Poisson quasi-Nijenhuis manifolds and the Toda system

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Plan of this talk

- Recall the definition of Poisson-Nijenhuis manifold (Magri-Morosi-Ragnisco, CMP, 1985) and its relation with (finite-dimensional) integrable systems
- Recall the definition of Poisson quasi-Nijenhuis manifold (Stiénon-Xu, CMP, 2007) $\longrightarrow$ no relation with integrable systems
- Present sufficient conditions on a Poisson quasi-Nijenhuis manifold such that a family of functions in involution can be easily constructed
- Frame in such a geometrical structure the closed (or periodic) n-particle Toda lattice, along with its relation with the open (or non periodic) Toda lattice

Poisson-Nijenhuis manifolds

A Poisson-Nijenhuis (PN) manifold $(\mathcal{M}, \pi, N)$ is a Poisson manifold $(\mathcal{M}, \pi)$ endowed with a $(1,1)$ tensor field $N : T\mathcal{M} \to T\mathcal{M}$ such that

1. the Nijenhuis torsion of N vanishes;
2. the Poisson tensor $\pi : T^*\mathcal{M} \to T\mathcal{M}$ and the tensor field N are compatible (in a suitable sense).

Recall that:

- a tensor $\pi : T^*\mathcal{M} \to T\mathcal{M}$ and a manifold $(\mathcal{M}, \pi)$ are said to be Poisson if
$$\{f, g\} = \langle df, \pi\, dg\rangle$$
has the usual properties of the Poisson bracket;
- the Nijenhuis torsion T_N of N is defined as
$$T_N(X, Y) = [NX, NY] - N\left([NX, Y] + [X, NY] - N[X, Y]\right) ,$$
where X and Y are vector fields on $\mathcal{M}$.
- the tensor $\pi' = N\,\pi : T^*\mathcal{M} \to T\mathcal{M}$ is Poisson and compatible with π, so that $\mathcal{M}$ is a bi-Hamiltonian manifold.

PN manifolds and integrable systems

Given a PN manifold $(\mathcal{M}, \pi, N)$, it is well known that the functions

$$I_k = \frac{1}{k}\,\mathrm{Tr}(N^k)\,, \qquad k = 1, 2, \dots\,,$$

satisfy

$$dI_{k+1} = N^* dI_k\,,$$

where $N^* : T^*\mathcal{M} \to T^*\mathcal{M}$ is the transpose of N. This entails the so-called Lenard-Magri relations

$$\pi\, dI_{k+1} = \pi'\, dI_k$$

and therefore the involutivity of the I_k (with respect to both Poisson brackets induced by π and π'). Moreover, if $X_k = \pi\, dI_k$, then

$$X_{k+1} = N\, X_k\,,$$

so that N is sometimes called a recursion operator.

Example: the open Toda lattice

Das and Okubo showed in 1989 that the open (or non periodic) n-particle Toda lattice can be studied in the context of PN manifolds.

The manifold is $\mathcal{M} = \mathbb{R}^{2n} \ni (q_1, \ldots, q_n, p_1, \ldots, p_n)$, the Poisson tensor π is the canonical one, and the tensor field N is (in the 3-particle case)

$$N = \left[\begin{array}{ccc|ccc} p_1 & 0 & 0 & 0 & 1 & 1 \\ 0 & p_2 & 0 & -1 & 0 & 1 \\ 0 & 0 & p_3 & -1 & -1 & 0 \\ \hline 0 & -e^{q_1-q_2} & 0 & p_1 & 0 & 0 \\ e^{q_1-q_2} & 0 & -e^{q_2-q_3} & 0 & p_2 & 0 \\ 0 & e^{q_2-q_3} & 0 & 0 & 0 & p_3 \end{array}\right].$$

It is compatible with π and its Nijenhuis torsion vanishes.
Hence $(\mathbb{R}^6, \pi, N)$ is a PN manifold.

The traces I_k of the powers of N are the usual integrals of motion of the open Toda lattice.

For example,

$$I_1 = \mathrm{Tr}(N) = 2(p_1 + p_2 + p_3)$$
$$I_2 = \frac{1}{2}\,\mathrm{Tr}(N^2) = p_1^2 + p_2^2 + p_3^2 + 2\mathrm{e}^{q_1 - q_2} + 2\mathrm{e}^{q_2 - q_3}$$

are respectively twice the total momentum and the Hamiltonian.

The involutivity of the I_k is a consequence of the relations

$$dI_{k+1} = N^* dI_k\,.$$

We will see that for Poisson quasi-Nijenhuis manifolds these relations do not hold, so that one needs additional assumptions to prove that the I_k are in involution.

Poisson quasi-Nijenhuis manifolds

A Poisson quasi-Nijenhuis (PqN) manifold $(\mathcal{M}, \pi, N, \phi)$ is a Poisson manifold $(\mathcal{M}, \pi)$ endowed with a $(1,1)$ tensor field $N : T\mathcal{M} \to T\mathcal{M}$ and a closed 3-form ϕ such that

- the Poisson tensor π and the $(1,1)$ tensor field N are compatible;
- the 3-form $i_N\phi$, defined as

  $$i_N\phi(X, Y, Z) = \phi(NX, Y, Z) + \phi(X, NY, Z) + \phi(X, Y, NZ)\,,$$

  is closed;
- $T_N(X, Y) = \pi\,(i_{X\wedge Y}\phi)$ for all vector fields X and Y, where $i_{X\wedge Y}\phi$ is the 1-form defined as $\langle i_{X\wedge Y}\phi, Z\rangle = \phi(X, Y, Z)$.

Notice that

$$\phi = 0 \implies T_N = 0 \implies \text{the manifold is PN}$$

In the paper where Stiénon and Xu introduced and studied PqN manifolds, they wrote:

> *Poisson Nijenhuis structures arise naturally in the study of integrable systems. It would be interesting to find applications of Poisson quasi-Nijenhuis structures in integrable systems as well.*

We will see that the integrability of the closed n-particle Toda lattice, along with its relation with the open one, can be interpreted in the framework of PqN manifolds. Before doing this, we will

1. check that the PqN structure is too general to ensure the involutivity of the traces I_k of the powers of N;
2. present a general scheme to deform a PN manifold into a PqN manifold;
3. give sufficient conditions for the involutivity of the traces I_k of a PqN manifold (obtained as a deformation of a PN manifold).

Examples of non-involutive PqN manifolds

We call involutive a PqN manifold $(\mathcal{M}, \pi, N, \phi)$ if

$$\{I_j, I_k\} = 0 \quad \text{for all } j, k \geq 1, \text{ where } I_k = \frac{1}{k}\operatorname{Tr}(N^k).$$

Generalizing the Das-Okubo example, we can easily construct non-involutive PqN manifolds.
Consider $\mathcal{M} = \mathbb{R}^6 \ni (q_1, q_2, q_3, p_1, p_2, p_3)$ with the canonical Poisson tensor π and the $(1,1)$ tensor field given by

$$N = \left[\begin{array}{ccc|ccc} p_1 & 0 & 0 & 0 & 1 & 1 \\ 0 & p_2 & 0 & -1 & 0 & 1 \\ 0 & 0 & p_3 & -1 & -1 & 0 \\ \hline 0 & -V(q_1 - q_2) & -V(q_3 - q_1) & p_1 & 0 & 0 \\ V(q_1 - q_2) & 0 & -V(q_2 - q_3) & 0 & p_2 & 0 \\ V(q_3 - q_1) & V(q_2 - q_3) & 0 & 0 & 0 & p_3 \end{array}\right],$$

where V is an arbitrary (differentiable) function of one variable.

It can be checked that

- $(\mathbb{R}^6, \pi, N, \phi)$ is a PqN manifold if

$$\begin{aligned}\phi = & \left(V'(q_1 - q_2) - V(q_1 - q_2)\right) d(p_1 + p_2) \wedge dq_2 \wedge dq_1 \\ & + \left(V'(q_2 - q_3) - V(q_2 - q_3)\right) d(p_2 + p_3) \wedge dq_3 \wedge dq_2 \\ & - \left(V'(q_3 - q_1) + V(q_3 - q_1)\right) d(p_1 + p_3) \wedge dq_3 \wedge dq_1 \\ & - 2V'(q_3 - q_1) dp_2 \wedge dq_3 \wedge dq_1 \,.\end{aligned}$$

- $\{I_1, I_2\} = \{I_1, I_3\} = 0$, while the Poisson bracket

$$\begin{aligned}\{I_2, I_3\} = & 4V(q_1 - q_2)\left(V'(q_2 - q_3) - V'(q_3 - q_1)\right) \\ & + 4V(q_2 - q_3)\left(V'(q_3 - q_1) - V'(q_1 - q_2)\right) \\ & + 4V(q_3 - q_1)\left(V'(q_1 - q_2) - V'(q_2 - q_3)\right)\end{aligned}$$

  does not vanish for any function V. However, involutivity holds in the cases $V(x) = \mathrm{e}^x$ (corresponding to the closed Toda lattice) and $V(x) = 1/x^2$ (corresponding to the Calogero model).

In conclusion, given a PqN manifold, further conditions on (π, N, ϕ) are needed to guarantee that the functions I_k are in involution.

Relations between PN and PqN manifolds

To deform a PN structure into a PqN one, and to give conditions on the deformation entailing that the PqN manifold is involutive, we need two ingredients.

1. Given a tensor field $N : T\mathcal{M} \to T\mathcal{M}$, the usual Cartan differential can be modified as follows,

$$\begin{aligned}(d_N\alpha)(X_0,\ldots,X_q) = & \sum_{j=0}^{q}(-1)^j L_{NX_j}\left(\alpha(X_0,\ldots,\hat{X}_j,\ldots,X_q)\right) \\ & + \sum_{i<j}(-1)^{i+j}\alpha([X_i,X_j]_N,X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_q)\,,\end{aligned}$$

where α is a q-form, the X_i are vector fields, L_Y is the Lie derivative along the vector field Y, and

$$[X,Y]_N = [NX,Y] + [X,NY] - N[X,Y].$$

The torsion of N vanishes if and only if $d_N^2 = 0$.

2. Given a Poisson tensor π on a manifold $\mathcal{M}$, one can define a Lie bracket between the 1-forms as

$$[\alpha, \beta]_\pi = L_{\pi\alpha}\beta - L_{\pi\beta}\alpha - d\langle \beta, \pi\alpha \rangle \,.$$

It can be uniquely extended to all forms on $\mathcal{M}$ in such a way that

- $[\eta, \eta']_\pi = -(-1)^{(q-1)(q'-1)}[\eta', \eta]_\pi$ if η is a q-form and η' is a q'-form;
- $[\alpha, f]_\pi = \langle \alpha, \pi\, df \rangle$ for all $f \in C^\infty(\mathcal{M})$ and for all 1-forms α;
- if η is a q-form, then $[\eta, \cdot]_\pi$ is a derivation of degree $q-1$ of the wedge product, that is,

$$[\eta, \eta' \wedge \eta'']_\pi = [\eta, \eta']_\pi \wedge \eta'' + (-1)^{(q-1)q'} \eta' \wedge [\eta, \eta'']_\pi$$

  if η' is a q'-form and η'' is any differential form.

This extension is a graded Lie bracket. An elegant way to define the compatibility between π and N is to ask (Kosmann-Schwarzbach, LMP, 1996) that d_N is a derivation of $[\cdot, \cdot]_\pi$, that is,

$$d_N[\eta, \eta']_\pi = [d_N\eta, \eta']_\pi + (-1)^{(q-1)}[\eta, d_N\eta']_\pi$$

if η is a q-form and η' is any differential form.

From PN to PqN manifolds: a deformation theorem

Theorem (deformation)

Given a PN manifold $(\mathcal{M}, \pi, N)$*, suppose that* Ω *is a closed 2-form such that*

$$[d_N\Omega, \Omega]_\pi = 0 \,.$$

Let $\Omega^\flat : T\mathcal{M} \to T^*\mathcal{M}$ *be defined as usual by* $\langle \Omega^\flat(X), Y\rangle = \Omega(X, Y)$*. If*

$$\hat{N} = N - \pi\,\Omega^\flat \quad \text{and} \quad \phi = d_N\Omega + \frac{1}{2}[\Omega, \Omega]_\pi \,,$$

then $(\mathcal{M}, \pi, \hat{N}, \phi)$ *is a PqN manifold.*

Many features of the usual picture of PN manifolds are lost in the PqN case.
In particular, the functions $\hat{I}_k = \frac{1}{k}\,\mathrm{Tr}(\hat{N}^k)$ do not fulfill the relations $\hat{N}^* d\hat{I}_k = d\hat{I}_{k+1}$, so that they may not be in involution (as seen before).
However, the involutivity of the $\hat{I}_k$ can be proved under additional hypotheses.

An involutivity theorem

In the following theorem we identify a suitable set of compatibility conditions between π, N and Ω, implying the involutivity of the traces $\hat{I}_k$ of the powers of the deformed tensor field $\hat{N}$.

Theorem (involutivity)

Given a PN manifold $(\mathcal{M}, \pi, N)$, let Ω be a closed 2-form on $\mathcal{M}$ such that

$$[\Omega, \Omega]_\pi = 0\,.$$

Define $\hat{N} = N - \pi\,\Omega^\flat$ and $\hat{I}_k = \frac{1}{k}\,\mathrm{Tr}(\hat{N}^k)$. Suppose that

1. $d_N\Omega = d\hat{I}_1 \wedge \Omega$;
2. $\Omega(Y_k, \cdot) = 0$, *where* $Y_k = (\hat{N})^{k-1}X_1 - X_k$ *and* $X_k = \pi\, d\hat{I}_k$;
3. $\{\hat{I}_1, \hat{I}_k\} = 0$ *for all* $k \geq 2$.

Then $(\mathcal{M}, \pi, \hat{N}, d_N\Omega)$ is an involutive PqN manifold, that is,

$$\{\hat{I}_j, \hat{I}_k\} = 0 \qquad \text{for all } j, k \geq 1.$$

From PN to PqN manifolds: the Toda case

Let us consider the PN structure of the 3-particle open Toda system and let $\Omega = \mathrm{e}^{q_3 - q_1} dq_3 \wedge dq_1$. It is closed and satisfies

$$[\Omega, \Omega]_\pi = 0\,,$$

so that $[d_N\Omega, \Omega]_\pi = 0$ and the deformation theorem can be applied (the generalization to the n-particle case being obvious). It turns out that

$$\hat{N} = N - \pi\,\Omega^\flat = \left[\begin{array}{ccc|ccc} p_1 & 0 & 0 & 0 & 1 & 1 \\ 0 & p_2 & 0 & -1 & 0 & 1 \\ 0 & 0 & p_3 & -1 & -1 & 0 \\ \hline 0 & -\mathrm{e}^{q_1-q_2} & -\mathrm{e}^{q_3-q_1} & p_1 & 0 & 0 \\ \mathrm{e}^{q_1-q_2} & 0 & -\mathrm{e}^{q_2-q_3} & 0 & p_2 & 0 \\ \mathrm{e}^{q_3-q_1} & \mathrm{e}^{q_2-q_3} & 0 & 0 & 0 & p_3 \end{array}\right].$$

The resulting PqN structure describes the closed Toda lattice. Indeed, if $\hat{I}_k = \frac{1}{k}\,\mathrm{Tr}(\hat{N}^k)$ and $X_k = \pi\, d\hat{I}_k$, then

$$\hat{I}_2 = \frac{1}{2}\,\mathrm{Tr}(\hat{N}^2) = p_1^2 + p_2^2 + p_3^2 + 2\mathrm{e}^{q_1 - q_2} + 2\mathrm{e}^{q_2 - q_3} + 2\mathrm{e}^{q_3 - q_1}$$

is (twice) the Hamiltonian, so that X_2 is (twice) the vector field associated with the closed Toda lattice.

Involutivity theorem: the Toda case

One can check that also the involutivity theorem can be applied to the (deformed) PqN structure associated with the closed n-particle Toda lattice.

In detail, if $\Omega = \mathrm{e}^{q_n - q_1} dq_n \wedge dq_1$, one can check that:

1. $d_N \Omega = d\hat{I}_1 \wedge \Omega$ (easy);
2. $\Omega(Y_k, \cdot) = 0$, where $Y_k = (\hat{N})^{k-1} X_1 - X_k$ (not so easy);
3. $\{\hat{I}_1, \hat{I}_k\} = 0$ for all $k \geq 2$ (easy).

This gives a geometric proof of the fact that the functions

$$\hat{I}_k = \frac{1}{k} \mathrm{Tr}(\hat{N}^k)$$

are integrals of motion (in involution) of the closed Toda lattice.

Résumé

- We presented a general result showing how a PN structure can be deformed into a PqN structure by means of a suitable 2-form Ω.
- We gave conditions on the deformation such that the PqN manifold turns out to be involutive.
- We applied these results to the Toda system.
- More precisely, we interpreted the (well known) integrals of motion of the closed Toda system as involutive deformations of the traces of the powers of the recursion operator of the open Toda system.