Integrable Systems around the world

Correlators of classical unitary ensembles and enumerative problems

Massimo Gisonni

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Overview and goals of the work

Joint work with T. Grava and G. Ruzza
Laguerre Ensemble: Correlators, Hurwitz Numbers and Hodge Integrals.
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1. Give an effective way to compute correlators of the classical unitary ensembles
   - GUE [Dubrovin & Yang, 2016] cracked this old unsolved problem
   - LUE [G., Grava, Ruzza] with somewhat different techniques
   - JUE Work in progress can be dealt with the same approach used for LUE

2. Coefficients of these correlators have a combinatorial interpretation
   - GUE $\to$ *Ribbon Graphs*
   - LUE $\to$ *Monotone Double Hurwitz Numbers*
   - JUE $\to$ *Monotone Triple Hurwitz Numbers*

3. They also have a role in the Enumerative Geometry side
   - GUE $\to$ *Gromov-Witten Invariants of* $\mathbb{P}^1$
   - LUE $\to$ *Triple Hodge Integrals*
   - JUE $\to$ *?*

Generating series for GUE correlators

The Gaussian Unitary Ensemble

The GUE is the datum of a measure on the N^2-dimensional vector space $\mathcal{H}_N$ of *Hermitian matrices* which can be written as

$$d\mu = Z_{GUE}^{-1}(N)\exp\{-\frac{1}{2}\mathrm{tr}\, M^2\}dM$$

where

$$dM = \prod_{i=1}^{N} dM_{ii} \prod_{i<j} d\mathrm{Re}\,(M_{ij})\, d\mathrm{Im}\,(M_{ij}), \qquad Z_{GUE}(N) = 2^{-\frac{N}{2}}\pi^{-\frac{N^2}{2}}.$$

The objects of interest in our study are the so called *multi-point correlators*, which is

$$\left\langle \mathrm{tr}\, M^{i_1} \cdots \mathrm{tr}\, M^{i_k} \right\rangle := \frac{\int_{\mathcal{H}_N} \mathrm{tr}\, M^{i_1} \ldots \mathrm{tr}\, M^{i_k}\, e^{-\frac{1}{2}\mathrm{tr}\, M^2} dM}{\int_{\mathcal{H}_N} e^{-\frac{1}{2}\mathrm{tr}\, M^2} dM}.$$

Or rather, the *connected* correlators,

$$\left\langle \mathrm{tr}\, M^{k_1} \cdots \mathrm{tr}\, M^{k_r} \right\rangle_{\mathsf{c}} := \sum_{\mathcal{P} \text{ partition of } \{1,\ldots,r\}} (-1)^{|\mathcal{P}|-1}(|\mathcal{P}|-1)! \prod_{I\in\mathcal{P}} \left\langle \prod_{i\in I} \mathrm{tr}\, M^{k_i} \right\rangle,$$

$$\left\langle \mathrm{tr}\, M^{k_1} \right\rangle_{\mathsf{c}} := \left\langle \mathrm{tr}\, M^{k_1} \right\rangle, \qquad \left\langle \mathrm{tr}\, M^{k_1}\mathrm{tr}\, M^{k_2} \right\rangle_{\mathsf{c}} := \left\langle \mathrm{tr}\, M^{k_1}\mathrm{tr}\, M^{k_2} \right\rangle - \left\langle \mathrm{tr}\, M^{k_1} \right\rangle \left\langle \mathrm{tr}\, M^{k_2} \right\rangle.$$

GUE correlators and Ribbon Graphs

It is an old and beautiful result by [Bessis, Itzykson & Zuber, 1980] that the GUE correlators admit a *genus expansions* in terms of *ribbon graphs*

$$\left\langle \prod_{j=1}^{\nu} \left(\operatorname{tr} M^j\right)^{n_j} \right\rangle = \sum_g \left\{ \begin{array}{c} g\text{-ribbon graphs with } n_j \\ j\text{-valent vertices} \end{array} \right\} N^{2-2g+\sum_{j=1}^{\nu}(\frac{j}{2}-1)n_j}.$$

- Harer and Zagier computed the generating series for the one-point correlators (1986)
- Morozov and Shakirov extended the construction for the two-point correlators (2009)
- For higher orders no explicit formula was known

The breakthrough came with the paper of [Dubrovin & Yang, 2016] using the *resolvent method* elaborated by [Bertola, Dubrovin & Yang, 2015]. The authors were able to find generating functions for these objects,

$$C_k(N, \lambda_1, \dots, \lambda_k) = \sum_{i_1,\dots,i_k=1}^{\infty} \frac{\langle \mathrm{Tr} M^{i_1} \dots \mathrm{Tr} M^{i_k} \rangle_c}{\lambda_1^{i_1+1} \dots \lambda_k^{i_k+1}},$$

in terms of sums of products of Gauss hypergeometric functions, e.g.

$$C_1(N, \lambda) = N \sum_{j \geq 0} \frac{(2j-1)!!}{\lambda^{2j+1}} {}_2F_1\left(\begin{array}{c} -j, \ 1-N \\ 2 \end{array} \middle| \, 2 \right).$$

One consider the *GUE partition function*

$$Z_N(\mathbf{t} = t_1, t_2, \dots) := \int_{\mathcal{H}_N} \exp \operatorname{tr} \left(-\frac{1}{2} M^2 + \sum_{k \geq 1} t_k M^k \right) \mathrm{d}M = h_0(\mathbf{t}) h_1(\mathbf{t}) \dots h_{N-1}(\mathbf{t}),$$

the $h_k(\mathbf{t})$'s are the norming constants of the corresponding system of (*Hermite* deformed) monic orthogonal polynomials. Notice that connected correlators can be recovered as

$$\left. \frac{\partial^r \log Z_N(\mathbf{t})}{\partial t_{i_1} \cdots \partial t_{i_r}} \right|_{\mathbf{s}=\mathbf{0}} = \left\langle \operatorname{tr} M^{i_1} \cdots \operatorname{tr} M^{i_r} \right\rangle_{\mathbf{c}}.$$

Two lines idea of the proof

- There exists a matrix $R_n(\lambda) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \mathcal{O}\left(\lambda^{-1}\right) \in \mathrm{Mat}\left(2, \mathbb{Z}[\mathbf{v}, \mathbf{w}]\left[\left[\lambda^{-1}\right]\right]\right)$ called the *matrix resolvent* which can be determined from a system of recursion relations
- Multi-point correlators can be recovered via the *master formula*

$$\sum_{i_1, \dots, i_k = 0}^{\infty} \frac{1}{\lambda_1^{i_1+2} \dots \lambda_k^{i_k+2}} \frac{\partial^k \log Z_N(\mathbf{t})}{\partial t_{i_1} \dots \partial t_{i_k}} = -\frac{1}{k} \sum_{\sigma \in S_k} \frac{\operatorname{tr}\ [R_n(\mathbf{t}, \lambda_{\sigma_1}) \dots R_n(\mathbf{t}, \lambda_{\sigma_k})]}{(\lambda_{\sigma_1} - \lambda_{\sigma_2}) \dots (\lambda_{\sigma_{k-1}} - \lambda_{\sigma_k})(\lambda_{\sigma_k} - \lambda_{\sigma_1})}.$$

The LUE case

The Laguerre partition function

Over the cone $\mathcal{H}_N^+$ of positive definite hermitian matrices of size N endowed with the probability measure $\det^\alpha M \exp \mathrm{tr}\,(-M) \mathrm{d}M$, define

$$Z_N^{LUE}(\alpha; \mathbf{t}) = \int_{\mathcal{H}_N^+} \det^\alpha M \; e^{-\mathrm{tr}\,\left(M + \sum_{k\neq 0} t_k M^k\right)} dM = h_0^\alpha(\mathbf{t}) h_1^\alpha(\mathbf{t}) \dots h_{N-1}^\alpha(\mathbf{t})$$

Trying to follow similar steps for the Laguerre Unitary Ensemble, one considers the (deformed) Laguerre Polynomials

$$x\pi_\ell^\alpha(x; \mathbf{t}) = \pi_{\ell+1}^\alpha(x; \mathbf{t}) + v_\ell^\alpha(\mathbf{t})\pi_\ell^\alpha(x; \mathbf{t}) + w_\ell^\alpha(\mathbf{t})\pi_{\ell-1}^\alpha(x; \mathbf{t}),$$

$$\int_{-\infty}^{\infty} \pi_\ell^\alpha(x; \mathbf{t})\pi_{\ell'}^\alpha(x; \mathbf{t}) e^{-V_\alpha(x,\mathbf{t})} dx = h_\ell(\mathbf{t})\delta_{\ell\ell'}, \quad wrt \quad V_\alpha(x, \mathbf{t}) = x - \alpha \log x + \sum_{k\neq 0} t_k x^k.$$

and the LUE multi-point correlators. defined akin the GUE ones as

$$\left\langle \mathrm{tr}\, M^{i_1} \cdots \mathrm{tr}\, M^{i_k} \right\rangle := \frac{\int_{\mathcal{H}_N^+} \mathrm{tr}\, M^{i_1} \dots \mathrm{tr}\, M^{i_k} \det^\alpha M \; e^{-\mathrm{tr}\, M} dM}{\int_{\mathcal{H}_N^+} e^{-\frac{1}{2}\mathrm{tr}\, M^2} dM}.$$

Combinatorial intepretation of the LUE correlators

Strictly and weakly monotone double Hurwitz numbers

Let μ, ν be partitions of the same integer $d = |\mu| = |\nu|$, and denote with $\mathfrak{C}_\mu \subset \mathfrak{S}_d$ the conjugacy class of permutations whose disjoint cycle factorization contains cycles of lengths $\mu_1, \dots, \mu_\ell$. Define $h_g^{>}(\mu;\nu)$ (resp. $h_g^{\geq}(\mu;\nu)$), the *strictly* (resp. *weakly*) *monotone double Hurwitz numbers*, as the number of tuples $(\rho, \eta, \tau_1, \dots, \tau_r)$ satisfying

(i) $r = \ell(\mu) + \ell(\nu) + 2g - 2$

(ii) $\rho \in \mathfrak{C}_\mu, \eta \in \mathfrak{C}_\nu$,

(iii) $\tau_i = (a_i, b_i)$ are transpositions, with $a_i < b_i$ and $b_1 < \cdots < b_r$ (resp. $b_1 \leq \cdots \leq b_r$),

(iv) $\rho\tau_1 \cdots \tau_r = \eta$,

(v) the subgroup generated by $\rho, \tau_1, \dots, \tau_r$ acts transitively on $\{1, \dots, d\}$.

For example for $\mu = \{1,1,1\}$ and $\nu = \{3\}$ we have

$$h_g^{>}(\{1,1,1\};\{3\}) = 2, \qquad h_g^{\geq}(\{1,1,1\};\{3\}) = 4$$

Since a 3-cycle in $\mathfrak{S}_3$ can be factorized with $r = 2$ transpositions as

$$(12)(13) = (132), \quad (12)(23) = (123)$$

in the strictly monotone case, and rather in the weakly monotone case

$$(12)(13) = (132), \quad (12)(23) = (123), \quad (23)(13) = (132), \quad (13)(23) = (123)$$

Like in the GUE case, the genus expansion of the LUE correlators can be written in terms of combinatorial objects, a result of [Cunden, Dahlqvist & O'Connell, 2018]. Namely

Genus expansion of the LUE correlators

In the scaling $\alpha = N(c-1)$ we have

$$N^{\ell-|\mu|-2} \left\langle \operatorname{tr} M^{\mu_1} \cdots \operatorname{tr} M^{\mu_\ell} \right\rangle_c = \sum_{g \geq 0} \frac{1}{N^{2g}} \sum_{s=1}^{1-2g+|\mu|-\ell} H_g^{>}(\mu; s) c^s, \qquad c > 1 - \frac{1}{N},$$

$$N^{\ell+|\mu|-2} \left\langle \operatorname{tr} M^{-\mu_1} \cdots \operatorname{tr} M^{-\mu_\ell} \right\rangle_c = \sum_{g \geq 0} \frac{1}{N^{2g}} \sum_{s \geq 1} \frac{H_g^{\geq}(\mu; s)}{(c-1)^{2g-2+|\mu|+\ell+s}}, \qquad c > 1 + \frac{|\mu|}{N},$$

Here above,

$$H_g^{>}(\mu; s) = \frac{z_\mu}{|\mu|!} \sum_{\nu \text{ of length } s} h_g^{>}(\mu; \nu), \qquad H_g^{\geq}(\mu; s) = \frac{z_\mu}{|\mu|!} \sum_{\nu \text{ of length } s} h_g^{\geq}(\mu; \nu),$$

are so called *multiparametric single Hurwitz numbers*, $|\mu| := \mu_1 + \cdots + \mu_\ell$ and $z_\mu := \prod_{i \geq 1} (i^{m_i})\, m_i!$, is the cardinality of the conjugacy class $\mathfrak{C}_\mu$.

Computing the correlators

Unfortunately, the method used for the GUE case does not fit very well for the LUE, since

$$v_\ell^\alpha(\mathbf{t}=\mathbf{0}) = 2\ell+\alpha+1, \qquad w_\ell^\alpha(\mathbf{t}=\mathbf{0}) = \ell(\ell+\alpha),$$

the initial conditions are much more complicated than the ones for Hermite.

But! logarithmic derivatives can be retrieved also analyzing the Riemann-Hilbert problem of the associated orthogonal polynomials.

The Riemann-Hilbert Problem

From work that dates back to [Its, Kitaev & Fokas, 1990] the matrix

$$Y(x;\mathbf{t}) := \begin{pmatrix} \pi_N^\alpha(x;\mathbf{t}) & \widehat{\pi}_N^\alpha(x;\mathbf{t}) \\ -\frac{2\pi\mathrm{i}}{h_{N-1}(\mathbf{t})}\pi_{N-1}^\alpha(x;\mathbf{t}) & -\frac{2\pi\mathrm{i}}{h_{N-1}(\mathbf{t})}\widehat{\pi}_{N-1}^\alpha(x;\mathbf{t}) \end{pmatrix}$$

solves the RHP for the associated orthogonal polynomials. Notice that it is analytic for $x \in \mathbb{C} \setminus [0,\infty)$ and continuous up to the boundary $(0,\infty)$ where it satisfies the *jump condition*

$$Y_+(x;\mathbf{t}) = Y_-(x;\mathbf{t}) \begin{pmatrix} 1 & \mathrm{e}^{-V_\alpha(x;\mathbf{t})} \\ 0 & 1 \end{pmatrix}, \qquad x \in (0,\infty).$$

Introduce the 2×2 matrix $\Psi(x;\mathbf{t})$, analytic for $x \in \mathbb{C} \setminus [0,\infty)$,

$$\Psi(x;\mathbf{t}) := Y(x;\mathbf{t}) \exp\left(-V_\alpha(x;\mathbf{t})\frac{\sigma_3}{2}\right).$$

which has *constant* jump matrix along $x \in (0,\infty)$. It then satisfies a compatible system of linear 2×2 matrix ODEs with rational coefficients,

$$\frac{\partial \Psi(x;\mathbf{t})}{\partial x} = \mathcal{A}(x;\mathbf{t})\Psi(x;\mathbf{t}), \qquad \frac{\partial \Psi(x;\mathbf{t})}{\partial t_k} = \Omega_k(x;\mathbf{t})\Psi(x;\mathbf{t}), \quad k \neq 0.$$

- $\Omega_k(x;\mathbf{t})$ being a polynomial in $x^{\pm 1}$ of degree $|k|$, for k positive/negative
- $\mathcal{A}(x;\mathbf{t})$ is a Laurent polynomial in x, modulo a proper truncation fo times
- Existence of the solution $\Psi(x;\mathbf{t})$ implies compatibility and hence the zero curvature conditions

$$\frac{\partial \mathcal{A}}{\partial t_k} - \frac{\partial \Omega_k}{\partial x} = [\Omega_k, \mathcal{A}], \qquad k \neq 0.$$

Isomonodromic tau function, [Bertola, Eynard & Harnad, CMP 2006]

The key fact is that we can identify the LUE partition funcion with the *isomonodromic tau function* of the monodromy-preserving deformation system above.

In particular, defining the matrix

$$R(x;\mathbf{t}) := \Psi(x;\mathbf{t})\mathrm{E}_{11}\Psi^{-1}(x;\mathbf{t}) = Y(x;\mathbf{t})\mathrm{E}_{11}Y^{-1}(x;\mathbf{t}),$$

Logarithmic derivatives of $Z_N(\alpha;\mathbf{t})$ can be expressed in terms of residues of computable objects

$$\frac{\partial \log Z_N(\alpha;\mathbf{t})}{\partial t_k} = -\operatorname*{res}_{x}\left(\mathrm{tr}\,\left(\mathcal{A}(x;\mathbf{t})R(x;\mathbf{t})\right) + \frac{1}{2}\frac{\partial}{\partial x}V_\alpha(x;\mathbf{t})\right)x^k\mathrm{d}x,$$

$$\frac{\partial^2 \log Z_N(\alpha;\mathbf{t})}{\partial t_{k_2}\partial t_{k_1}} = \operatorname*{res}_{x_1}\operatorname*{res}_{x_2}\frac{\mathrm{tr}\left(R(x_1;\mathbf{t})R(x_2;\mathbf{t})\right)-1}{(x_1-x_2)^2}x_1^{k_1}x_2^{k_2}\mathrm{d}x_1\mathrm{d}x_2,$$

$$\frac{\partial^r \log Z_N(\alpha;\mathbf{t})}{\partial t_{k_r}\cdots\partial t_{k_1}} = \operatorname*{res}_{x_1}\cdots\operatorname*{res}_{x_r}\left[\sum_{(i_1,\dots,i_r)\in\mathcal{C}_r}\frac{(-1)^{r+1}\mathrm{tr}\,\left(R(x_{i_1};\mathbf{t})\cdots R(x_{i_r};\mathbf{t})\right)}{(x_{i_1}-x_{i_2})\cdots(x_{i_{r-1}}-x_{i_r})(x_{i_r}-x_{i_1})} + \right.$$
$$\left. -\frac{\delta_{r,2}}{(x_1-x_2)^2}\right]x_1^{k_1}\cdots x_r^{k_r}\mathrm{d}x_1\cdots\mathrm{d}x_r$$

where the symbol $\operatorname*{res}_{x_i}$ denotes $\operatorname*{res}_{x_i=\infty}$ (resp. $\operatorname*{res}_{x_i=0}$) if $k_i > 0$ (resp. $k_i < 0$).

Positive, negative and mixed type correlators

Evaluating at $\mathbf{t} = 0$ and computing the asymptotic expansions of $R(x;\mathbf{0})$ at 0 and ∞, we get straightforwardly formulæ for multi-point connected correlators of *general type*

$$\left\langle \mathrm{tr}\,M^{\mu_1}\cdots\mathrm{tr}\,M^{\mu_\ell}\mathrm{tr}\,M^{-\nu_1}\cdots\mathrm{tr}\,M^{-\nu_{\ell'}}\right\rangle_{\mathsf{c}}$$

Asymptotics for the matrix R

$$R_+(x) := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sum_{\ell\geq 0} \frac{1}{x^{\ell+1}} \begin{pmatrix} \ell A_\ell(N, N+\alpha) & B_\ell(N+1, N+\alpha+1) \\ -N(N+\alpha)B_\ell(N, N+\alpha) & -\ell A_\ell(N, N+\alpha) \end{pmatrix}$$

$$R_-(x) := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sum_{\ell\geq 0} \frac{x^\ell}{(\alpha-\ell)_{2\ell+1}} \begin{pmatrix} (\ell+1)A_\ell(N, N+\alpha) & -B_\ell(N+1, N+\alpha+1) \\ N(N+\alpha)B_\ell(N, N+\alpha) & -(\ell+1)A_\ell(N, N+\alpha) \end{pmatrix}$$

with $M = N + \alpha$ and

$$A_\ell(N,M) := \begin{cases} N, & \ell = 0, \\ \frac{1}{\ell}\sum_{j=0}^{\ell-1}(-1)^j \frac{(N-j)_\ell (M-j)_\ell}{j!(\ell-1-j)!}, & \ell \geq 1, \end{cases} \qquad B_\ell(N,M) := \sum_{j=0}^{\ell}(-1)^j \frac{(N-j)_\ell(M-j)_\ell}{j!(\ell-j)!}.$$

These are obtained from the Lax differential equation satisfied by $R(x)$

$$\frac{\partial}{\partial x}R(x) = [\mathcal{A}(x), R(x)], \qquad \mathcal{A}(x) = -\frac{1}{2}\sigma_3 + \frac{1}{x}\begin{pmatrix} N + \frac{\alpha}{2} & -\frac{h_N}{2\pi \mathrm{i}} \\ \frac{2\pi\mathrm{i}}{h_{N-1}} & -N - \frac{\alpha}{2} \end{pmatrix}.$$

We also retrieve two important properties of the moments which first appeared in [Cunden, Mezzadri, O'Connell & Simm, 2019]

- $A_\ell(N,M), B_\ell(N,M)$ satisfy a three-term recurrence relation, they are *Hahn orthogonal polynomials*, and $\langle \mathrm{tr}\, M^k \rangle = A_k(N, N+\alpha)$
- Reciprocity laws:

$$\langle \mathrm{tr}\, M^{-k-1} \rangle = \frac{\langle \mathrm{tr}\, M^k \rangle}{(\alpha-k)_{2k+1}}, \quad \langle \mathrm{tr}\, X^k \mathrm{tr}\, X \rangle_{\mathrm{c}} = kA_k(N, N+\alpha) = \alpha(\alpha-k)_{2k+1} \langle \mathrm{tr}\, X^{-k}\mathrm{tr}\, X^{-1} \rangle_{\mathrm{c}}$$

Hence multi-point correlators are themselves combination of orthogonal polynomials.

Hurwitz numbers and Hodge Integrals

Hodge–GUE Correspondence

In [Dubrovin, Liu, Yang & Zhang, 2016] is shown that defining the generating function for special cubic Hodge integrals

$$\mathcal{H}_{cubic}(\mathbf{p};\epsilon) = \sum_{g\geq 0} \epsilon^{2g-2} \sum_{n\geq 0} \sum_{k_1,\dots,k_n\geq 0} \frac{p_{k_1}\cdots p_{k_n}}{n!} \int_{\overline{\mathcal{M}}_{g,n}} \Lambda^2(-1)\Lambda\left(\frac{1}{2}\right) \prod_{i=1}^{n} \psi_i^{k_i},$$

its exponential $Z_{cubic}(\mathbf{p};\epsilon) := e^{\mathcal{H}(\mathbf{p};\epsilon)}$ *uniqely* factorize the GUE partition function with even couplings only, which is

$$Z_N^{even}(\mathbf{s};\epsilon) = c \cdot Z_{cubic}\left(\mathbf{p}(x+\frac{\epsilon}{2},\mathbf{s});\sqrt{2\epsilon}\right) Z_{cubic}\left(\mathbf{p}(x-\frac{\epsilon}{2},\mathbf{s});\sqrt{2\epsilon}\right)$$

Otos! the LUE partition function for $\alpha = \pm\frac{1}{2}$ also does the job by almost trivial arguments

$$Z_{2N}^{\text{even}}(\mathbf{s}) = D_N Z_N\left(-\frac{1}{2};\mathbf{t}_+\right) Z_N\left(\frac{1}{2};\mathbf{t}_+\right)$$

Hodge–LUE Correspondence

By *uniqueness* of the factorization for Z_N^{even}, we are able to identify the partition functions Z_{cubic} and $Z_{LUE}^{(\alpha=-\frac{1}{2})}$, in particular unveiling the existence of a **matrix model** for the former.

Most interestingly, this carries a relation between cubic Hodge integrals and double monotone Hurwitz numbers, namely

$$\sum_{\gamma=0}^{g} 2^{4\gamma} \sum_{s\geq 1} \left[\sum_{p\geq 0} (-1)^p \binom{2-2\gamma+|\mu|-\ell-s}{p} \binom{s}{2g-2\gamma-p} \right] H_{\gamma}^{>}(\mu;s)$$

$$= 2^{3g+1-\ell} \int_{\overline{\mathcal{M}}_{g,\ell}} \Lambda^2(-1)\Lambda\left(\frac{1}{2}\right) \exp\left(-\sum_{d\geq 1} \frac{\kappa_d}{d}\right) \prod_{a=1}^{\ell} \frac{\mu_a \binom{2\mu_a}{\mu_a}}{1-\mu_a\psi_a}$$

which for example in genus zero reads

$$\mathscr{H}_{0,\mu} = 2^{\ell-2}\lambda^{|\mu|+2-\ell} \sum_{s=1}^{|\mu|+1-\ell} H_0^{>}(\mu;s).$$

The JUE case [Work in progress]

The Jacobi partition function

On the set $\mathcal{H}_N^{(0,1)}$ of hermitian matrices of size N with spectrum in the real interval $(0,1)$, endowed with the probability measure $Z_N(\alpha_1,\alpha_2;\mathbf{0})^{-1}\det^{\alpha_1}(\mathbf{1}-M)\det^{\alpha_2}M\,\mathrm{d}M$, define

$$Z_N(\alpha_1,\alpha_2;\mathbf{t}_+,\mathbf{t}_-)=\int_{\mathcal{H}^{(0,1)}}\det^{\alpha_1}(\mathbf{1}-M)\det^{\alpha_2}M\exp\mathrm{tr}\left(\sum_{k\neq 0}t_kM^k\right)\mathrm{d}M$$

Then, JUE multi-point connected correlators

$$\left.\frac{\partial^\ell\log Z_N(\alpha_1,\alpha_2;\mathbf{t}_+,\mathbf{t}_-)}{\partial t_{k_1}\cdots\partial t_{k_\ell}}\right|_{\mathbf{t}_+=\mathbf{t}_-=\mathbf{0}}=\left\langle\mathrm{tr}\,M^{k_1}\cdots\mathrm{tr}\,M^{k_\ell}\right\rangle_{\mathsf{c}},$$

where $k_1,k_2,\dots,k_\ell\in\mathbb{Z}$, can be computed adapting the exact same procedure, working now with the *Jacobi orthogonal Polynomials*,

$$\int_0^1\pi_\ell^{(\alpha_1,\alpha_2)}(x)\pi_{\ell'}^{(\alpha_1,\alpha_2)}(x)\mathrm{e}^{-V_{\alpha_1,\alpha_2}(x)}\mathrm{d}x=\delta_{\ell,\ell'}h_\ell,\qquad V_{\alpha_1,\alpha_2}(x):=-\alpha_1\log(1-x)-\alpha_2\log x.$$

Asymptotics for the matrix R

$$R^{\infty}(x) := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \sum_{\ell \geq 0} \frac{1}{x^{\ell+1}} \frac{1}{\alpha_1 + \alpha_2 + 2N} \begin{pmatrix} \ell A_\ell(N, \alpha_1, \alpha_2) & c_{N,\alpha} \cdot B_\ell(N+1, \alpha_1, \alpha_2) \\ -B_\ell(N, \alpha_1, \alpha_2) & -\ell A_\ell(N, \alpha_1, \alpha_2) \end{pmatrix}.$$

Similar expansions hold at the other singular points $x = 0$ and $x = 1$.

As in the LUE case, massaging the Lax equation for R we deduce a number of facts

- the A_ℓ and B_ℓ's are *Wilson orthogonal polynomials*, a fact already observed in [Cunden, Mezzadri, O'Connell & Simm, 2019]. For example

$$A_\ell = d_{N,\alpha} \cdot W_{N-1}\left(-\left(\ell + \frac{1}{2}\right)^2; \frac{3}{2}, \frac{1}{2}, \alpha_2 + \frac{1}{2}, \frac{1}{2} - \alpha_1 - \alpha_2 - 2N\right)$$

- a Reciprocity law holds in this case as well,

$$\left(\left\langle \operatorname{tr} M^{-k-1}\right\rangle - \left\langle \operatorname{tr} M^{-k}\right\rangle\right) = \left(\prod_{j=-k}^{k} \frac{\alpha_1 + \alpha_2 + 2n - j}{\alpha_2 - j}\right)\left(\left\langle \operatorname{tr} M^{k}\right\rangle - \left\langle \operatorname{tr} M^{k+1}\right\rangle\right),$$

  notice that, in this case, the *differences* of moments are the correct quantities to consider.

Combinatorial intepretation of the JUE correlators

Weakly monotone triple Hurwitz numbers

Given three partitions $\lambda, \mu, \nu \vdash d$, for any integer $g \geq 0$ define $h_g(\lambda, \mu, \nu)$ to be the number of tuples $(\pi_1, \pi_2, \tau_1, \dots, \tau_r)$ of permutations in $\mathfrak{S}_d$ such that

(i) $r = \ell(\mu) + \ell(\nu) + \ell(\lambda) - d + 2g - 2$,

(ii) $\pi_1 \in \mathfrak{C}_\mu$, $\pi_2 \in \mathfrak{C}_\nu$,

(iii) $\tau_i = (a_i, b_i)$ are transpositions, with $a_i < b_i$ and $b_1 \leq \cdots \leq b_r$,

(iv) $\pi_1 \pi_2 \tau_1 \cdots \tau_r \in \mathfrak{C}_\lambda$,

(v) the subgroup generated by $\pi_1, \pi_2, \tau_1, \dots, \tau_r$ acts transitively on $\{1, \dots, d\}$.

In the scaling limit $N \to +\infty$ with $\alpha_1 = (c_1 - 1)N$, $\alpha_2 = (c_2 - 1)N$, we have the following expansions for any partition $\lambda = (\lambda_1, \dots, \lambda_{\ell(\lambda)})$

$$N^{\ell - |\mu| - 2} \left\langle \prod_{j=1}^{\ell(\lambda)} \operatorname{tr} X^{\lambda_j} \right\rangle_{\mathsf{c}} = \sum_{g \geq 0} N^{2-2g} \sum_{\mu, \nu \vdash |\lambda|} c_2^{\ell^*(\nu)} \left(- \frac{c_2}{c_1 + c_2} \right)^{\ell(\mu) + \ell(\nu) - \ell^*(\lambda) + 2g - 2} h_g(\lambda, \mu, \nu).$$

A similar expansion also holds for negative multi-point correlators.

The Jacobi partition function is a KP hypergeometric τ-function [Okounkov, Harnad, Orlov]

KP hypergeometric τ-function are in correspondence with families of *multiparametric Hurwitz numbers*, from which the above expansion can be deduced.

Summarizing

What has been done

- Effective computation of correlators for the LUE and JUE ensemble
- The multi-point correlators of ·UE ensemble are combinations of hypergeometric OP
- Computation of monotone double hurwitz numbers
- A matrix model for the cubic Hodge partition function
- Link between Hodge integrals and monotone double Hurwitz numbers

What is to be done

- Combinatorial interpretation (if any) of the mixed JUE and LUE correlators
- A combinatorial proof that JUE correlators are linked to triple monotone Hurwitz numbers
- Extend the approach to the $\beta = 1, 4$ Orthogonal and Symplectic ensembles
- Orthogonality properties for multi-point ·UE correlators, [Jonnadula, Mezzadri & Keating, 2020]
- Enumerative Geometry invariants associated to the Jacobi partition function

Fin

Thank you!

Introduce the weight generating function $G(z) = \prod_{k=1}^{\infty}(1 + c_k z)$. For any $N \in \mathbb{Z}$, partition λ and non-vanishing parameter β define the *content product*

$$r_\lambda^{(G,\beta)}(N) := r_0^{(G,\beta)}(N) \prod_{(i,j)\in\lambda} G(\beta(N+j-i)), \qquad r_0^{(G,\beta)}(N) := \prod_{j=1}^{N} G(\beta(N-j))^j.$$

These coefficients determine a 2D-Toda τ-function [Okounkov, Harnad, Orlov]

$$\tau^{(G,\beta)}(N,\mathbf{t},\mathbf{s}) := \sum_\lambda r_\lambda^{(G,\beta)}(N) s_\lambda(\mathbf{t}) s_\nu(\mathbf{s})$$

On the other side, define the *multiparametric double Hurwitz numbers*

$$H_G^d(\mu,\nu) := \sum_{k=0}^{\infty} \sum_{\substack{\mu^{(1)},\dots,\mu^{(k)} \\ \sum_{i=1}^{k} \ell^*(\mu^{(i)})=d}} W_G(\mu^{(1)},\dots,\mu^{(k)}) H(\mu^{(1)},\dots,\mu^{(k)},\mu,\nu),$$

$$W_G(\mu^{(1)},\dots,\mu^{(k)}) := \frac{1}{|\mathrm{aut}(\lambda)|} \sum_{\sigma\in\mathfrak{S}_k} \sum_{1\le i_1<\dots<i_k} c_{i_{\sigma(1)}}^{\ell^*(\mu^{(1)})} \cdots c_{i_{\sigma(k)}}^{\ell^*(\mu^{(k)})}.$$

These are the coefficients of the *same* τ-function in a different basis of symmetric polynomials

$$\tau^{(G,\beta)}(N,\mathbf{t},\mathbf{s}) = \sum_{d=0}^{\infty} \sum_{\mu,\nu,\ |\mu|=|\nu|} \beta^d H_G^d(\mu,\nu) p_\mu(\mathbf{t}) p_\nu(\mathbf{s}).$$

Notice that evaluation at $\mathbf{s} = \mathbf{s}_\infty = (1,0,0,\dots)$ turns the above into *KP τ-functions*.

- In particular, the parametric function $G(z)$ uniquely determines the "species" of the corresponding Hurwitz numbers, and it can be retrieved once the $r_\lambda^{(G,\beta)}(N)$ are known,
- The Jacobi partition function, restricting to positive times only, is a *KP τ-function* itself,
- By standard arguments, matrix models for $\beta = 2$ can be rewritten in a Schur expansion as

$$\tau_N(\mathbf{t}) = \int_{\mathbf{R}^N} \Delta^2(\mathbf{x}) \prod_{j=1}^{N} \exp\Big(\sum_{k\geq 1} t_k x^k\Big) dm(x_j) = \sum_{\ell(\lambda)\leq N} c_{\lambda,N} s_\lambda(\mathbf{t})$$

$$c_{\lambda,N} = (-1)^{\frac{N(N-1)}{2}} N! \det[\mathcal{M}_{\lambda_i+N-i,j-1}], \qquad \mathcal{M}_{i,j} = \int_{\mathbf{R}} x^{i+j} dm(x),$$

in terms of the moments of the measure only.

Carrying out the above computation, one can correctly identify the function G corresponding to the Jacobi partition function, and in turn realize which kind of multiparametric Hurwitz numbers are associated to it.