1, r_2 \dots\} \subseteq R$:

$$\#\{r_i \in K\} < \infty \text{ for all compact } K \subseteq R.$$

For any $B \subseteq R$ bounded Borel set and any $n \geq 0$, let $C_{B,n}$ the set of configurations $\{r_1, r_2 \dots\} \subseteq R$ such that $\#\{r_i \in B\} = n$.

Let $\mathfrak{S}$ be the $\sigma$-algebra of subsets of $\mathrm{Conf}(R)$ generated by the $C_{B,n}$'s.

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**DEFINITION.** *A* **point process on $R$** *is a probability measure on* $(\mathrm{Conf}(R), \mathfrak{S})$.

# Correlation functions

For any $B \subseteq R$ bounded Borel, let

$$\begin{array}{rccc} \#_B : & \mathrm{Conf}(R) & \to & \mathbb{Z}_{\geq 0} \\ & \{r_1, r_2, \dots\} & \mapsto & \#\{r_i \in B\}. \end{array}$$

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From now on, fix a reference measure $\mathrm{d}r$ on $R$.

**DEFINITION.** *Given a point process on $R$ and let $k \geq 1$. If there exists $\rho_k \in L^1_{\mathsf{loc}}(R^k, \mathrm{d}r^{\otimes k})$, $\rho_k > 0$ are such that:*

- *for all $B_1, \dots, B_m \subseteq R$ disjoint bounded Borel, and*
- *for all integers $k_1, \dots, k_m \geq 1$ with $k_1 + \dots + k_m = k$,*

$$\mathbb{E}\left[\frac{\#_{B_1}!}{(\#_{B_1} - k_1)!} \cdots \frac{\#_{B_m}!}{(\#_{B_m} - k_m)!}\right] = \int_{B_1^{k_1} \times \dots \times B_m^{k_m}} \rho_k(r_1, ..., r_k)\, \mathrm{d}r_1 \dots \mathrm{d}r_k,$$

*then $\rho_k$ is called* **k-point correlation function**.

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Under mild analytic assumptions, the correlation functions $\{\rho_k\}_{k \geq 1}$ uniquely define a point process [Lenard (1970s)].

# Determinantal point processes

A point process is called **determinantal** if it has correlation functions

$$\rho_k(r_1, \ldots, r_k) = \det \left[K(r_i, r_j)\right]_{i,j=1}^{k}$$

for some **correlation kernel** $K(r_1, r_2)$.

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**THEOREM.** *Let $K(r_1, r_2)$ be the correlation kernel of a determinantal point process on $R$. If $\varphi : R \to \mathbb{R}$ is such that the formula*

$$\mathcal{K}_\varphi : f(r) \mapsto \int_R K(r, r')\varphi(r')f(r')\mathrm{d}r'$$

*defines a trace-class operator, i.e. $\int_R |\varphi(r)K(r,r)|\mathrm{d}r < +\infty$, then*

$$\mathbb{E}\left[\prod_{j\geq 1}(1-\varphi(r_j))\right] = \sum_{k\geq 0}\frac{(-1)^k}{k!}\int_{R^k}\det\left[K(r_i, r_j)\varphi(r_j)\right]_{i,j=1}^k \mathrm{d}r_1 \cdots \mathrm{d}r_k$$

$$= \sum_{k\geq 0}(-1)^k \operatorname{tr}\left(\mathcal{K}_\varphi^{\wedge k}\right) =: \det_{L^2(R)}(I - \mathcal{K}_\varphi).$$

# Airy process

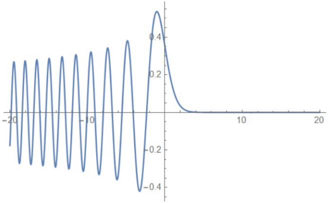
Recall the **Airy function**

$$\mathrm{Ai}(r) := \frac{1}{\pi}\int_0^{+\infty} \cos\left(\frac{s^3}{3} + rs\right) \mathrm{d}s$$

$$\mathrm{Ai}''(r) = r\mathrm{Ai}(r)$$

$$\mathrm{Ai}(r) = \frac{\exp\left(-\frac{2}{3}r^{3/2}\right)}{2\sqrt{\pi}r^{1/4}}\left(1+\mathcal{O}\left(r^{-3/2}\right)\right), \qquad r\to+\infty$$

$$\mathrm{Ai}(r) = \frac{\sin\left(\frac{2}{3}|r|^{3/2}+\frac{\pi}{4}\right)}{\sqrt{\pi}|r|^{1/4}}\left(1+\mathcal{O}\left(r^{-3/2}\right)\right), \quad r\to-\infty$$



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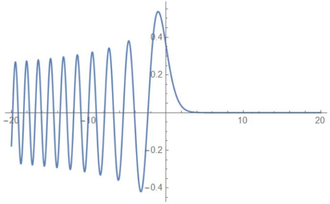


The **Airy process** is the determinantal point process on $\mathbb{R}$ with kernel

$$K^{\mathrm{Ai}}(r_1, r_2) = \frac{\mathrm{Ai}(r_1)\mathrm{Ai}'(r_2) - \mathrm{Ai}(r_2)\mathrm{Ai}'(r_1)}{r_1 - r_2}$$

The Airy process characterizes in various instances the generic **universal edge behavior of eigenvalues in random matrices**

# Tracy-Widom distribution

In the general Theorem on determinantal point processes, take $K = K^{\mathrm{Ai}}$ and $\varphi_s(r) = \gamma\chi_{[s,+\infty)}(r)$ with $\gamma \in [0,1]$. *(Note $\int_s^{+\infty} |K^{\mathrm{Ai}}(r,r)|\mathrm{d}r < +\infty$ for all $s$)*

$$F_{\mathrm{TW}}(s;\gamma) := \mathbb{E}\left[\prod_{j\geq 1}(1-\gamma\chi_{[s,+\infty)}(r_j))\right] = \det{}_{L^2(s,+\infty)}(I-\gamma\mathcal{K}^{\mathrm{Ai}})$$

where $\mathcal{K}^{\mathrm{Ai}}$ is the operator with kernel $K^{\mathrm{Ai}}$.

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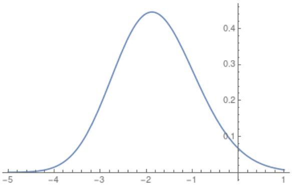
**THEOREM [C. Tracy and H. Widom (1993)].**

$$F_{\mathrm{TW}}(s;\gamma) = \exp\left(-\int_s^{+\infty}(s'-s)y_\gamma^2(s')\mathrm{d}s'\right)$$

*where $y_\gamma(s)$ is the Ablowitz-Segur PII solution:*

$$\begin{cases} y_\gamma''(s) = s y_\gamma(s) + 2y_\gamma^3(s), \\ y_\gamma(s) \sim \sqrt{\gamma}\mathrm{Ai}(s), \qquad s \to +\infty. \end{cases}$$




The Tracy-Widom distribution $F_{\mathrm{TW}}$ has then been recognized as a **universal** distribution appearing in the study of many strongly correlated systems.

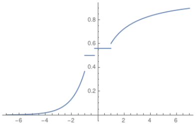
# Airy multiplicative statistics and KdV

Now take $K = K^{\mathrm{Ai}}$ and $\varphi_{x,t}(r) = \sigma(t^{-2/3}(r + xt^{-1/3}))$ with:

- $\sigma = \sigma(z)$ in $\mathscr{C}^{\infty}(\mathbb{R} \setminus \{z_1, \dots, z_k\})$, for some points $z_1 < \cdots < z_k$,
- $\sigma^{(n)}(z_j \pm)$ exist for all $n \geq 0$ and $j = 1, \dots, k$,
- $z < z' \Rightarrow \sigma(z) \leq \sigma(z')$, $\sigma(-\infty) = 0$, $\sigma(+\infty) = \gamma \in [0, 1]$,
- $\int_{\mathbb{R}} \varphi_{x,t}(r) K^{\mathrm{Ai}}(r, r) \mathrm{d}r < +\infty$. *(E.g. $\sigma(z) = \mathcal{O}(|z|^{-3/2-\delta}),\ z \to -\infty,\ \delta > 0$.)*

Introduce

$$Q_\sigma(x,t) := \det_{L^2(\mathbb{R})}\left(I - \mathcal{K}^{\mathrm{Ai}}_{\varphi_{x,t}}\right),$$
$$u_\sigma(x,t) := \partial_x^2 \log Q_\sigma(x,t) + \frac{x}{2t}.$$



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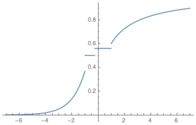


**THEOREM** (Cafasso-Claeys-R, 2020). *For all* $x \in \mathbb{R}, t > 0$, *the function* $u_\sigma = u_\sigma(x,t)$ *satisfies the KdV equation;*

$$\partial_t u_\sigma + 2u_\sigma \partial_x u_\sigma + \frac{1}{6}\partial_x^3 u_\sigma = 0.$$

See also Quastel-Remenik: *KP governs random growth of a one dimensional substrate*

# KdV equation

The KdV equation was originally introduced by **Boussinesq** in 1877, and considered again by **Korteweg** and **de Vries** shortly after, in 1895.

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In the 1960s it has been recognized as a prototypical example of **integrable PDE**.
It then appeared in the most disparate branches of Mathematical Physics: beyond the original setting of fluid dynamics, let us mention the *Fermi-Pasta-Ulam-Tsingou problem*, *quantum gravity*, *enumerative geometry*, and now *integrable probability*.

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Techniques of **direct/inverse scattering** (based on the Lax pair representation) allow to study the initial value ($t = 0$) problem for a broad class of KdV solutions.
The class of KdV solutions introduced here **exit the realm of this theory**.
In particular, the initial datum blows up at $t = 0$ for $x < 0$: in this case we are closer to results of Its and Sukhanov for the **cylindrical KdV** equation.

# Lax pair representation

**THEOREM [P. D. Lax (1968)].** *If the function* $\psi(x,t;z)$ *("wave function") satisfies the Schrödinger equation*

$$\mathcal{L}\psi(x,t;z) = z\psi(x,t;z), \qquad \mathcal{L} = \partial_x^2 + 2u(x,t),$$

*and simultaneously*

$$\partial_t\psi(x,t;z) = \mathcal{M}\psi(x,t;z), \qquad \mathcal{M} = -\frac{2}{3}\partial_x^3 - 2u(x,t)\partial_x - \partial_x u(x,t),$$

*then the potential* $u(x,t)$ *of the Schroedinger equation satisfies the KdV equation*

$$\boxed{\partial_t u(x,t) + 2u(x,t)\partial_x u(x,t) + \frac{1}{6}\partial_x^3 u(x,t) = 0.}$$

# Integro-differential Painlevé II

**THEOREM** (Cafasso-Claeys-R, 2020). *The wave function* $\psi_\sigma(x,t;z)$ *associated with the KdV solution* $u_\sigma(x,t) = \partial_x^2 \log Q_\sigma(x,t) + \frac{x}{2t}$ *satisfies*

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*and has the asymptotic behavior*

$$\psi_\sigma(x,t;z) \sim \frac{\mathrm{e}^{-\frac{2}{3}tz^{3/2}+xz^{1/2}}}{2\sqrt{\pi}z^{1/4}},$$

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*as* $z \to \infty$ *with* $|\arg z| < \pi - \delta$, *for any* $\delta > 0$.

Substituting in the Schrödinger equation we quickly get Amir-Corwin-Quastel's **integro-differential Painlevé II equation**;

$$\boxed{\partial_x^2\psi_\sigma(x,t;z) = \left(z - \frac{x}{t} + \frac{2}{t}\int_{\mathbb{R}} \psi_\sigma^2(x,t;z')\mathrm{d}\sigma(z')\right)\psi_\sigma(x,t;z).}$$

## Step function $\sigma$: Painlevé II and Tracy-Widom

When $\sigma(z) = \gamma\chi_{[0,+\infty)}(z)$ we have

$$\varphi_{x,t}(r) = \sigma(t^{-2/3}(r + xt^{-1/3})) = \gamma\chi_{[-xt^{-1/3},+\infty)}(r).$$

By definition $Q_\sigma(x,t) = F_{\mathrm{TW}}(-xt^{-1/3};\gamma)$, so by the Tracy-Widom theorem

$$u_\sigma(x,t) = \partial_x^2 F_{\mathrm{TW}}(-xt^{-1/3};\gamma) + \frac{x}{2t} = \frac{x}{2t} - t^{-2/3} y_\gamma^2\left(-xt^{-1/3}\right)$$

and $u_\sigma(x,t)$ is a **self-similar KdV solution**.

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Indeed, the integro-differential PII reduces for $\sigma(z) = \gamma\chi_{[0,+\infty)}(z)$ to

$$\partial_x^2\psi_\sigma(x,t;z) = \left(z - \frac{x}{t} + \frac{2\gamma}{t}\psi_\sigma^2(x,t;0)\right)\psi_\sigma(x,t;z).$$

which implies (assuming $\gamma \neq 0$) the **PII equation**

$$y''(s) = sy(s) + 2y^3(s)$$

where $s = -xt^{-1/3}$, $y(s) = \gamma^{1/2}t^{-1/6}\psi_\sigma(-st^{1/3},t;0)$.

# Piecewise constant $\sigma$: coupled Painlevé II system

When $\sigma(z) = \sum_{\ell=1}^{k} \mu_\ell \chi_{[z_\ell,+\infty)}(z)$ ($\mu_\ell > 0$, $\gamma = \mu_1 + \cdots + \mu_k \in [0,1]$) the integro-differential PII implies

$$\begin{cases} \partial_s^2 y_1(s,t) = (s + z_1 t^{2/3}) y_1(s,t) + 2 y_1(s,t) \sum_{\ell=1}^{k} y_\ell^2(s,t), \\ \qquad \vdots \\ \partial_s^2 y_k(s,t) = (s + z_k t^{2/3}) y_k(s,t) + 2 y_k(s,t) \sum_{\ell=1}^{k} y_\ell^2(s,t), \end{cases}$$

for the dependent variables $y_1(s,t), \dots, y_k(s,t)$, where

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This system was first derived by Claeys and Doeraene (2017).

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The KdV solution is

$$u_\sigma(x,t) = \frac{x}{2t} - t^{-2/3} \sum_{\ell=1}^{k} y_\ell^2(-xt^{-1/3}, t).$$

# Logistic $\sigma$: narrow wedge solution to the KPZ equation

Another interesting case follows from the remarkable identity (Amir, Corwin, and Quastel (2011) - Borodin and Gorin (2016))

$$\mathbb{E}_{\mathrm{KPZ}}\left[\mathrm{e}^{-\mathrm{e}^{T^{1/3}(\Upsilon(T)-s)}}\right] = \mathbb{E}_{\mathrm{Ai}}\left[\prod_{j\geq 1}\frac{1}{1+\mathrm{e}^{T^{1/3}\left(r_j-s\right)}}\right].$$

It expresses the **narrow wedge solution** to the **Kardar-Parisi-Zhang** stochastic PDE

$$\Upsilon(T) = \frac{\mathcal{H}(2T,0)+\frac{T}{12}}{T^{1/3}}, \qquad \begin{cases}\partial_T\mathcal{H}(T,X) = \frac{1}{2}\partial_X^2\mathcal{H}(T,X) + \frac{1}{2}\left(\partial_X\mathcal{H}(T,X)\right)^2 + \xi(T,X),\\ \mathcal{H}(0,X) = \log\delta_{X=0}, \qquad (\xi = \textit{space-time white noise})\end{cases}$$

in terms of a multiplicative Airy statistics.

# LOGISTIC $\sigma$: NARROW WEDGE SOLUTION TO THE KPZ EQUATION

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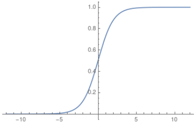
The right side of this identity coincides with the Fredholm determinant $Q_\sigma(x,t)$, provided

$$\sigma(z) = \frac{\mathrm{e}^z}{1+\mathrm{e}^z},\ s = -xt^{-1/3},\ T = t^{-2}.$$



# Small time asymptotics for $u_\sigma(x,t)$

Decay properties: $\begin{cases} \left|\sigma(z) - \gamma\chi_{[0,+\infty)(z)}\right| \le c_1 e^{-c_2|z|} & z \in \mathbb{R}, \\ |\sigma'(z)| \le c_3 |z|^{-2} & |z| > C. \end{cases}$

**THEOREM** (Cafasso-Claeys-R, 2020). *Assuming $\sigma$ satisfies the decay properties we have the following uniform estimates.*

- $\forall t_0 > 0\ \exists M, c > 0$ *such that for* $x < -Mt^{1/3}$, $0 < t < t_0$

$$u_\sigma(x,t) = x/(2t) + \mathcal{O}(\exp(-c|x|t^{-1/3})).$$

- $\exists \epsilon > 0$ *such that* $\forall M > 0$ *and for* $|x| \le Mt^{1/3}$, $0 < t < \epsilon$*:*

$$u_\sigma(x,t) = x/(2t) - t^{-2/3} y_\gamma^2(-xt^{-1/3}) + \mathcal{O}(1).$$

- *If* $\gamma = 1$*:* $\exists v_\sigma(x)$ *such that* $\exists \epsilon, M > 0$ *such that* $\forall K > 0$ *and for* $Mt^{1/3} < x < K$, $0 < t < \epsilon$ *that*

$$u_\sigma(x,t) = v_\sigma(x)\left(1 + \mathcal{O}(x^{-1}t^{1/3})\right).$$

# Integro-differential Painlevé V

**PROPOSITION** (Cafasso-Claeys-R, 2020). *Assuming $\sigma$ satisfies the decay properties and $\gamma = 1$, the function $v_\sigma$ has the asymptotics*

$$v_\sigma(x) = \frac{1}{8x^2} + \frac{1}{2}\int_{\mathbb{R}} \left(\chi_{[0,+\infty)}(z) - \sigma(z)\right) \mathrm{d}z + \mathcal{O}\left(x^2\right), \qquad \text{as } x \to 0.$$

*Moreover, it can be expressed for all $x > 0$ as*

$$v_\sigma(x) = \frac{1}{x}\int_{\mathbb{R}} \left(z\phi_\sigma^2(x;z) + \left(\partial_x \phi_\sigma(x;z)\right)^2\right) \mathrm{d}\sigma(z),$$

*where $\phi_\sigma(x;z)$ solves the Schrödinger equation*

$$\partial_x^2 \phi_\sigma(x;z) = \left(z - 2v_\sigma(x)\right)\phi_\sigma(x;z),$$

*and satisfies $\int_{\mathbb{R}} \phi_\sigma^2(x;z)\,\mathrm{d}\sigma(z) = x/2$ and has the asymptotics*

$$\phi_\sigma(x;z) \sim \sqrt{x/2}\,\mathrm{I}_0\left(x\sqrt{z}\right), \text{ as } z \to \infty \text{ with } |\arg z| < \pi - \delta, \text{ for any } \delta > 0,$$

*where $\mathrm{I}_0$ is the modified Bessel function of the first kind.*

# Small time asymptotics for $\log Q_\sigma(x,t)$

**THEOREM** (Cafasso-Claeys-R, 2020). *Assuming $\sigma$ satisfies the decay properties we have the following uniform estimates.*

- $\forall t_0 > 0\ \exists M, c > 0$ *such that for* $x < -Mt^{1/3}$, $0 < t < t_0$

$$\log Q_\sigma(x,t) = \mathcal{O}\left(\exp\left(-c|x|t^{-1/3}\right)\right).$$

- $\exists \epsilon > 0$ *such that* $\forall M > 0$ *and for* $|x| \leq Mt^{1/3}$, $0 < t < \epsilon$*:*

$$\log Q_\sigma(x,t) = \log F_{\mathrm{TW}}\left(-xt^{-1/3}\right) + \mathcal{O}\left(t^{1/3}\right).$$

- *If $\gamma = 1$:* $\exists v_\sigma(x)$ *such that* $\exists \epsilon, M > 0$ *such that* $\forall K > 0$ *and for* $Mt^{1/3} < x < K$, $0 < t < \epsilon$ *that*

$$\log Q_\sigma(x,t) = -\frac{x^3}{12t} - \frac{1}{8}\log\left(xt^{-1/3}\right) + \frac{\log 2}{24} + \log\zeta'(-1) + \int_0^x (x-x')\left(v_\sigma(x') - \frac{1}{8x'^2}\right)\mathrm{d}x' + \mathcal{O}\left(x^{-1}t^{1/3}\right).$$

# Strategy of the proofs

Its-Izergin-Korepin-Slavnov theory of integrable operators

↓

Riemann-Hilbert problem

↓

KdV equation

Integro-differential PII

Deift-Zhou asymptotic analysis

# Its-Izergin-Korepin-Slavnov theory

An operator $\mathcal{K}$ on $L^2(\mathbb{R})$ is called **integrable** if it is of the form

$$\mathcal{K} : f(r) \mapsto \int_{\mathbb{R}} \frac{\mathbf{g}^\top(r)\mathbf{h}(r')}{r - r'} f(r')\mathrm{d}r'$$

for $\mathbf{g}, \mathbf{h} \in L^\infty(\mathbb{R}) \otimes \mathbb{C}^n$ satisfying $\mathbf{g}^\top(r)\mathbf{h}(r) = 0$.

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Consider the following **Riemann-Hilbert problem** for a sectionally analytic matrix $Y = Y(r) \in \mathrm{Mat}_{n\times n}(\mathbb{C})$.

- $Y(r)$ *analytic for* $r \in \mathbb{C} \setminus \mathbb{R}$*, and* $\forall r \in \mathbb{R}\ \exists Y_\pm(r) = \lim_{\epsilon\to 0+} Y(r \pm \mathrm{i}\epsilon)$.
- $Y_+(r) = Y_-(r)(I - 2\pi\mathrm{i}\,\mathbf{g}(r)\mathbf{h}^\top(r))$ *for* $r \in \mathbb{R}$.
- $Y(r) = I + \mathcal{O}(r^{-1})$ *as* $r \to \infty$ *uniformly in the* $\mathbb{C} \setminus \mathbb{R}$.

# ITS-IZERGIN-KOREPIN-SLAVNOV THEORY

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**THEOREM [A. R. Its, A. G. Izergin, V. E. Korepin, and N. A. Slavnov (1990)].** *Let* $\mathcal{K}$ *be an integrable operator on* $L^2(\mathbb{R})$*. The operator* $I - \mathcal{K}$ *on* $L^2(\mathbb{R})$ *is invertible if and only if the Riemann-Hilbert problem for* $Y$ *is solvable, in which case the resolvent operator is also integrable:*

$$(I-\mathcal{K})^{-1}\mathcal{K} : f(r) \mapsto \int_{\mathbb{R}} \frac{\mathbf{g}^\top(r)Y_+^\top(r)Y_+^{-\top}(r')\mathbf{h}(r')}{r-r'} f(r')\mathrm{d}r'.$$

## Application of the Its-Izergin-Korepin-Slavnov theorem

The relevant operator $\mathcal{K}^{\mathrm{Ai}}_{\varphi_{x,t}}$ is of **integrable form** with

$$\mathbf{g}(r) = \sqrt{\sigma(z(r))}\begin{pmatrix} -\mathrm{i}\mathrm{Ai}'(r) \\ \mathrm{Ai}(r) \end{pmatrix}, \quad \mathbf{h}(r) = \sqrt{\sigma(r(z))}\begin{pmatrix} -\mathrm{i}\mathrm{Ai}(r) \\ \mathrm{Ai}'(r) \end{pmatrix},$$

where $z(r) = t^{-2/3}(r + xt^{-1/3})$.
Moreover, the assumptions on $\sigma(z)$ imply that

$$\det_{L^2(\mathbb{R})}\left(1 - \mathcal{K}^{\mathrm{Ai}}_{\varphi_{x,t}}\right) = \mathbb{E}\left[\prod_{j\geq 1}\left(1 - \sigma\left(t^{-2/3}\left(r_j - xt^{-1/3}\right)\right)\right)\right] > 0$$

and so there exists unique the $2 \times 2$ matrix $Y$ satisfying the RH characterization of the Its-Izergin-Korepin-Slavnov theorem.

# Airy model Riemann-Hilbert problem

We can write

$$\mathbf{g}(r) = \mathrm{i}\sqrt{\sigma(z(r))}\Phi_-^{\mathrm{Ai}}(r)\begin{pmatrix}1\\0\end{pmatrix}, \qquad \mathbf{h}^\top(r) = \frac{\sqrt{\sigma(z(r))}}{2\pi}(0,-1)\left(\Phi_+^{\mathrm{Ai}}\right)^{-1}(r),$$

in terms of the $2\times 2$ matrix solution $\Phi^{\mathrm{Ai}} = \Phi^{\mathrm{Ai}}(r)$

$$\Phi^{\mathrm{Ai}}(r) = \begin{cases} -\begin{pmatrix}\mathrm{Ai}'(r) & -\omega\mathrm{Ai}'\left(\omega^2 r\right)\\ \mathrm{i}\mathrm{Ai}(r) & -\mathrm{i}\omega^2\mathrm{Ai}\left(\omega^2 r\right)\end{pmatrix} & \text{for } \operatorname{Im} r > 0,\\ -\begin{pmatrix}\mathrm{Ai}'(r) & \omega^2\mathrm{Ai}'(\omega r)\\ \mathrm{i}\mathrm{Ai}(r) & \mathrm{i}\omega\mathrm{Ai}(\omega r)\end{pmatrix} & \text{for } \operatorname{Im} r < 0,\end{cases}$$

of the following **Airy model Riemann-Hilbert problem**.

- $\Phi^{\mathrm{Ai}}(r)$ is analytic for $r\in\mathbb{C}\setminus\mathbb{R}$, and $\forall r\in\mathbb{R}\ \exists\Phi_\pm^{\mathrm{Ai}}(r) = \lim_{\epsilon\to 0+}\Phi^{\mathrm{Ai}}(r\pm\mathrm{i}\epsilon)$.
- $\Phi_+^{\mathrm{Ai}}(r) = \Phi_-^{\mathrm{Ai}}(r)\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}, \quad r\in\mathbb{R}.$
- As $z\to\infty$,

$$\Phi^{\mathrm{Ai}}(r) = \left(I + \mathcal{O}\left(\frac{1}{r}\right)\right)r^{\frac{\sigma_3}{4}}\frac{\begin{pmatrix}1 & -\mathrm{i}\\ -\mathrm{i} & 1\end{pmatrix}}{2\sqrt{\pi}}\mathrm{e}^{-\frac{2}{3}r^{3/2}\sigma_3}\times\begin{cases} I, & |\arg r|<\pi-\delta,\\ \begin{pmatrix}1 & 0\\ \mp 1 & 1\end{pmatrix}, & \pi-\delta<\pm\arg r<\pi.\end{cases}$$

# Airy dressing

The aforementioned expressions

$$\mathbf{g}(r) = \mathrm{i}\sqrt{\sigma(z(r))}\Phi_-^{\mathrm{Ai}}(r)\begin{pmatrix}1\\0\end{pmatrix}, \qquad \mathbf{h}^\top(r) = \frac{\sqrt{\sigma(z(r))}}{2\pi}(0,-1)\left(\Phi_+^{\mathrm{Ai}}\right)^{-1}(r)$$

suggest to introduce the $2\times 2$ matrix

$$\boxed{\Psi(z) := \begin{pmatrix}1 & \frac{\mathrm{i}x^2}{4t}\\ 0 & 1\end{pmatrix} t^{-\sigma_3/6} Y(r(z))\Phi^{\mathrm{Ai}}(r(z)), \qquad r(z) = t^{2/3}z - xt^{-1/3},}$$

which solves the following **Riemann-Hilbert problem**.

- $\Psi(z)$ *is analytic for* $z \in \mathbb{C}\setminus\mathbb{R}$, *and* $\forall z\in\mathbb{R}\ \exists \Psi_\pm(z) = \lim_{\epsilon\to 0+}\Psi(z\pm \mathrm{i}\epsilon)$.
- $\Psi_+(z) = \Psi_-(z)\begin{pmatrix}1 & 1-\sigma(z)\\ 0 & 1\end{pmatrix}, \quad z\in\mathbb{R}.$
- *As* $z\to\infty$,

$$\Psi(z) = \left(I + \frac{\Psi^{(1)}}{z} + \mathcal{O}\left(\frac{1}{z^2}\right)\right) z^{\frac{\sigma_3}{4}} \frac{\begin{pmatrix}1 & -\mathrm{i}\\ -\mathrm{i} & 1\end{pmatrix}}{2\sqrt{\pi}} \mathrm{e}^{(-\frac{2}{3}tz^{3/2}+xz^{1/2})\sigma_3}$$

$$\times \begin{cases} I, & |\arg z| < \pi-\delta,\\ \begin{pmatrix}1 & 0\\ \mp 1 & 1\end{pmatrix}, & \pi-\delta < \pm\arg z < \pi.\end{cases}$$

# Matrix Lax pair

The jump of $\Psi(z)$ is **independent of $x, t$** and so by (almost standard) methods we obtain the **matrix Lax pair**

$$\partial_x \Psi(z) = \begin{pmatrix} p & \mathrm{i}z + 2\mathrm{i}q \\ -\mathrm{i} & -p \end{pmatrix} \Psi(z), \quad \partial_t \Psi(z) = \begin{pmatrix} -\frac{2}{3}zp + \frac{2}{3}\partial_x q & -\frac{2}{3}z^2 - \frac{4}{3}\mathrm{i}zq - \frac{2}{3}\mathrm{i}\partial_x r \\ \frac{2}{3}\mathrm{i}z - \frac{4}{3}\mathrm{i}q - \frac{2}{3}\mathrm{i}p^2 & \frac{2}{3}zp - \frac{2}{3}q \end{pmatrix} \Psi(z)$$

where we denote $\Psi^{(1)}(x,t) = \begin{pmatrix} q(x,t) & -\mathrm{i}r(x,t) \\ \mathrm{i}p(x,t) & -q(x,t) \end{pmatrix}$.

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The gauge transformation

$$\widehat{\Psi}(z) := \mathrm{e}^{\frac{\pi \mathrm{i}}{4}\sigma_3} \begin{pmatrix} 1 & -\mathrm{i}p \\ 0 & 1 \end{pmatrix} \Psi(z)\, \mathrm{e}^{-\frac{\pi \mathrm{i}}{4}\sigma_3}$$

satisfies the **matrix Lax pair**

$$\partial_x \widehat{\Psi}(z) = \begin{pmatrix} 0 & -z+2u \\ -1 & 0 \end{pmatrix} \widehat{\Psi}(z), \quad \partial_t \widehat{\Psi}(z) = \begin{pmatrix} -\frac{1}{3}\partial_x u & \frac{2}{3}z^2-\frac{2}{3}zu-\frac{4}{3}u^2-\frac{1}{3}\partial_x^2 u \\ \frac{2}{3}z+\frac{2}{3}u & \frac{1}{3}\partial_x u \end{pmatrix} \widehat{\Psi}(z),$$

in terms of $\boxed{u(x,t) := \partial_x p(x,t).}$

# Korteweg-de Vries equation

In particular can be written as

$$\widehat{\Psi}(z) = \begin{pmatrix} \partial_x \psi\,(z) & -\partial_x \widetilde{\psi}\,(z) \\ -\psi\,(z) & \widetilde{\psi}\,(z) \end{pmatrix},$$

with the **wave function** $\psi$ satisfying

$$\boxed{\partial_x^2 \psi(z) = (z - 2u)\psi(z), \qquad \psi(z) \sim \frac{\mathrm{e}^{-\frac{2}{3}tz^{3/2} + xz^{1/2}}}{2\sqrt{\pi}z^{1/4}}.}$$

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As in **Lax's theorem**, an elementary computation shows that the **compatibility** of the Lax pair implies the **KdV equation**

$$u_t + 2uu_x + \frac{1}{6}u_{xxx} = 0.$$

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The proof of the KdV equation is complete if we show that

$$\partial_x^2 \log Q_\sigma(x,t) + \frac{x}{2t} = \partial_x p(x,t), \qquad Q_\sigma(x,t) = \det_{L^2(\mathbb{R})}\left(I - \mathcal{K}^{\mathrm{Ai}}_{\varphi_{x,t}}\right).$$

# Integral identities

**PROPOSITION 1**. $\partial_x \log Q_\sigma = \frac{1}{t}\int_{\mathbb{R}}(\psi(z)\psi_{xz}(z) - \psi_x(z)\psi_z(z))\mathrm{d}\sigma(z)$.

**PROPOSITION 2**. $\int_{\mathbb{R}} \psi^2(z)\mathrm{d}\sigma(z) = \frac{x}{2} - t\partial_x p$.

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**PROPOSITION 2**. $\int_{\mathbb{R}} \psi^2(z) \mathrm{d}\sigma(z) = \frac{x}{2} - t\partial_x p$.

Taking one more $x-$derivative in Prop. 1 and simplifying the RHS with the aid of the Schrödinger equation for $\psi$ we obtain

$$\partial_x^2 \log Q_\sigma = -\frac{1}{t} \int_{\mathbb{R}} \psi^2(z) \mathrm{d}\sigma(z)$$

which shows that $\partial_x^2 \log Q_\sigma(x,t) + \frac{x}{2t} = \partial_x p(x,t)$, and so this proves the KdV equation.
The integro-differential PII equation follows directly from these integral identities.

# Proof of Proposition 1

**PROPOSITION 1.** $\partial_x \log Q_\sigma = \frac{1}{t}\int_{\mathbb{R}}(\psi(z)\psi_{xz}(z) - \psi_x(z)\psi_z(z))\mathrm{d}\sigma(z)$.

The Jacobi variational formula gives

$$\partial_x \log Q_\sigma(x,t) = \partial_x \det{}_{L^2(\mathbb{R})}(1-\mathcal{K}^{\mathrm{Ai}}_{\varphi_{x,t}}) = -\mathrm{tr}\left(\left(1-\mathcal{K}^{\mathrm{Ai}}_{\varphi_{x,t}}\right)^{-1}\partial_x\mathcal{K}^{\mathrm{Ai}}_{\varphi_{x,t}}\right)\right)$$

$$= -t^{-1/3}\int_{\mathbb{R}} L_{x,t}\left(r\left(z\right), r\left(z\right)\right)\frac{\mathrm{d}\sigma\left(z\right)}{\sigma\left(z\right)},$$

where we bear in mind that $\sigma$ may have singularities and we have denoted

$$L_{x,t}(r_1,r_2) = \frac{\mathbf{g}\left(r_1\right)Y_+^{\top}\left(r_1\right)Y_+^{-\top}\left(r_2\right)\mathbf{h}^{\top}\left(r_2\right)}{r_1-r_2}$$

the kernel of the resolvent operator of $\mathcal{K}^{\mathrm{Ai}}_{\varphi_{x,t}}$.
Further algebraic manipulations complete the proof.

# Proof of Proposition 2

**PROPOSITION 2**. $\int_{\mathbb{R}} \psi^2(z)\mathrm{d}\sigma(z) = \frac{x}{2} - t\partial_x p$.

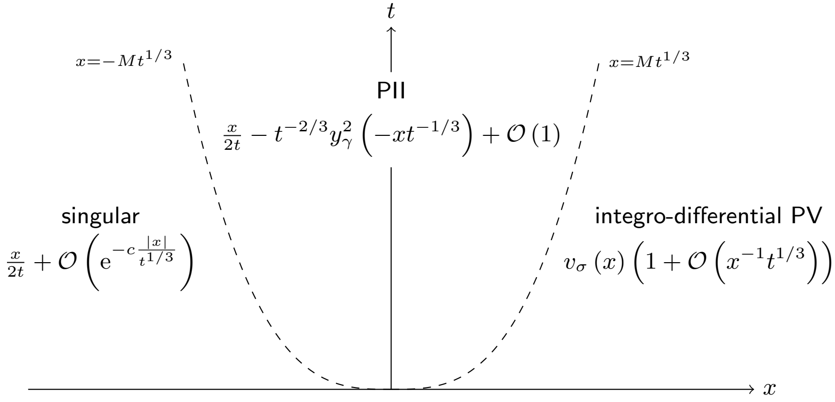
It follows from a simple residue computation of the integral

$$\int_{\mathbb{R}\setminus\{z_1,\dots,z_k\}} \Psi(z) \begin{pmatrix} 0 & -\sigma'(z) \\ 0 & 0 \end{pmatrix} \Psi^{-1}\mathrm{d}z = \int_{\mathbb{R}\setminus\{z_1,\dots,z_k\}} \left(\partial_z\Psi_+\Psi_+^{-1} - \partial_z\Psi_-\Psi_-^{-1}\right)\mathrm{d}z$$

$$= -\operatorname*{res}_{z=\infty} \partial_z\Psi\Psi^{-1} - \sum_{\ell=1}^{k} \operatorname*{res}_{z=z(r_\ell)} \partial_z\Psi\Psi^{-1}$$

by taking the $(2,1)$-entry.

# Small time asymptotics of $u_\sigma$: the three regimes



# Asymptotic analysis of the RH problems

When $x < -Mt^{1/3}$, jump of $Y$ is close to $I \Rightarrow \boxed{\Psi(z) \sim \Phi^{\mathrm{Ai}}(r(z))}$.

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When $|x| < Mt^{1/3}$, we perform a rescaling of variables

$$w = zt^{2/3} \Rightarrow \sigma(z) = \sigma(wt^{-2/3}) \to \gamma\chi_{[0,+\infty)}(w) = \gamma\chi_{[0,+\infty)}(z)$$

and so for $w \neq 0$ the jump of $\Psi(z)$ is close to that of $\Psi_{\sigma=\gamma\chi_{[0,+\infty)}}(z)$; near $w = 0$ we construct a local parametrix in terms of elementary functions, and so $\boxed{\Psi(z) \sim \Psi_{\sigma=\gamma\chi_{[0,+\infty)}}(z)}$ for large $z$.

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When $x > Mt^{1/3}$ we have to consider a **model RH problem** (for $\gamma = 1$) which is the formal reduction $(x,t) = (\xi, 0)$ of the RH problem for $\Psi$.

- $\Phi(z)$ *is analytic for* $z \in \mathbb{C} \setminus \mathbb{R}$, *and* $\forall z \in \mathbb{R}\ \exists \Phi_\pm(z) = \lim_{\epsilon \to 0+} \Phi(z \pm \mathrm{i}\epsilon)$.
- $\Phi_+(z) = \Phi_-(z) \begin{pmatrix} 1 & 1-\sigma(z) \\ 0 & 1 \end{pmatrix}, \quad z \in \mathbb{R}.$
- *As* $z \to \infty$,

$$\Phi(z) = \left(I + \mathcal{O}\left(\frac{1}{z}\right)\right) z^{\frac{\sigma_3}{4}} \frac{\begin{pmatrix} 1 & -\mathrm{i} \\ -\mathrm{i} & 1 \end{pmatrix}}{2\sqrt{\pi}} \mathrm{e}^{\xi z^{1/2}\sigma_3} \times \begin{cases} I, & |\arg z| < \pi - \delta, \\ \begin{pmatrix} 1 & 0 \\ \mp 1 & 1 \end{pmatrix}, & \pi - \delta < \pm \arg z < \pi. \end{cases}$$

Thank you!