On the mean Density of States of some matrices related to the beta ensembles and an application to the Toda lattice

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Overview

The talk is based on the following articles:

- G.M., *On the mean Density of States of some matrices related to the beta ensembles and an application to the Toda lattice*, arXiv preprint:2008.04604. Submitted to Ann. Henri Poincaré.
- T. Grava, A. Maspero, G. M., and A. Ponno, *Adiabatic invariants for the FPUT and Toda chain in the thermodynamic limit*, arXiv preprint:2001.08070, (2020). Accepted in Commun. Math. Phys.

Let's start from the beginning

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The classical Toda chain is the dynamical system described by the following Hamiltonian:

$$H_T(p,q) := \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \sum_{j=1}^{N} V_T(q_{j+1}-q_j)\ , \quad V_T(x) = e^{-x} + x - 1\,,$$

with periodic boundary conditions $p_{j+N} = p_j$, $q_{j+N} = q_j \quad \forall\, j \in \mathbb{Z}$.

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with periodic boundary conditions $p_{j+N} = p_j\ ,\ q_{j+N} = q_j \quad \forall\, j \in \mathbb{Z}$.
Its equations of motion take the form

$$\dot{q}_j = \frac{\partial H_T}{\partial p_j} = p_j, \quad \dot{p}_j = -\frac{\partial H_T}{\partial q_j} = V_T'(q_{j+1}-q_j) - V_T'(q_j - q_{j-1}),\ \ j = 1,\dots,N\,.$$

One realize that these equations are equivalent to the Lax pair

$$\dot{L} = [L, B]$$

where:

$$L(a,b) = \begin{pmatrix} b_1 & a_1 & 0 & \dots & a_N \\ a_1 & b_2 & a_2 & \ddots & \vdots \\ 0 & a_2 & b_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{N-1} \\ a_N & \dots & 0 & a_{N-1} & b_N \end{pmatrix}, \qquad \begin{cases} b_j = -p_j \\ a_j = e^{\frac{q_j - q_{j+1}}{2}} \end{cases},$$

$$B = L_+ - L_- \,.$$

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$$B = L_+ - L_- .$$

This implies that the eigenvalues $\lambda_1^{(N)} \leqslant \ldots \leqslant \lambda_N^{(N)}$ are constants of motion, so the system is integrable.

Long story short

While we were trying to study some statistical properties of the Toda lattice, we started wondering "What is the typical distribution of the eigenvalues $\lambda_1, \dots, \lambda_N$?"

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We endow the phase space

$$\mathcal{M} := \left\{ (p,q) \in \mathbb{R}^N \times \mathbb{R}^N \ : \quad \sum_{j=1}^N q_j = \sum_{j=1}^N p_j = 0 \right\} ,$$

with the Gibbs measure of the periodic Toda lattice:

$$\mathrm{d}\mu_{Toda} := \frac{1}{Z_{Toda}(\beta)} \; e^{-\beta H_T(p,q)} \delta_{\sum_{j=1}^N p_j} \delta_{\sum_{j=1}^N q_j} \mathrm{d}p \, \mathrm{d}q \, ,$$

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Definition

"Typical" := "Mean value with respect to the Gibbs measure"

The proper question:

Question

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The mean *Density of States* $\overline{\mathrm{d}\nu}$ of a random matrix T_N is the non random probability distribution, provided it exists, defined as

$$\left\langle \overline{\mathrm{d}\nu}, f \right\rangle := \lim_{N\to\infty} \mathbf{E}\left[\left\langle \frac{1}{N} \sum_{j=1}^{N} \delta_{\lambda_j^{(N)}}, f \right\rangle \right],$$

for all continuous and bounded functions f, here $\langle \mathrm{d}\sigma, f\rangle := \int_{\mathbb{R}} f \mathrm{d}\sigma$.

$$d\mu_{Toda} := \frac{1}{Z_{Toda}(\beta)} \prod_{j=1}^{N} a_j^{2\beta-1} e^{-\beta \sum_{j=1}^{N} \left(\frac{b_j^2}{2} + a_j^2 \right)} \delta_{\sum_{j=1}^{N} b_j} \delta_{\prod_{j=1}^{N} a_j - 1} da\, db\,, \quad a_j \geqslant 0 \; \forall\, j\,,$$

$$L(a,b) = \begin{pmatrix} b_1 & a_1 & 0 & \dots & a_N \\ a_1 & b_2 & a_2 & \ddots & \vdots \\ 0 & a_2 & b_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{N-1} \\ a_N & \dots & 0 & a_{N-1} & b_N \end{pmatrix}.$$

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So L became a random periodic Jacobi matrix. Moreover one immediately notice that all b_js are almost normal random variables and the a_js are almost χ_β distributed.

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Problem: The variables are **dependent**.

Get rid of the dependence

Technical result

The mean density of states $\overline{\mathrm{d}\nu}_{Toda}$ of the random Lax operator L with the entries distributed according to $\mathrm{d}\mu_{Toda}$ coincides, in the large N limit, to the mean density of states of $\frac{1}{\sqrt{\beta}}H_\alpha$ where:

$$H_\alpha \sim \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{N}(0,2) & \chi_{2\alpha} & & \\ \chi_{2\alpha} & \mathcal{N}(0,2) & \chi_{2\alpha} & \\ & \ddots & \ddots & \ddots \\ & & \chi_{2\alpha} & \mathcal{N}(0,2) \end{pmatrix},$$

$\alpha = \beta + \theta$ and θ is chosen in such a way that

$$\int_{\mathbb{R}} r e^{-\beta e^{-r} - (\beta+\theta)r} \mathrm{d}r = 0 \, .$$

H_α is a particular kind random Jacobi matrix.

Almost there!

The matrix H_α already appeared in the literature and it's mean *spectral measure* $\overline{\mu}_{H_\alpha}(x)\mathrm{d}x$ was known explicitly thanks to Duy and Shirai.

The mean *spectral measure* $\overline{\mathrm{d}\mu}_{H_\alpha}$ of a random Jacobi matrix T_N is the non random probability distribution, provided it exists, defined as

$$\left\langle \overline{\mathrm{d}\mu}_{H_\alpha}, f \right\rangle := \lim_{N\to\infty} \mathbf{E}\left[\left\langle \frac{1}{N} \sum_{j=1}^{N} q_j^2 \delta_{\lambda_j^{(N)}}, f \right\rangle \right],$$

for all continuous and bounded functions f, here $q_j = |\langle v_j^{(N)}, e_1 \rangle|$ and $v_1^{(N)}, \dots, v_N^{(N)}$ are the orthonormal eigenvectors of H_α and $\langle \mathrm{d}\sigma, f\rangle := \int_{\mathbb{R}} f \mathrm{d}\sigma$.

How they got the result

Duy and Shirai proved that the mean spectral measure $\overline{\mu}_{H_\alpha}(x)\mathrm{d}x$ of H_α coincides, in the large N limit, to the one of the Gaussian β ensemble in the high temperature regime.

$$H_\alpha \sim \frac{1}{\sqrt{2}}\begin{pmatrix} \mathcal{N}(0,2) & \chi_{2\alpha} & & \\ \chi_{2\alpha} & \mathcal{N}(0,2) & \chi_{2\alpha} & \\ & \ddots & \ddots & \ddots \\ & & \chi_{2\alpha} & \mathcal{N}(0,2) \end{pmatrix} \quad H_\beta \sim \frac{1}{\sqrt{2}}\begin{pmatrix} \mathcal{N}(0,2) & \chi_{\beta(N-1)} & & & \\ \chi_{\beta(N-1)} & \mathcal{N}(0,2) & \chi_{\beta(N-2)} & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \chi_\beta \\ & & & \chi_\beta & \mathcal{N}(0,2) \end{pmatrix}$$

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The high temperature regime means that $\lim_{N\to\infty} N\beta = 2\alpha$.

Remark

For fixed k the upper right $k \times k$ block of H_β weakly converge in the high temperature regime to the one of H_α.

The key identity

Thanks to this result I was able to prove that:

$$w_\alpha^{(l)} = \partial_\alpha(\alpha u_\alpha^{(l)}),$$

where $w_\alpha^{(l)}$ is the l^{th} moment of the mean density of states of H_α and $u_\alpha^{(l)}$ is the one of the mean spectral measure of H_α.
The l^{th} moment of a measure $\mathrm{d}\sigma$ is defined to be $\int x^l \mathrm{d}\sigma$.

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The previous relations is important because it puts us in position to apply the so called *moment matching technique*.

Moment matching technique(in less then nutshell): if two measures $\mathrm{d}\sigma, \mathrm{d}\sigma'$ on $\mathbb{R}$ have the same moments and these moments grow fast enough, then the two measure are the same.

So, since

$$w_\alpha^{(l)} = \partial_\alpha(\alpha u_\alpha^{(l)})$$

and $u_\alpha^{(l)}$ are the moments of $\overline{\mu}_{H_\alpha}(x)\mathrm{d}x$, one can conclude that the mean density of states of H_α is

$$\overline{\mathrm{d}\nu}_{H_\alpha} = \partial_\alpha(\alpha\overline{\mu}_{H_\alpha}(x))\mathrm{d}x\,.$$

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In this way I got also the mean density of state of the periodic Toda lattice:

$$\overline{\mathrm{d}\nu_{Toda}} = \sqrt{\beta}\partial_\alpha(\alpha\mu_\alpha(\sqrt{\beta}x))_{|\alpha=\beta+\theta}\mathrm{d}x$$

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What is nice of this result is that we have a connection between the Gaussian β ensemble and the Toda lattice.

Remark

The result has been obtained in a non rigorous way and with a different technique by Spohn.

Remark

It is worth to mention that the measure with density $\overline{\mu}_{H_\alpha}$ already appeared in the literature as the orthogonality measure of the associate Hermite polynomials, i.e. the polynomials satisfying

$$xH_n^{(\alpha)}(x) = H_{n+1}^{(\alpha)}(x) + (n+\alpha)H_{n-1}^{(\alpha)}(x), \quad H_{-1}(x,\alpha) = 0, \ H_0(x) = 1 \,.$$

For $\alpha = 0$ one obtains the classical Hermite polynomials.

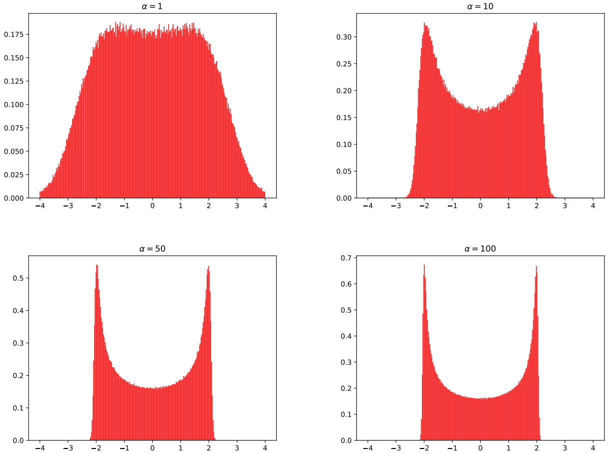
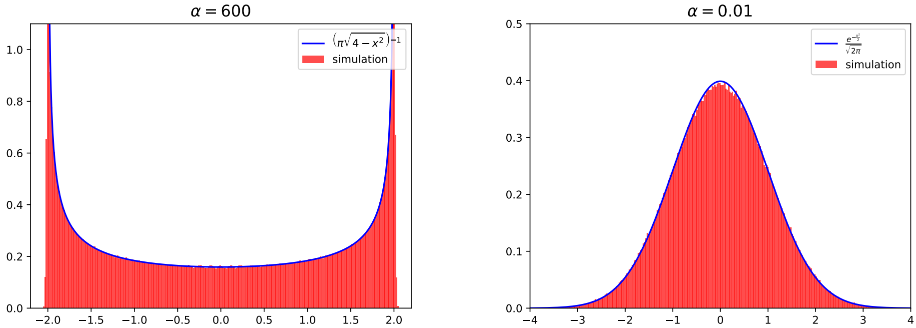


Figura: $H_\alpha/\sqrt{\alpha}$ empirical spectral density for different values of the parameters, $N = 500$, trials: 5000.

$$\lim_{\alpha\to\infty} \overline{\mathrm{d}\nu}_{H_\alpha}(\sqrt{\alpha}x) = \frac{\mathbb{1}_{(-2,2)}}{\pi\sqrt{4-x^2}}\mathrm{d}x\,,$$

$$\lim_{\alpha\to 0} \overline{\mathrm{d}\nu}_{H_\alpha}(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}\mathrm{d}x\,.$$



The Gaussian α ensemble

Theorem

Consider the matrix ensemble defined by H_α,

$$H_\alpha \sim \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{N}(0,2) & \chi_{2\alpha} & & \\ \chi_{2\alpha} & \mathcal{N}(0,2) & \chi_{2\alpha} & \\ & \ddots & \ddots & \ddots \\ & & \chi_{2\alpha} & \mathcal{N}(0,2) \end{pmatrix}.$$

Let $\overline{\mu}_{H_\alpha}(x)\mathrm{d}x$ be its mean spectral measure and $\overline{\mathrm{d}\nu}_{H_\alpha}$ its mean density of states then:

$$\overline{\mathrm{d}\nu}_{H_\alpha} = \partial_\alpha(\alpha\overline{\mu}_{H_\alpha}(x))\mathrm{d}x.$$

Moreover $\overline{\mu}_{H_\alpha}(x)$ is explicitly given by:

$$\overline{\mu}_{H_\alpha}(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \left|\widehat{f}_\alpha(x)\right|^{-2}, \quad \widehat{f}_\alpha(x) := \sqrt{\frac{\alpha}{\Gamma(\alpha)}} \int_0^\infty t^{\alpha-1} e^{-\frac{t^2}{2}} e^{ixt} \mathrm{d}t\,.$$

We notice that $\overline{\mu}_{H_\alpha}(x)$ is a parabolic cylinder function.

Main ideas:

- Prove that the spectral measure of H_α and H_β in the high temperature regime ($\lim_{N\to\infty} \beta N \to 2\alpha$) coincides.
- Use the moment matching technique.

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- Use the moment matching technique.

Natural question: Can we apply the same reasoning to the other β ensembles? Is is possible to define a Laguerre and Jacobi α ensemble as in the Gaussian one and compute their mean density of states?

Laguerre β ensemble

$$L_{\beta,\gamma} = B_{\beta,\gamma} B_{\beta,\gamma}^{\intercal}, \quad B_{\beta,\gamma} = \frac{1}{\sqrt{2}} \begin{pmatrix} x_1 & & & \\ y_1 & x_2 & & \\ & \ddots & \ddots & \\ & & y_{N-1} & x_N \end{pmatrix}$$

$$B_{\beta,\gamma} \in \mathrm{Mat}(N \times M),\ M \geqslant N, \quad \lim_{N\to\infty} M/N = \gamma \in (0,1)$$

$$x_n \sim \chi_{\beta(M-n+1)} \quad n = 1, \ldots, N$$

$$y_n \sim \chi_{\beta(N-n)} \quad n = 1, \ldots, N-1\,.$$

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$$y_n \sim \chi_{\beta(N-n)} \quad n = 1, \dots, N-1\,.$$

In the high temperature regime $\lim_{N\to\infty} \beta N = 2\alpha$, for any fixed k, the upper left $k \times k$ block of $B_{\beta,\gamma}$ weakly converge to:

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \chi_{\frac{2\alpha}{\gamma}} & & & \\ \chi_{2\alpha} & \chi_{\frac{2\alpha}{\gamma}} & & \\ & \ddots & \ddots & \\ & & \chi_{2\alpha} & \chi_{\frac{2\alpha}{\gamma}} \end{pmatrix}$$

Laguerre α ensemble

$$L_{\alpha,\gamma} = B_{\alpha,\gamma} B^{\intercal}_{\alpha,\gamma}, \quad B_{\alpha,\gamma} = \frac{1}{\sqrt{2}} \begin{pmatrix} \chi_{\frac{2\alpha}{\gamma}} & & & & \\ \chi_{2\alpha} & \chi_{\frac{2\alpha}{\gamma}} & & & \\ & \ddots & \ddots & & \\ & & \chi_{2\alpha} & \chi_{\frac{2\alpha}{\gamma}} & \end{pmatrix}, \quad B_{\alpha,\gamma} \in \mathrm{Mat}(N \times M),\, M \geqslant N.$$

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Theorem

Consider the matrix $L_{\alpha,\gamma}$, let $\overline{\mu}_{L_{\alpha,\gamma}}(x)\mathrm{d}x$ be its mean spectral measure and $\overline{\mathrm{d}\nu}_{L_{\alpha,\gamma}}$ its mean density of states then:

$$\overline{\mathrm{d}\nu}_{L_{\alpha,\gamma}} = \partial_\alpha(\alpha \overline{\mu}_{L_{\alpha,\gamma}}(x))\mathrm{d}x.$$

Moreover $\overline{\mu}_{L_{\alpha,\gamma}}(x)$ is explicity given by:

$$\overline{\mu}_{L_{\alpha,\gamma}}(x) := \frac{1}{\Gamma(\alpha+1)\Gamma\left(1+\frac{\alpha}{\gamma}+\alpha\right)} \frac{x^{\frac{\alpha}{\gamma}} e^{-x}}{\left|\psi\left(\alpha, -\frac{\alpha}{\gamma}; x e^{-i\pi}\right)\right|^2} \quad x \geqslant 0\,,$$

Remark

The measure with density $\overline{\mu}_{L_{\alpha,\gamma}}(x)$ already appeared in the literature, indeed H. D. Trinh and K. D. Trinh proved that it is the mean spectral measure of the Laguerre β ensemble in the high temperature regime. Moreover this measure is the orthogonality measure of the so called associate Laguerre polynomials of type II, i.e. the polynomials satisfying

$$L_0(x) = 1\,, \qquad L_1(x) = \frac{\alpha + \frac{\alpha}{\gamma} + 1 - x}{\alpha + 1}\,,$$

$$-xL_n^{\alpha,\gamma}(x) = (n+1+\alpha)L_{n+1}^{\alpha,\gamma} - \left(2n + \frac{\alpha}{\gamma} + \alpha + 1\right)L_n^{\alpha,\gamma}(x) + \left(n + \alpha + \frac{\alpha}{\gamma}\right)L_{n-1}^{\alpha,\gamma}(x)\,.$$

Jacobi β ensemble

$$J_\beta = D_\beta D_\beta^\intercal, \quad D_\beta = \begin{pmatrix} s_1 & & & \\ t_1 & s_2 & & \\ & \ddots & \ddots & \\ & & t_{N-1} & s_N \end{pmatrix}$$

$$t_n = \sqrt{q_n(1-p_n)},\ s_n = \sqrt{p_n(1-q_{n-1})}$$

$$q_n \sim \text{Beta}\left(\frac{\beta(N-n)}{2}, \frac{\beta(N-n)}{2} + a + b + 2\right) \quad (q_0 = 0)$$

$$p_n \sim \text{Beta}\left(\frac{\beta(N-n)}{2} + a + 1, \frac{\beta(N-n)}{2} + b + 1\right)$$

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In the high temperature regime $\lim_{N\to\infty} \beta N = 2\alpha$, for any fixed k

$$\begin{aligned} q_n &\rightharpoonup \text{Beta}\left(\alpha, \alpha + a + b + 2\right) \\ p_n &\rightharpoonup \text{Beta}\left(\alpha + a + 1, \alpha + b + 1\right) \end{aligned}, \quad n = 0, \dots, k.$$

Jacobi α ensemble

$$J_\alpha = D_\alpha D_\alpha^\intercal, \quad D_\alpha = \begin{pmatrix} s_1 & & & \\ t_1 & s_2 & & \\ & \ddots & \ddots & \\ & & t_{N-1} & s_N \end{pmatrix}$$

$$t_n = \sqrt{q_n(1-p_n)},\ s_n = \sqrt{p_n(1-q_{n-1})}$$

$$q_n \sim \operatorname{Beta}(\alpha, \alpha + a + b + 2) \quad (q_0 = 0),$$

$$p_n \sim \operatorname{Beta}(\alpha + a + 1, \alpha + b + 1)$$

Jacobi α ensemble

Theorem

Consider the matrix J_α, let $\overline{\mu}_{J_\alpha}(x)\mathrm{d}x$ be its mean spectral measure and $\overline{\mathrm{d}\nu}_{J_\alpha}$ its mean density of states then:

$$\overline{\mathrm{d}\nu}_{J_\alpha} = \partial_\alpha(\alpha\overline{\mu}_{J_\alpha}(x))\mathrm{d}x.$$

Moreover, for $a, b > -1$ and $a \notin \mathbb{N}$, $\overline{\mu}_{J_\alpha}(x)$ is explicity given by:

$$\overline{\mu}_{J_\alpha}(x) := \frac{\Gamma(\alpha+1)\Gamma(\alpha+a+b+2)}{\Gamma(\alpha+a+1)\Gamma(\alpha+b+1)} \frac{x^a(1-x)^b}{\left|U(x)+e^{i\pi b}V(x)\right|^2} \qquad 0 \leqslant x \leqslant 1,$$

where

$$U(x) := \frac{\Gamma(\alpha+1)\Gamma(a+1)}{\Gamma(1+\alpha+a)} {}_2F_1(\alpha, -\alpha-a-b-1, -a; x)\,,$$

$$V(x) := \frac{-\pi\alpha\Gamma(\alpha+a+b+2)}{\sin(\pi a)\Gamma(1+\alpha+b)\Gamma(a+2)} (1-x)^{b+1}x^{a+1}.$$

$${}_2F_1(1-\alpha, \alpha+a+b+2, 2+a; x)\,,$$

Remark

The measure with density $\overline{\mu}_{J_\alpha}(x)$ already appeared in the literature, indeed H. D. Trinh and K. D. Trinh proved that it is the mean spectral measure of the Jacobi β ensemble in the high temperature regime. Moreover this measure is the orthogonality measure of the so called associate Jacobi polynomials of type III, i.e. the polynomials satisfying

$$xJ_n^{\alpha,a,b}(x) = \sqrt{\xi_n\mu_{n+1}}J_{n+1}^{\alpha,a,b}(x) + (\xi_n + \eta_n)J_n^{\alpha,a,b}(x) + \sqrt{\xi_{n-1}\mu_n}J_{n-1}^{\alpha,a,b}(x)\,,$$

where

$$\begin{cases} \xi_0(\alpha) = \frac{\alpha+a+1}{2\alpha+a+b+2} \\ \xi_n(\alpha) = \frac{n+\alpha+a+1}{2n+2\alpha+a+b+2}\,\frac{n+\alpha+a+b+1}{2n+2\alpha+a+b+1}, \quad n>0 \\ \eta_n(\alpha) = \frac{n+\alpha}{2n+2\alpha+a+b+1}\,\frac{n+\alpha+b}{2n+2\alpha+a+b}, \quad n>0 \end{cases}, \qquad \alpha \geqslant 0,\ a,b>-1\,,$$

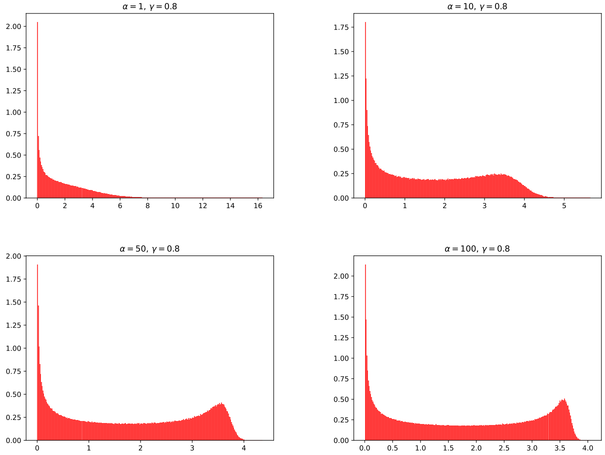


Figura: $L_{\alpha,\gamma}/\sqrt{\alpha}$ empirical spectral density for different values of the parameters, $N = 500$, trials: 5000.

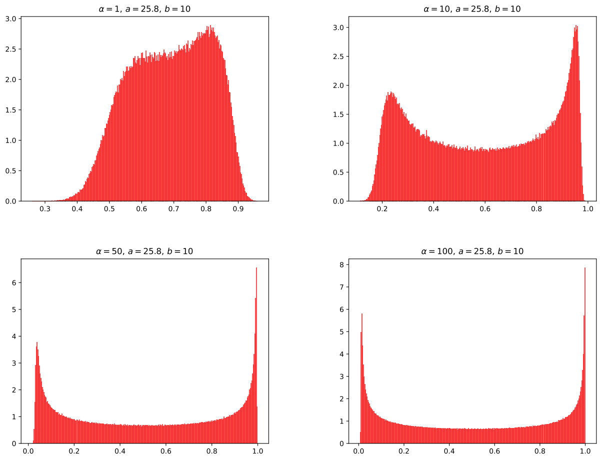
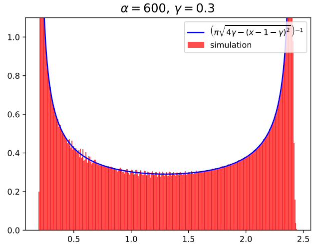


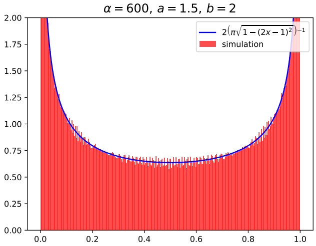
Figura: J_α empirical spectral density for different values of the parameters, $N = 500$, trials: 5000.

Parameter limit

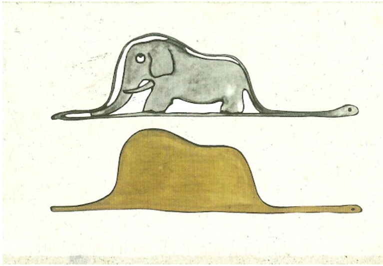
$$\lim_{\alpha\to\infty} \overline{\mathrm{d}\nu}_{L_{\alpha,\gamma}}\left(\frac{\alpha x}{\gamma}\right) = \frac{\mathbb{1}_{((1-\sqrt{\gamma})^2,(1+\sqrt{\gamma})^2)}}{\pi\sqrt{4\gamma-(x-1-\gamma)^2}}\mathrm{d}x\,,$$

$$\lim_{\alpha\to\infty} \overline{\mathrm{d}\nu}_{J_{\alpha}}(x) = \frac{2\mathbb{1}_{(0,1)}}{\pi\sqrt{1-(2x-1)^2}}\mathrm{d}x\,.$$





Thank you for the attention!



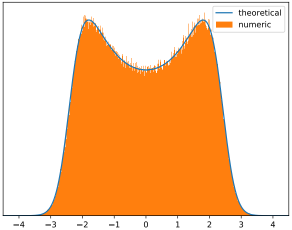


Figura: A hat, a snake eating an elephant and Toda's DOS look the same!