Sorting networks, staircase Young tableaux and last passage percolation

Fabio Deelan Cunden (SISSA)

based on joint works arXiv:2003.03331, 2005.02043 with

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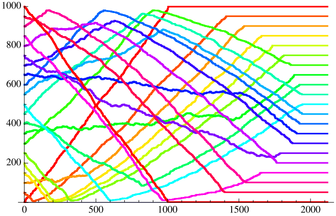
Shane Gibbons (University College Dublin)
Dan Romik (University of California, Davis)

Integrable systems around the world
14-16 September 2020

Outline

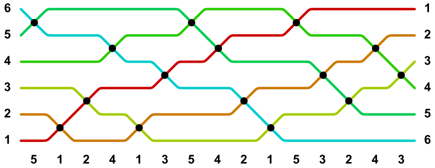
I will discuss a few fun random processes related to **random Young tableaux** and **random sorting networks**.

1	3	4	7	11	16
2	6	8	14	17	
5	12	15	19		
9	13	18			
10	20				
21					

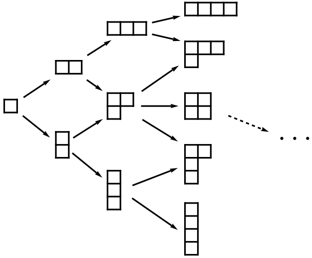


- The main new phenomenon is a set of distributional identities between the **finishing times** of the different processes.
- The study of these identities leads to interesting algebraic combinatorics and involves the **RSK**, **Burge**, and **Edelman-Greene** correspondences.
- Emergent **random matrix distributions**.

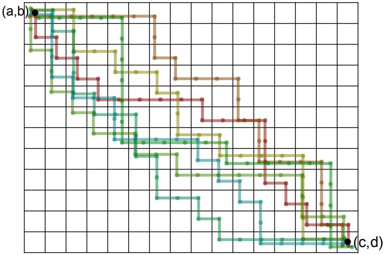
Three continuous time random processes



Oriented Swap Process (OSP)



Corner Growth Process (CGP)



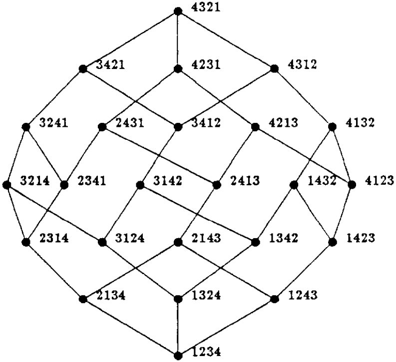
Last Passage Percolation (LPP)

Background: sorting networks

The symmetric group S_n with Coxeter generators $\tau_i = (i\ \ i+1), \quad i = 1, \ldots, n-1.$

(Every permutation σ can be written as a product of **adjacent swaps** $\sigma = \tau_{i_1}\tau_{i_2}\cdots\tau_{i_k}$.)

Consider the Cayley graph (*permutahedron*):



Background: sorting networks

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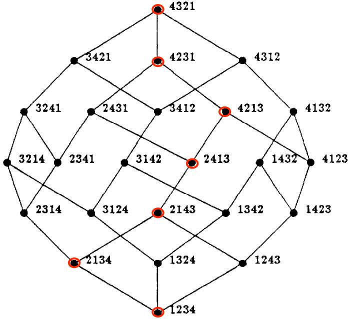
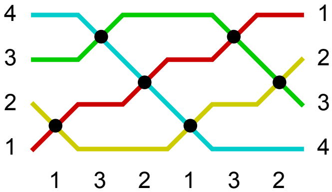
A sorting network of order n is a path of minimal length in the permutahedron from $\mathrm{id}_n = [1, 2, \ldots, n]$ to $\mathrm{rev}_n = [n, \ldots, 2, 1]$.

Can be encoded as reduced decompositions of rev_n.

Ex. $[1,2,3,4] \xrightarrow{\tau_1} [2,1,3,4] \xrightarrow{\tau_3} [2,1,4,3] \xrightarrow{\tau_2} [2,4,1,3] \xrightarrow{\tau_1} [4,2,1,3] \xrightarrow{\tau_3} [4,2,3,1] \xrightarrow{\tau_2} [4,3,2,1].$





$\mathrm{rev}_4 = [4,3,2,1] = \tau_1\tau_3\tau_2\tau_1\tau_3\tau_2 (= 132132)$

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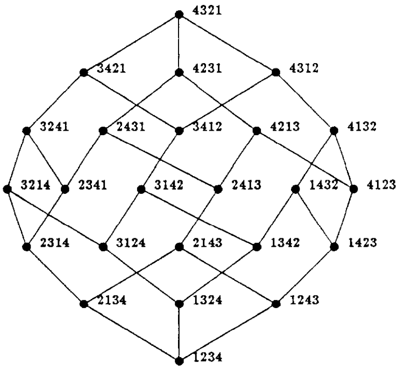
Ex.



$$\mathrm{SN}_4 = \{123121, 121321, 212321, 231231, 213231, 123212, 312312, 132312, 312132, 132132, 321232, 231213, 213213, 232123, 323123, 321323\}$$

Background: staircase shape Young tableaux

Identify integer partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ with Young diagrams.
Denote by $\mathrm{SYT}(\lambda)$ the set of standard[†] Young tableaux of shape λ.

Ex. $\lambda = (4, 3, 1) \vdash 8$.

1	2	4	8
3	6	7	
5			

[†] labelling of a diagrams, strictly increasing along rows and columns.

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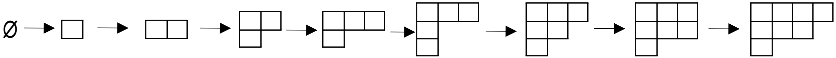
Standard Young tableaux encode growths of Young diagrams. In the example:



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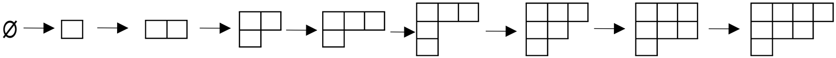

Theorem (Hook-length formula) [Frame-Robinson-Thrall, 1953]

$$|\mathrm{SYT}(\lambda)| = \frac{|\lambda|!}{\prod_{(i,j)\in\lambda} h_{ij}}$$

[†] labelling of a diagrams, strictly increasing along rows and columns.

Background: staircase shape Young tableaux

In this talk, $\mathrm{SYT}(\delta_n)$ denote the set of standard Young tableaux of shape $\delta_n = (n-1, n-2, \ldots, 2, 1)$, aka **staircase shape Young tableaux** of order n.

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Sorting networks and staircase shape Young tableaux

Theorem [Stanley, 1984]

$$|\mathrm{SN}_n| = |\mathrm{SYT}(\delta_n)|.$$

Sorting networks and staircase shape Young tableaux

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Theorem [Edelman-Greene, 1987]

Combinatorial bijection

$$\mathrm{SN}_n \overset{1:1}{\longleftrightarrow} \mathrm{SYT}(\delta_n).$$

Based on a generalised RSK algorithm (Coxeter-Knuth insertion).

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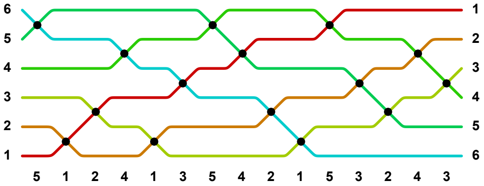
$\overset{\mathrm{EG}}{\longmapsto}$



Random sorting networks

The oriented swap process (Angel-Holroyd-Romik, 2009):

- At $t = 0$ start from particles labelled $\mathrm{id}_n = (1, 2, \ldots, n)$;
- Independent Poisson clocks between adjacent positions;
- When a clock rings, the adjacent particles attempt to swap position. If they are in increasing order, they swap, otherwise, they do not;
- The absorbing state is $\mathrm{rev}_n = (n, n-1, \ldots, 2, 1)$;
- Continuous time random walk on the permutahedron.

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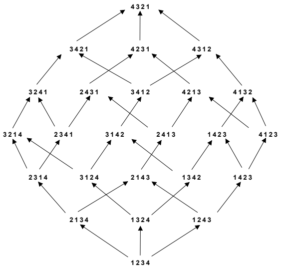
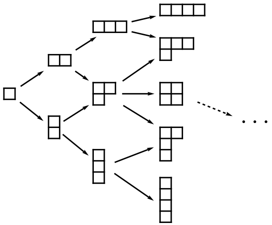


To each transition edge in the graph is associated an Exp(1)-distributed waiting time, all such times being independent.

Randomly growing Young staircase shape

Our model for randomly growing Young diagrams is the CGP (Rost 1980, …), in a continuous time version where each new box that can be added to the existing Young diagram gets added at a random time following a Poisson clock (independently of all other clocks).

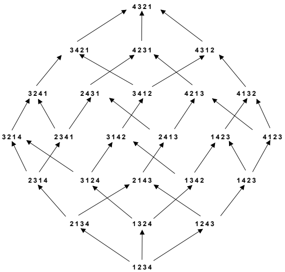


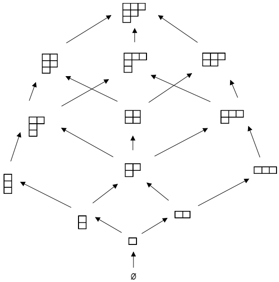
- continuous time random walk on $\mathcal{Y}$ (to each edge is associated an Exp(1) random variable).

We stop the process when the staircase shape $\delta_n = (n-1, n-2, \ldots, 2, 1)$ is reached.

A mysterious equidistribution phenomenon

Consider the continuous-time simple random walks on these graphs:





$\mathcal{U}_n$ = absorbing time of the random walk on $\mathcal{P}_n$.

$\mathcal{V}_n$ = absorbing time of the random walk on $\mathcal{Y}(\delta_n)$.

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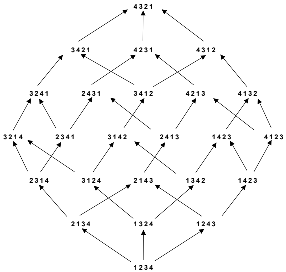

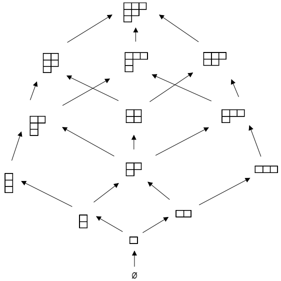

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$\mathcal{V}_n$ = absorbing time of the random walk on $\mathcal{Y}(\delta_n)$.

Conjecture [Bisi-Cunden-Gibbons-Romik]

The equality in distribution $\mathcal{U}_n \stackrel{\mathrm{d}}{=} \mathcal{V}_n$ holds for all $n \geq 2$.

(Later in this talk: connections to RMT and recent progresses)

Finishing times

The conjecture $\mathcal{U}_n \overset{\mathrm{d}}{=} \mathcal{V}_n$ follows from a more detailed equidistribution phenomenon involving the random finishing times

$$\boldsymbol{U}_n = (U_n(1), \ldots, U_n(n-1)) \quad \text{and} \quad \boldsymbol{V}_n = (V_n(1), \ldots, V_n(n-1)).$$

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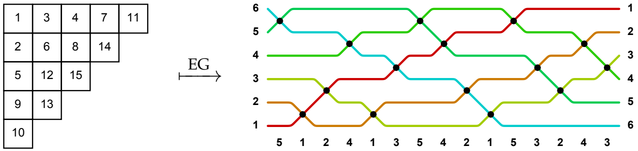
<u>Ex.</u> In a simpler discrete setting:



$\boldsymbol{U}_n = (10, 13, 15, 14, 11)$ and $\boldsymbol{V}_n = (10, 13, 15, 14, 11)$
(in this case the two vectors are equal by Edelman-Greene).

Finishing times

$$\boldsymbol{U}_n = (U_n(1), \ldots, U_n(n-1)) \quad \text{and} \quad \boldsymbol{V}_n = (V_n(1), \ldots, V_n(n-1)).$$

- $U_n(k)$ = time of last swap between positions k and $k+1$ in OSP;
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Conjecture [BCGR]

The equidistribution $\boldsymbol{U}_n \overset{\mathrm{d}}{=} \boldsymbol{V}_n$ holds for all $n \geq 2$.

Finishing times

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Conjecture [BCGR]

The equidistribution $\boldsymbol{U}_n \overset{\mathrm{d}}{=} \boldsymbol{V}_n$ holds for all $n \geq 2$.

Note: $\mathcal{U}_n = \max_{1\le k\le n-1} U_n(k)$, and $\mathcal{V}_n = \max_{1\le k\le n-1} V_n(k)$ are the absorbing times.

Sanity check:

Theorem [Angel-Holroyd-Romik, 2009]

$U_n(k) \overset{\mathrm{d}}{=} V_n(k)$, for all n,k.

Proof based on a coupling of the oriented swap process to a family of TASEPs.
(This is a much weaker statement.)

Combinatorial identity

By taking a Fourier transform (basically), the conjecture can be recast as a purely combinatorial identity - a kind of weighted, vector-valued version of Stanley's equi-enumeration result.

Theorem [BCGR]

The equidistribution conjecture $\boldsymbol{U}_n \stackrel{\mathrm{d}}{=} \boldsymbol{V}_n$ is equivalent to the identity of $\mathbb{C}[x_1,\ldots,x_{n-1}]S_{n-1}$-valued generating functions

$$F_n(x_1,\ldots,x_{n-1}) = G_n(x_1,\ldots,x_{n-1})$$

$$F_n = \sum_{t \in \mathrm{SYT}(\delta_n)} f_t(x_1,\ldots,x_{n-1})\,\sigma_t$$

$f_t(x_1,\ldots,x_{n-1})$ = the **generating factor** of t
σ_t = the **finishing permutation** of t

$$G_n = \sum_{s \in \mathrm{SN}_n} g_s(x_1,\ldots,x_{n-1})\,\pi_s$$

$g_s(x_1,\ldots,x_{n-1})$ = the **generating factor** of s
π_s = the **finishing permutation** of s

(See the paper for precise definitions.)

Example

For the tableau t and the sorting network $s = \mathrm{EG}(t)$ in the running example,

1	3	4	7	11
2	6	8	14	
5	12	15		
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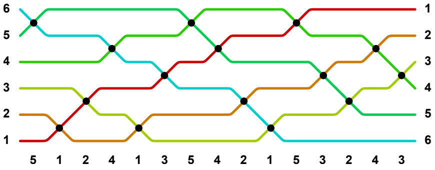
10				

$\xmapsto{\mathrm{EG}}$



$$\sigma_t = (1, 3, 5, 4, 2) \quad \text{the order of the `finishing times' of } t$$

$$f_t = \frac{1}{(x_1+1)(x_1+2)^2(x_1+3)^3(x_1+4)^4} \cdot \frac{1}{x_2+3} \cdot \frac{1}{(x_3+2)(x_3+3)} \cdot \frac{1}{x_4+2} \cdot \frac{1}{x_5+1}.$$

$$\pi_s = (1, 3, 5, 4, 2) \quad \text{the order of the `finishing times' of } s$$

$$g_s = \frac{1}{(x_1+5)(x_1+4)(x_1+3)^5(x_1+2)^3} \cdot \frac{1}{x_2+2} \cdot \frac{1}{(x_3+1)(x_3+2)} \cdot \frac{1}{x_4+1} \cdot \frac{1}{x_5+1}.$$

Note that $\sigma_t = \pi_s$ (by Edelman-Green) but $f_t \neq g_s$.

Combinatorial version of the conjecture

Using the combinatorial reformulation we were able to compute the generating functions and verify the identity for small values of n (computer-assisted proof).

Theorem [BCGR]

The generating function equality

$$F_n(x_1, \ldots, x_{n-1}) = G_n(x_1, \ldots, x_{n-1})$$

and hence the equidistribution relation

$$\boldsymbol{U}_n \stackrel{\mathrm{d}}{=} \boldsymbol{V}_n$$

hold for $n \leq 6$.

The absorbing time of the OSP

Angel-Holroyd-Romik (2009) posed the question:

An interesting open problem would be to find sequences of scaling constants (a_n), (b_n) *and a distribution function* F *such that [...]*

$$a_n(\mathcal{U}_n - b_n) \xrightarrow[n\to\infty]{d} F.$$

The absorbing time of the OSP

- $\mathcal{U}_n$ = absorbing time of the OSP, $\mathcal{V}_n$ = point-to-line LPP time.

Corollary [BCGR]

Assuming the conjecture,

(i) $P\left(\mathcal{U}_n \le t\right) = \dfrac{1}{C_n} \displaystyle\int_0^t \cdots \int_0^t \prod_{1 \le i < j \le n-1} \left| y_i - y_j \right| \prod_{i=1}^{n-1} \mathrm{e}^{-y_i} \, \mathrm{d}y_i \,,$

(ii) $P\left(\dfrac{\mathcal{U}_n - 2n}{(2n)^{1/3}} \le t \right) \xrightarrow[n \to \infty]{} F_1(t)$

- $\mathcal{U}_n$ is distributed as the max eigenvalue of LOE;
- F_1: Tracy-Widom $\beta = 1$ distribution.

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Theorem [Bufetov-Gorin-Romik, 2020]

$\mathcal{U}_n \stackrel{\mathrm{d}}{=} \mathcal{V}_n$ (hence (i) and (ii) are true unconditionally).
(Proof based on results by Borodin-Gorin-Wheeler (2019) for multicoloured TASEPs.)

Last passage percolation

The empirical discovery that $\boldsymbol{U}_n \overset{\mathrm{d}}{=} \boldsymbol{V}_n$ led us to to discover yet another vector $\boldsymbol{W}_n$ with the same distribution. This vector can be thought of as a kind of dual to $\boldsymbol{V}_n$.

To define $\boldsymbol{W}_n$, we first need to reinterpret the model of randomly growing Young diagrams as a last passage percolation model.

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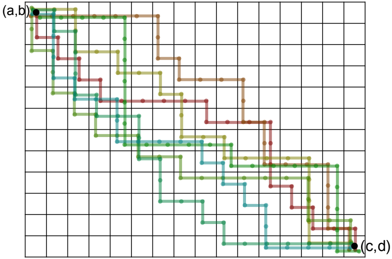
To define $\boldsymbol{W}_n$, we first need to reinterpret the model of randomly growing Young diagrams as a last passage percolation model.

Last passage percolation (LPP) from (a,b) to (c,d):

$$L(a,b;c,d) = \max_{\pi:\,(a,b)\to(c,d)} \sum_{(i,j)\in\pi} X_{i,j}$$

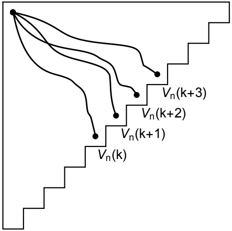


- the max is over directed paths π from (a,b) to (c,d);
- $(X_{i,j})_{i,j}$ is a random environment of i.i.d. Exp(1) waiting times.

Last passage percolation

From the standard theory we can redefine V_n in terms of point-to-line LPP times by

$$V_n = (L(1,1;n-1,1), L(1,1;n-2,2), \dots, L(1,1;1,n-1)) =$$



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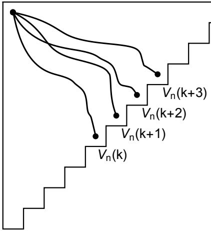
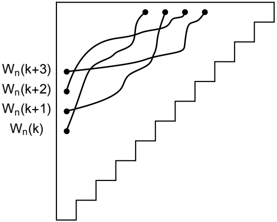


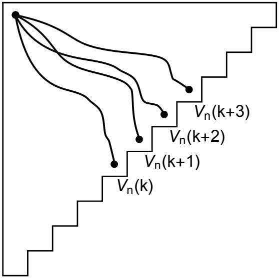
and, by analogy, we define W_n as line-to-line LPP times

$$W_n = (L(n-1,1;1,1), L(n-2,1;1,2), \dots, L(1,1;1,n-1)) =$$

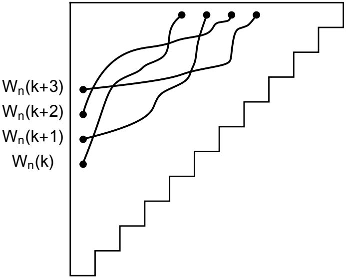


LPP and LPP*

- $V_n = (L(1,1;n-k,k))_{k=1}^{n-1}$
- $W_n = (L(k,1;1,n-k)))_{k=1}^{n-1}$



Point−to−line LPP



Line−to−line LPP

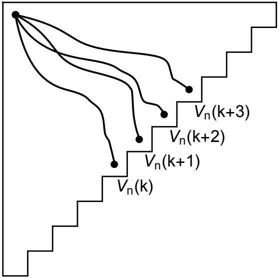
It is clear that $V_n(k) \stackrel{\mathrm{d}}{=} W_n(k)$ (by definition + symmetry).
A much stronger and non-trivial result holds...

Theorem [BCGR]

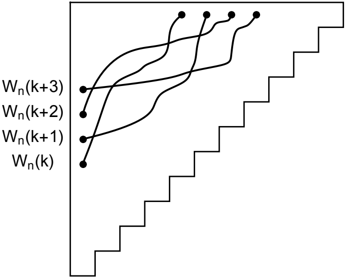
The equality in distribution $\boldsymbol{V}_n \stackrel{\mathrm{d}}{=} \boldsymbol{W}_n$ holds for all n.

LPP and LPP*

- $V_n = (L(1,1;n-k,k))_{k=1}^{n-1}$
- $W_n = (L(k,1;1,n-k)))_{k=1}^{n-1}$



Point−to−line LPP



Line−to−line LPP

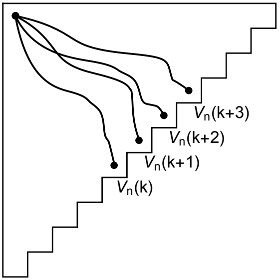
Theorem [BCGR]

The equality in distribution $V_n \stackrel{d}{=} W_n$ holds for all n.

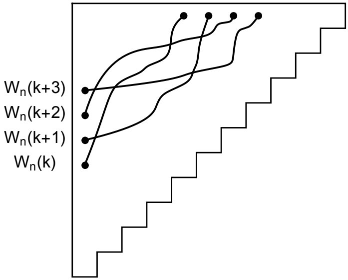
Proof based on RSK and Burge correspondences (Krattenthaler (2006), Bisi-O'Connell-Zygouras (2020)). In a discrete setting of i.i.d. geometric weights, V_n and W_n arise as border entries of the RSK and Burge output tableaux. A short calculation shows that these two tableaux have the same distribution. Take the limit to get the result for exponential weights.

LPP and LPP*

- $\boldsymbol{V}_n = (L(1,1;n-k,k))_{k=1}^{n-1}$
- $\boldsymbol{W}_n = (L(k,1;1,n-k)))_{k=1}^{n-1}$



Point−to−line LPP



Line−to−line LPP

Theorem [BCGR]

The equality in distribution $\boldsymbol{V}_n \stackrel{\mathrm{d}}{=} \boldsymbol{W}_n$ holds for all n.

This is a special case of ‘hidden’ distributional symmetries for LPP conjectured by Borodin-Gorin-Wheeler. Recent proof of general conjecture by Dauvergne (2020).

Take-home messages

We discovered new connections between the models described above. Specifically, an interesting pair of distributional identities

$$\boldsymbol{U}_n \stackrel{\mathrm{d}}{=} \boldsymbol{V}_n \stackrel{\mathrm{d}}{=} \boldsymbol{W}_n$$

of random $(n-1)$-vectors of times for these random processes:

$\boldsymbol{U}_n$	$\boldsymbol{V}_n$	$\boldsymbol{W}_n$
OSP	CGP/LPP	LPP*

Take-home messages

We discovered new connections between the models described above. Specifically, an interesting pair of distributional identities

$$\boldsymbol{U}_n \stackrel{\mathrm{d}}{=} \boldsymbol{V}_n \stackrel{\mathrm{d}}{=} \boldsymbol{W}_n$$

of random $(n-1)$-vectors of times for these random processes:

$\boldsymbol{U}_n$	$\boldsymbol{V}_n$	$\boldsymbol{W}_n$
OSP	CGP/LPP	LPP*

Results:

- We proved that $\boldsymbol{V}_n \stackrel{\mathrm{d}}{=} \boldsymbol{W}_n$;
    - Proof based on the duality between the RSK and Burge correspondences;
    - Cf. 'hidden distributional symmetries' for LPP, polymers and related models.
- We conjectured that $\boldsymbol{U}_n \stackrel{\mathrm{d}}{=} \boldsymbol{V}_n$;
    - It implies that the absorbing time of the OSP has a random matrix distribution;
    - It can be expressed as a purely combinatorial identity of generating functions (related to Edelman-Greene correspondence);
    - Strong evidence supporting the conjecture: true for the marginals [AHR09]; true for $n \leq 6$ [BCGR19]; true for the maxima [BGR20].

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Appendix

Coupling between the oriented swap process and TASEPs

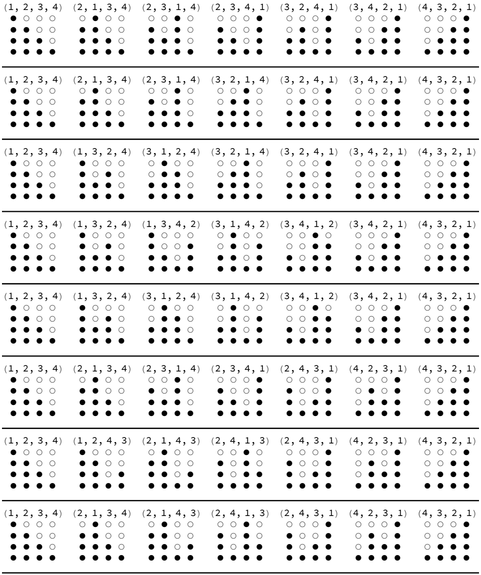
$n = 3$:

(1, 2, 3)	(2, 1, 3)	(2, 3, 1)	(3, 2, 1)
● ○ ○	○ ● ○	○ ○ ●	○ ○ ●
● ● ○	● ● ○	● ○ ●	○ ● ●
● ● ●	● ● ●	● ● ●	● ● ●

(1, 2, 3)	(1, 3, 2)	(3, 1, 2)	(3, 2, 1)
● ○ ○	● ○ ○	○ ● ○	○ ○ ●
● ● ○	● ○ ●	○ ● ●	○ ● ●
● ● ●	● ● ●	● ● ●	● ● ●

Coupling between the oriented swap process and TASEPs

$n = 4$:

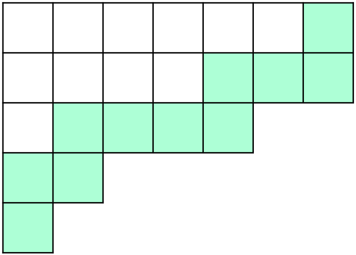


Robinson-Schensted-Knuth and Burge correspondences

RSK and Bur can be seen as bijections on (real) tableaux of shape λ:

$$x = \{x_{i,j} : (i,j) \in \lambda\} \overset{\mathrm{RSK}}{\longmapsto} r = \{r_{i,j} : (i,j) \in \lambda\}$$

$$x = \{x_{i,j} : (i,j) \in \lambda\} \overset{\mathrm{Bur}}{\longmapsto} b = \{b_{i,j} : (i,j) \in \lambda\}$$



If (m,n) is on the border strip:

$$r_{m,n} = \max_{\pi : (1,1) \to (m,n)} \sum_{(i,j) \in \pi} x_{i,j}$$

$$b_{m,n} = \max_{\pi : (m,1) \to (1,n)} \sum_{(i,j) \in \pi} x_{i,j}$$

$\longrightarrow$ “deterministic” LPP times!

Robinson-Schensted-Knuth and Burge correspondences

Ex.

$$\begin{array}{cccc} 0 & 1 & 5 & 3 \\ 0 & 0 & 2 & \\ 1 & 1 & 2 & \\ 3 & 0 & 1 & \\ 4 & & & \end{array} \quad \xmapsto{\text{RSK}} \quad \begin{array}{cccc} 0 & 0 & 6 & 9 \\ 1 & 2 & 8 & \\ 2 & 4 & 10 & \\ 4 & 4 & 11 & \\ 8 & & & \end{array}$$

$$\begin{array}{cccc} 0 & 1 & 5 & 3 \\ 0 & 0 & 2 & \\ 1 & 1 & 2 & \\ 3 & 0 & 1 & \\ 4 & & & \end{array} \quad \xmapsto{\text{Bur}} \quad \begin{array}{cccc} 0 & 1 & 6 & 9 \\ 0 & 1 & 7 & \\ 0 & 2 & 11 & \\ 4 & 6 & 14 & \\ 8 & & & \end{array}$$

Point-to-line and line-to-line LPP vectors

Lemma [BCGR]

If X is a random tableau of shape λ with i.i.d. geometric or exponential entries, then $\mathrm{RSK}(X) \stackrel{\mathrm{d}}{=} \mathrm{Bur}(X)$.

Taking $\lambda = \delta_n = (n-1, n-2, \ldots, 1)$ and $X_{i,j} \sim \mathrm{Exp}(1)$:

$$(\mathrm{RSK}(X)_{n-k,k})_k \stackrel{\mathrm{d}}{=} (\mathrm{Bur}(X)_{n-k,k})_k$$

$$(L(1,1;n-k,k))_k \stackrel{\mathrm{d}}{=} (L(n-k,1;1,k))_k$$

$$\boldsymbol{V}_n \stackrel{\mathrm{d}}{=} \boldsymbol{W}_n$$