Painlevé equations and anharmonic oscillators

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Overview

1. The problem relating:
   - Rational solutions of **Painlevé II**
   - Degenerate spectrum of **anharmonic oscillator**
2. Link between PII and the anharmonic oscillator
3. Our approach
4. Exact WKB Method

All in ≤ 30 minutes!

Painlevé II: the basics

$$\text{PII} \ : \quad v'' = 2v^3 + tv + \alpha, \quad \alpha \in \mathbb{C}$$

Key property:

- $\exists!$ rational solutions $\iff \alpha = N \in \mathbb{Z}$:

$$v(t) = \frac{\mathrm{d}}{\mathrm{d}t} \log\left(\frac{Y_{N-1}(t)}{Y_N(t)}\right)$$

where $Y_n(t)$ are **Yablonskii-Vorob'ev polynomials**:
$Y_0 = 1, Y_1 = t$ and

$$Y_{N+1} = \frac{tY_N^2 + 4({Y'_N}^2 - Y_N Y''_N)}{Y_{N-1}}$$

(YES, they're polynomials!)

Poles of rational sols of PII $\longleftrightarrow$ **Roots** of Yablonski-Vorob'ev poly

Roots of Y_N

Question: What do the roots of $Y_n(t)$ look like?

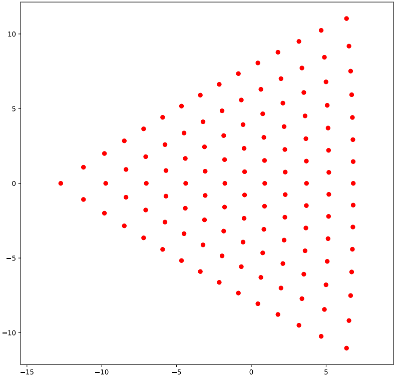

Figure: Roots of $Y_{15}(t)$

- Buckingham, Miller (2013): Large N asymptotic analysis via JM Lax pair of PII
- Bertola, Bothner (2015): Hankel determinant expression for Y_N^2, RHP analysis of (pseudo) orthogonal polynomials

Anharmonic oscillator

Eigenvalue problem:

$$y'' - \left(\frac{x^4}{4} - \frac{ax^2}{2} - (N+1)x\right) y = \lambda y, \quad y(re^{\pm i\pi/3}) \xrightarrow[r\to\infty]{} 0.$$

Anharmonic oscillator

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If $N \in \mathbb{N}$ there exists **quasi-polynomial** solutions:

$$y(x) := p(x)e^{-\frac{x^3}{6}+\frac{ax}{2}}, \quad p(x) \text{ polynomial of } \deg(p) \leq N.$$

Question: for which $a \in \mathbb{C}$ are these eigenvalues λ degenerate?
Answer: Obtain such $a \in \mathbb{C}$ as zeros of a discriminant :

$$D_n(a) = \text{poly of} \deg N(N+1)/2.$$

Roots of the discriminant

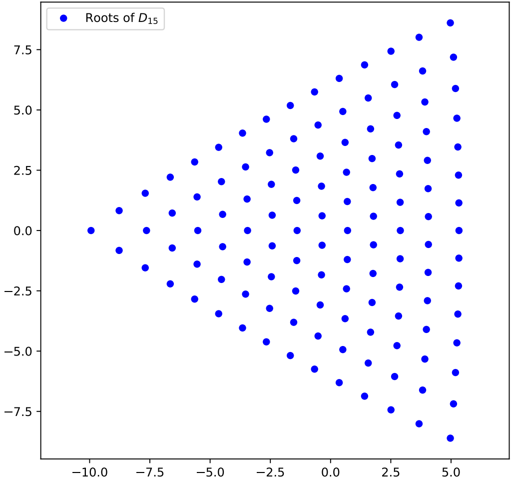


Figure: The answer seems familiar...

Suspicious coincidence

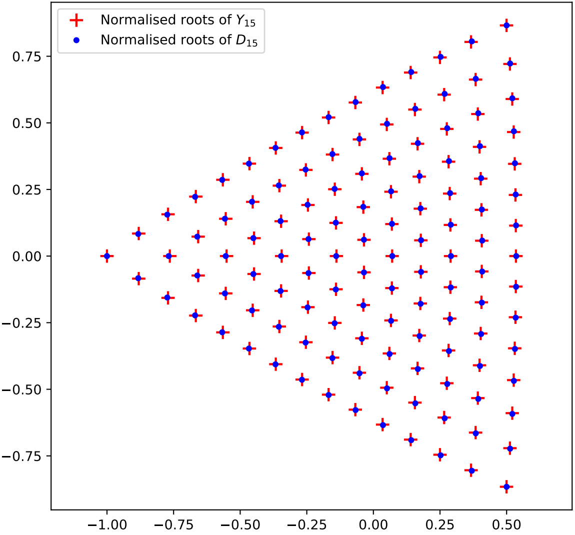


Figure: This image should surprise you

Shapiro-Tater conjecture

Shapiro-Tater conjecture (2014)

(After appropriate scaling) the sets

$$\{\text{roots of } D_N(a) = 0\} \text{ and } \{\text{roots of } Y_N(t) = 0\}$$

coincide as $N \to \infty$.

Shapiro-Tater results:

- Support of counting measure for the 'algebraic' eigenvalues.
- Partial results on monodromy of eigenvalues.
- Only numerical evidence towards conjecture.

Coincidence? Unlikely!

Searching for a link

Lax pair at the pole $t = a$ of transcendent $\longrightarrow$ Anharmonic oscillator

$$\begin{cases} \Phi_x = A(x,t)\Phi \\ \Phi_t = B(x,t)\Phi \end{cases} \quad \longrightarrow \quad y'' = Q(x)y$$

Previous works:

- Its, Novokshenov (1986)
  PII(0) - Flashcka-Newell $\longrightarrow \quad Q(x) = 16x^4 + 8ax^2 + \lambda$
- Masoero (2010)
  PI - tritronquée solution $\longrightarrow \quad Q(x) = 4x^3 - 2ax - 28b$

where a =pole.

Analysis of anharmonic oscillators leads to **asymptotic description of poles**.

Jimbo-Miwa Lax pair

Jimbo-Miwa (1981) gave a Lax pair for PII:

$$\Phi_x = \left(x^2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + x \begin{bmatrix} 0 & u \\ -2u^{-1}w & 0 \end{bmatrix} + \begin{bmatrix} w + \frac{t}{2} & -uv \\ -2u^{-1}(vw + \theta) & -w - \frac{t}{2} \end{bmatrix} \right) \Phi$$

$$\Phi_t = \left(\frac{x}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 0 & u \\ -2u^{-1}w & \end{bmatrix} \right) \Phi, \qquad \theta \in \mathbb{C}$$

where $u = u(t), v = v(t), w = w(t)$ are functions of t.
If $A_t - B_x + [A, B] = 0$, they satisfy:

$$u_t = -vu, \quad v_t = v^2 + w + \frac{t}{2}, \quad w_t = -2vw - \theta.$$

and $v(t)$ solves PII$(-\theta + 1/2)$.

Question: How to get anharmonic oscillator from here?

Reduction at pole

Transform the differential equation: $\Phi_x(x,t) = A(x,t)\Phi(x,t)$ by setting

$$\widehat{\Phi}(x,t) = G(x,t)\Phi(x,t)$$

such that the system becomes:

$$\widehat{\Phi}_x(x,t) = \begin{bmatrix} 0 & 1 \\ Q(x,t) & 0 \end{bmatrix} \widehat{\Phi}(x).$$

$\widehat{\Phi} = [y(x,t), y_x(x,t)]^\top \longrightarrow$ scalar ODE: $\quad y'' = Q(x,t)y.$

Near a pole of PII $t = a$, the potential simplifies:

$$\lim_{t \to a} Q(x;t) = x^4 - ax^2 + 2\theta x - \left(-10b + \frac{7}{36}a^2\right) =: Q_{\mathrm{JM}}(x).$$

Found link between PII and the anharmonic oscillator:

$$
\begin{aligned}
Q_{\mathrm{ST}} =& \frac{x^4}{4} - \frac{ax^2}{2} - (N+1)x - \Lambda \\
& \updownarrow \\
Q_{\mathrm{JM}} =& x^4 - ax^2 + 2\theta x - \lambda
\end{aligned}
$$

Only beginning of the story.

Need to understand the **quasi-polynomials** and **repeated eigenvalue** conditions.

Approach

Implement conditions for existence of

1. quasi-polynomials (QP),
2. repeated eigenvalues (RE)

asymptotically for large N, to obtain quantization conditions:

$$\pi i(2k+1) = N \oint_{\gamma} \sqrt{x^4 - ax^2 + 2x - \lambda}\,\mathrm{d}x$$

which implicitly determine (a, λ) such that we have QP and RE. Then compare with similar quantization conditions for Yablonski-Vorob'ev.

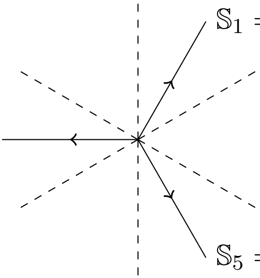
What are these conditions?

Quasi-polynomial condition

A necessary condition to have quasi-polynomial solutions of

$$y''(x) = \left(\frac{x^4}{4} - \frac{ax^2}{2} - (N+1)x - \Lambda \right) y(x).$$

is the **vanishing of the Stokes matrices**. The solutions $[P(x), Q_i(x)]$ give explicit Stokes phenomenon:



$$\mathbb{S}_1 = \begin{bmatrix} 1 & s_1 \\ 0 & 1 \end{bmatrix}$$

$$\mathbb{S}_3 = \begin{bmatrix} 1 & 0 \\ s_3 & 1 \end{bmatrix}$$

$$\mathbb{S}_5 = \begin{bmatrix} 1 & s_{-1} \\ 0 & 1 \end{bmatrix}$$

where the Stokes factors are:

$$s_i = \int \frac{1}{P_N(x)^2} e^{x^3/3 - ax} \mathrm{d}x$$

Repeated eigenvalue condition

If $p(x)e^{\theta(x)}$ is a quasi-polynomial solution, a necessary condition for the existence of repeated eigenvalues is:

$$\int_\Gamma p(x)^2 e^{2\theta(x)}\, \mathrm{d}x = 0,$$

where Γ is the contour:

Question: how to turn these *exact* conditions into *asymptotic* conditions?

Exact WKB method

The **exact WKB method** in the spirit of Voros (94) and Kawai, Takei (05).

$$y'' = n^2 \left(\frac{x^4}{4} - \frac{ax^2}{2} - x - \Lambda \right) f = n^2 Q(x) y$$

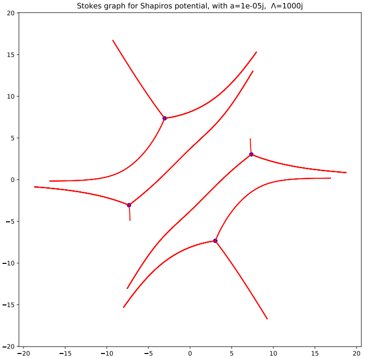
Ansatz $y(x) = e^{\int^x S(u,n)\mathrm{d}u}$ where $S(x,n) = nS_{-1} + S_0 + n^{-1}S_1 + \dots$

$$\implies \psi_{\pm}^{(\tau)}(x,n) = \frac{1}{S_{\mathrm{odd}}(x,n)^{1/2}} \exp\left(\pm \int_{\tau}^{x} S_{\mathrm{odd}}(u,n)\mathrm{d}u \right)$$
$$= \frac{n^{-1/2}}{V(x)^{1/4}} \exp\left(\pm n \int_{\tau}^{x} \sqrt{Q(u)}\mathrm{d}u \right) \left(1 + \mathcal{O}\left(n^{-1}\right)\right)$$

The WKB solutions $\psi_{\pm}$ are asymptotic to actual solutions **in certain regions**.

Generic Stokes graphs

Stokes lines are the level set $\mathrm{Im}\int_\tau^x \sqrt{Q(u)}\mathrm{d}u = 0$, $\tau =$root of $Q(u)$



Theorem

For every region $\mathcal{D}$ and turning point τ, there exists a unique vector solution $\boldsymbol{y}$ to ODE s.t. $\boldsymbol{y}(x) \sim [\psi_+^{(\tau)}(x,n), \psi_-^{(\tau)}(x,n)]$ in $\mathcal{D}$ as $n \to \infty$.

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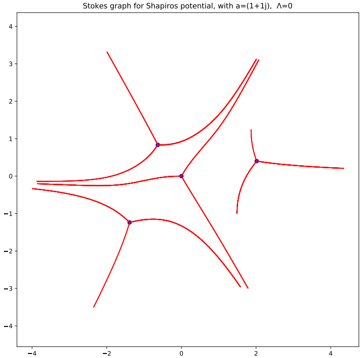

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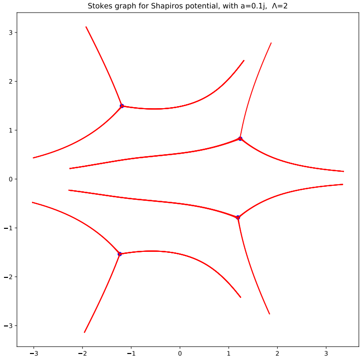

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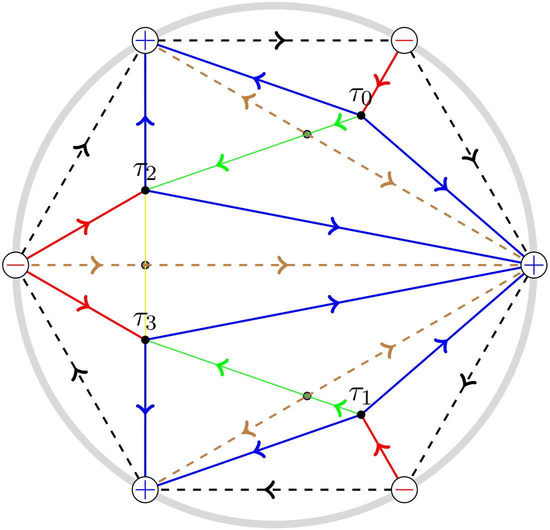
RHP for WKB solutions

Jumps across Stokes lines are known (Voros 83) so we can write a RHP.



Computation of **Stokes matrices** using:

$$\begin{bmatrix} 1 & 0 \\ -i & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix},$$

$$e^{\sigma_3 V_{a,b}}, \quad e^{\sigma_3 W_a},$$

$$V_{i,j} = \frac{1}{2} \int_{\tau_i}^{\tau_j} S_{\text{odd}}(x, n) \mathrm{d}x,$$

$$W_i = \frac{1}{2} \int_{\tau_i}^{\infty} S_{\text{odd}}^{\text{reg}}(x, n) \mathrm{d}x.$$

Stokes matrices

$$S_0 = \begin{bmatrix} 1 & 0 \\ -ie^{2W_0}(1+e^{-2V_{1,0}}) & 1 \end{bmatrix}, \quad S_3 = \begin{bmatrix} 1 & -ie^{-2W_3}X \\ 0 & 1 \end{bmatrix},$$

$$S_1 = \begin{bmatrix} 1 & -ie^{-2W_2}(1+e^{2V_{0,2}}) \\ 0 & 1 \end{bmatrix}, \quad S_4 = \begin{bmatrix} 1 & 0 \\ -ie^{2W_3} & 0 \end{bmatrix},$$

$$S_2 = \begin{bmatrix} 1 & 0 \\ -ie^{2W_2} & 0 \end{bmatrix}, \quad S_5 = \begin{bmatrix} 1 & -ie^{-2W_1}(1+e^{2V_{3,1}}) \\ 0 & 1 \end{bmatrix}.$$

where $X = 1 + e^{-2V_{3,1}} + e^{-2(V_{3,1}+V_{1,0})} + e^{-2(V_{3,1}+V_{1,0}+V_{0,2})}$.
Stokes matrices vanish if and only if

$$1 + e^{2V_{0,2}} = 0 \iff \pi i\,(2k+1) = V_{0,2} = \int_{\gamma_{0,2}} S_{\mathrm{odd}}(x,n)\mathrm{d}x,$$

$$1 + e^{2V_{1,3}} = 0 \iff \pi i\,(2l+1) = V_{1,3} = \int_{\gamma_{1,3}} S_{\mathrm{odd}}(x,n)\mathrm{d}x,$$

What's next? Implement (RE) condition asympotically with exact WKB. ***Work in progress***

Summary

1. The conjecture of Shapiro-Tater
2. Connection between Lax pairs near poles $\longleftrightarrow$ anharmonic oscillators
3. Study of quasi-polynomials and repeated eigenvalues.
4. Implement asymptotically via exact WKB
5. Work in progress - stay tuned!

Thank you for listening!