On the asymptotic analysis of bordered Toeplitz determinant

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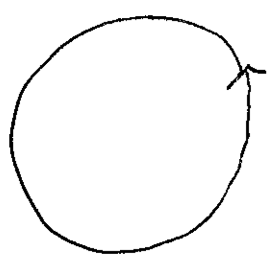
Toeplitz Determinants

Definition.

$\varphi(z) \in L_1(C)$

$C: \quad |z| = 1, \quad z = e^{i\theta}, \quad 0 \le \theta < 2\pi$



$$\varphi_k := \int_C \varphi(z) z^{-k} \frac{dz}{2\pi i z} = \frac{1}{2\pi} \int_0^{2\pi} e^{-ik\theta} \varphi(e^{i\theta}) d\theta$$

$$T_n[\varphi] = \{\varphi_{j-k}\}_{j,k=0,\dots,n-1}$$

$$= \begin{pmatrix} \varphi_0 & \varphi_{-1} & \varphi_{-2} & \cdots & \varphi_{-n+1} \\ \varphi_1 & \varphi_0 & \varphi_{-1} & \cdots & \varphi_{-n+2} \\ \varphi_2 & \varphi_1 & \varphi_0 & \varphi_{-1} \cdots & \varphi_{-n+3} \\ \vdots & & \ddots & \ddots & \vdots \\ & & & & \varphi_{-1} \\ \varphi_{n-1} & \cdots & \cdots & \varphi_1 & \varphi_0 \end{pmatrix}$$

$$\boxed{D_n[\varphi] = \det T_n[\varphi]}$$

Otto Toeplitz, 1907

(Habilitationsschrift)

Applications:

- Statistical mechanics, $\boxed{\text{Ising model}}$, exactly solvable quantum mechanical models (spin chains XXO, XY)

- Orthogonal polynomials, spectral theory of difference operators

- Random matrices, random permutations, growth processes, random Young tableaux tillings, packing.

Main questions of the theory

$$D_n[\varphi] \cong ?$$

$$n \to \infty$$

$$\varphi \equiv \varphi(z;\vec{t})$$

$$D_n[\varphi] \cong ?$$

$$n \to \infty, \quad \vec{t} \to \vec{t}_0$$

Classics: Strong Szegő Theorem.

- $\varphi(z) \neq 0$
- $\operatorname{Ind} \varphi(z) = 0$
- $\sum_{k=-\infty}^{\infty} |k| \left|(\log \varphi)_k\right|^2 < \infty$

$$(\log \varphi)_k = \frac{1}{2\pi i} \int_C \log \varphi(z)\, z^{-k} \frac{dz}{z}$$

Then

$$D_n[\varphi] = e^{n (\log \varphi)_0}\, e^{E[\varphi]} (1 + o(1)), \quad n \to \infty$$

$$E[\varphi] = \sum_{k=1}^{\infty} k\, (\log \varphi)_k (\log \varphi)_{-k}$$

- G. Szegö 1952 ($\varphi > 0$, $C^{1+\varepsilon}$)

- I. Ibragimov 1968

  B. Golinskii and I. Ibragimov 1971

  ($\varphi > 0$)

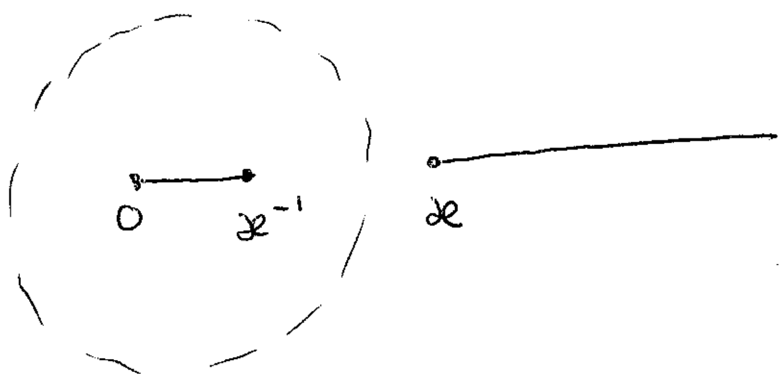
- K. Johansson 1988

  (φ - complex).

First non-trivial example

$$\varphi(z) = \sqrt{\frac{1 - z^{-1}\varkappa^{-1}}{1 - z\varkappa^{-1}}}$$

$$\varkappa > 1 \quad \Rightarrow \quad \operatorname{Ind}\varphi = 0$$



$$\log\varphi(z) = -\frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k\varkappa^k z^k} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{z^k}{k\varkappa^k}$$

Hence

$$(\log \varphi)_0 = 0$$

$$(\log \varphi)_k = \frac{1}{2k\varkappa^k}$$

$$k \geqslant 1$$

$$(\log \varphi)_{-k} = -\frac{1}{2k\varkappa^k}$$

$$\Downarrow$$

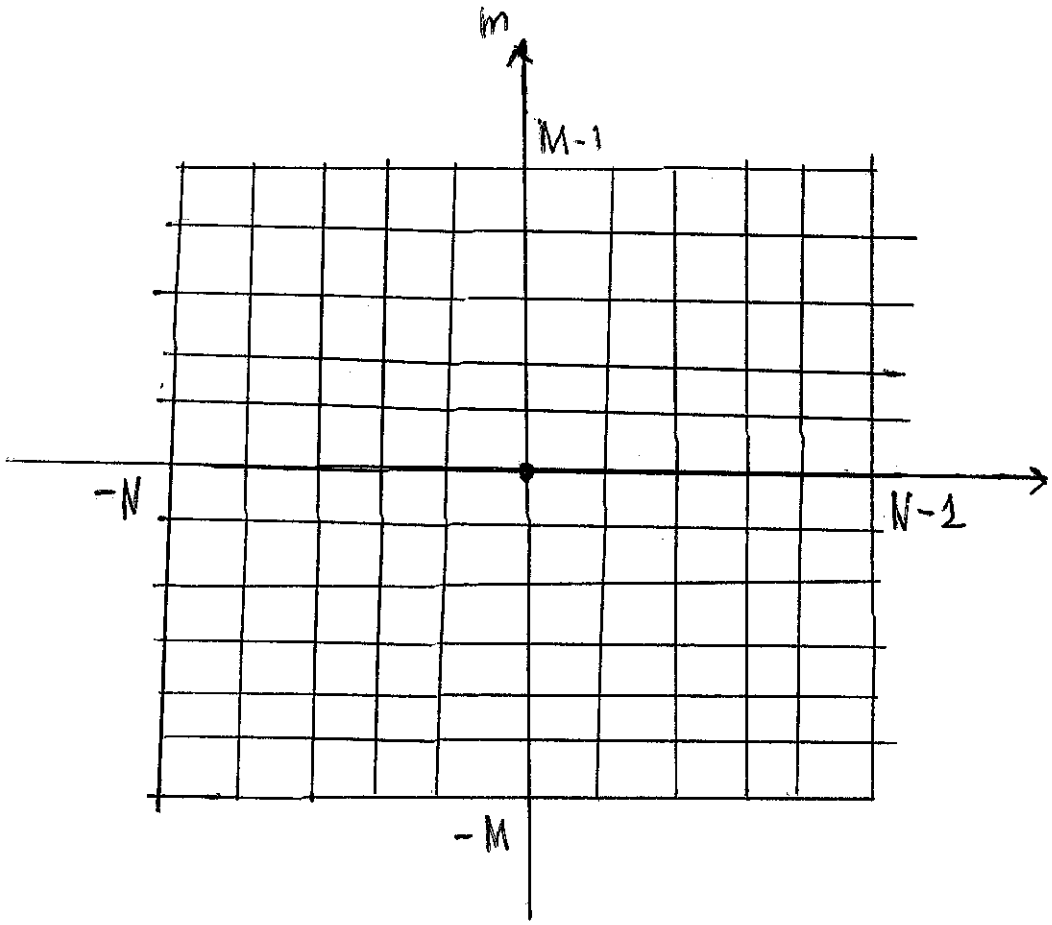
$$\sum_{k=1}^{\infty} k\,(\log\varphi)_k\,(\log\varphi)_{-k} = -\frac{1}{4}\sum_{k=1}^{\infty}\frac{1}{k\varkappa^{2k}}$$

$$= \frac{1}{4}\log\left(1-\varkappa^{-2}\right) \qquad (6.1)$$

Therefore:

$$\boxed{D_n[\varphi] \sim \left(1-\varkappa^{-2}\right)^{1/4} \quad n \to \infty}$$

A remarkable feature of this example is its relation to Ising model:



$$\Lambda_{MN} = \left\{ (m,n) \in \mathbb{Z}^2 : \begin{array}{l} -M \leq m \leq M-1 \\ -N \leq n \leq N-1 \end{array} \right\}$$

Spin configuration on Λ_{MN}

$$\sigma : \Lambda_{M,N} \to \{\pm 1\}$$

$$(m,n) \longmapsto \sigma_{mn} = \pm 1 .$$

Total Energy corresponding σ:

$$E(\sigma) = - \sum_{m=-M}^{M-1} \sum_{n=-N}^{N-1} \left(J_h \sigma_{mn} \sigma_{mn+1} + J_v \sigma_{mn} \sigma_{m+1n} \right)$$

A corresponding equilibrium Gibbs statistical mechanics is considered:

$$P_{r_{\Lambda_{MN}}} (\sigma) = \frac{1}{Z_{\Lambda_{MN}}} e^{-\frac{E(\sigma)}{K_B T}}$$

– Gibbs measure

$$Z_{\Lambda_{MN}} = \sum_{\mathfrak{z}}' e^{-\frac{E(\mathfrak{z})}{k_B T}}$$ - partition function.

Objects of principal interest

(a) Free Energy:

$$f = -k_B T \lim_{N \to \infty} \frac{1}{N^2} \log Z_{\Lambda_{NN}}$$

(b) Correlation functions

$$\left\langle \prod_{(m,n) \in A} \mathfrak{z}_{mn} \right\rangle = \lim_{M,N \to \infty} \sum_{\mathfrak{z}}' \prod_{(m,n) \in A} \mathfrak{z}_{mn} P_{\Gamma_{\Lambda_{MN}}}(\mathfrak{z})$$

Of special importance are:

(a) the presence of a phase transition $\equiv$ the singularity of $f(T)$ at some critical point T_c.

(b) a 2-point correlation function

$$\langle \sigma_{00}\sigma_{mn}\rangle = \lim_{M,N\to\infty} \sum_{\sigma}{}' \sigma_{00}\sigma_{mn} \frac{e^{-\frac{E(\sigma)}{k_B T}}}{Z_{\Lambda_{MN}}(T)}$$

Importance of $\langle \sigma_{00}\sigma_{mn}\rangle$:

$$\boxed{M = \sqrt{\lim_{n\to\infty} \langle \sigma_{00}\sigma_{nn}\rangle}}$$

- spontaneous magnetization.

Answers:

(a) T_c :

$$\sinh\frac{2J_h}{K_B T_c}\,\sinh\frac{2J_v}{K_B T_c} = 1$$

$\boxed{\text{L. Onsager, 1944}}$

(b)

$$\langle \sigma_{00}\sigma_{nn}\rangle = D_n[\varphi]$$

$$\varphi(z) = \sqrt{\frac{1 - z^{-1}\varkappa^{-1}}{1 - z\varkappa^{-1}}}$$

$$\varkappa = \sinh\frac{2J_h}{K_B T}\,\sinh\frac{2J_v}{K_B T}$$

Bruria Kaufman, Lars Onsager

≈ 1948

<u>Remark 1</u> condition $\varkappa > 1$ means that $T < T_c$. Asymptotics (6.1) then shows that

Spontaneous magnetization

$$\boxed{M = \sqrt{\lim_{n \to \infty} \langle \sigma_{00} \sigma_{nn} \rangle} = (1 - \varkappa^{-2})^{1/8} > 0 \ !}$$

– long-range order for $T < T_c$

($T \geq T_c$ – $M = 0$, no long-range order.
B. McCoy, T.T. Wu, 1973)

Remark 2. 1915 Szegő Theorem

$$D_n[\varphi] \underset{n\to\infty}{=} e^{n(\log\varphi)_0 + o(n)} \qquad (10.1)$$

This is not enough for Ising correlations.
Indeed,

$$(10.1) \Rightarrow \qquad \langle \sigma_{00}\sigma_{nn}\rangle \underset{n\to\infty}{=} e^{o(n)}$$

Hence the need of the Strong Szegő Theorem.

Bordered Toeplitz Determinants

Definitions

$$D_n^B[\varphi;\Psi] = \det \begin{pmatrix} \varphi_0 & \varphi_{-1} & \varphi_{-2} & \dots & \varphi_{-n+1} \\ \varphi_1 & \varphi_0 & \varphi_{-1} & \dots & \varphi_{-n+2} \\ \vdots & & & & \vdots \\ \varphi_{n-2} & \varphi_{n-3} & \dots & & \varphi_{-1} \\ \Psi_{n-1} & \Psi_{n-2} & \dots & & \Psi_0 \end{pmatrix}$$

$$\varphi_k = \int_C \varphi(z) z^{-k} \frac{dz}{2\pi i z}$$

$$\Psi_k = \int_C \Psi(z) z^{-k} \frac{dz}{2\pi i z}$$

Ising again!

Next-to-diagonal two point correlation function:

$$\langle \sigma_{00}\, \sigma_{n-1\,n} \rangle = D_n^B \, [\varphi ; \Psi]$$

$$\varphi(z) = \sqrt{\frac{1 - z^{-1}\varkappa^{-1}}{1 - z\varkappa^{-1}}}$$

$$\Psi(z) = q(z)\,\varphi(z) + \begin{cases} 0 & J_h < J_v \\ \dfrac{b}{z-c} & J_h > J_v \end{cases}$$

$$q(z) = \frac{r\, z}{z - c}$$

$$\boxed{\text{Helen Au-Yang, Jacques H.H. Perk} \quad 1987}$$

where, as before,

$$\varkappa = \sinh \frac{2J_h}{K_B T} \sinh \frac{2J_v}{K_B T}$$

and

$$c = - \sinh \frac{2J_h}{K_B T} \Big/ \sinh \frac{2J_v}{K_B T}$$

$$b = \cosh \frac{2J_h}{K_B T} \Big/ \sinh \frac{2J_v}{K_B T}$$

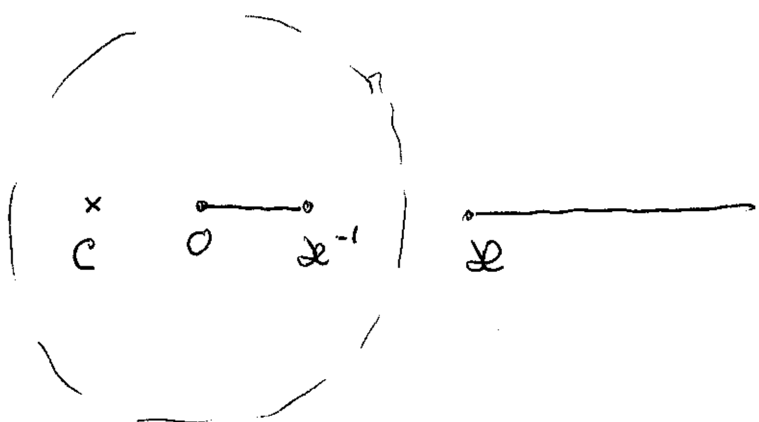
$$\tau = \cosh \frac{2J_v}{K_B T} \Big/ \sinh \frac{2J_v}{K_B T}$$

$$T < T_c \quad , \text{i.e.} \quad \varkappa > 1$$

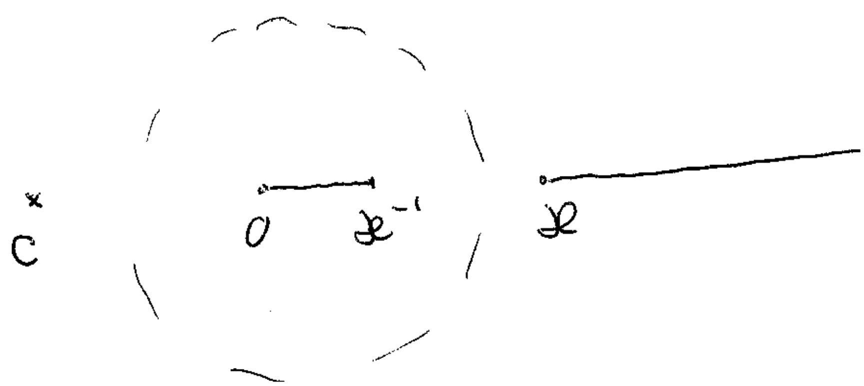
Also note:

$$J_h < J_v \quad \Rightarrow \quad |c| < 1$$

$$J_h > J_v \quad \Rightarrow \quad |c| > 1$$



$J_h < J_v$



$J_h > J_v$

This example suggests to start the study of bordered Toeplitz determinants with the following two types:

$$\Psi(z) = q(z)$$

$$\Psi(z) = q(z)\varphi(z)$$

$$q(z) = a + \sum_{j=0}^{m} \frac{b_j}{z - c_j}$$

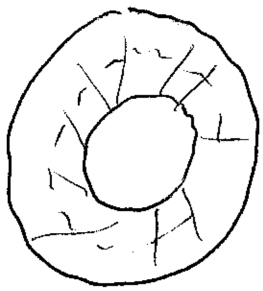
$$a, b_j, c_j \in \mathbb{C}$$

Theorem 1.

(E. Basor, T. Ehrhardt, R. Gharakhloo, Y. Li, A. I.)

Let

$$\varphi(z) \neq 0, \quad \mathrm{Ind}\,\varphi = 0$$

analytic on: 

Let also

$$\alpha(z) := \exp\left[\frac{1}{2\pi i}\int_C \frac{\log \varphi(\tau)}{\tau - z}\,d\tau\right]$$

Then:

- $$D_n^B\left[\varphi;\ a+\sum_{j=1}^{m}\frac{b_j}{z-c_j}\right] \qquad c_j \neq 0$$

$$= D_{n-1}[\varphi]\left(a + \sum_{\substack{1\leq j\leq m \\ |c_j|>1}} \frac{b_j}{c_j}\,\alpha(c_j) + O(e^{-\varepsilon n})\right)$$

$$n \to \infty \qquad \varepsilon > 0$$

and

$$D_n^B\left[\varphi;\left(a+\sum_{j=0}^{m}\frac{b_j}{z-c_j}\right)\varphi\right]$$

$$= D_n[\varphi]\left(a-\sum_{j=1}^{m}\frac{b_j}{c_j}\right)$$

$c_0 = 0$
$c_j \neq 0$
$j \geq 1$

(18.1)

$$+ D_{n-1}[\varphi]\left(\sum_{\substack{j=1 \\ 0<|c_j|<1}}^{m}\frac{b_j}{c_j}\,\alpha(c_j)\right.$$

$$\left. + \frac{b_0}{2\pi i}\int_C \log\varphi(z)\,dz + O\left(e^{-\varepsilon n}\right)\right)$$

$\varepsilon > 0$

$n \to \infty$

Remark 3. Suppose that

$$(\log \varphi)_0 = 0, \quad b_0 = 0,$$

and
$$a = \sum_{j=1}^{m} \frac{b_j}{c_j} \qquad (19.1)$$

Then the $\underbrace{\text{asymptotics}}_{(18.1)}$ become sensitive to the location of the poles c_j:

$$D_n^B \cong \begin{cases} O(e^{-\varepsilon n}) & \text{if all } |c_j| > 1 \\[2ex] D_{n-1}[\varphi] \displaystyle\sum_{\substack{j=1 \\ 0<|c_j|<1}}^{m} \frac{b_j}{c_j}\, \alpha(c_j) & \text{if there are } |c_j| < 1. \end{cases}$$

Note that conditions (19.1) take place for

$$\varphi(z) = \sqrt{\frac{1 - z^{-1}\varkappa^{-1}}{1 - z\varkappa^{-1}}} \qquad \varkappa > 1$$

$$\Psi(z) = q(z)\varphi(z)$$

$$q(z) = \frac{\tau z}{z - c} \qquad c \neq 0 .$$

<u>Remark 4.</u> Theorem 1

$$\Downarrow$$

$$\boxed{\lim_{n\to\infty} \langle \sigma_{00}\sigma_{n-1\,n}\rangle = (1-\varkappa^{-2})^{1/4}}$$

Outline of the proof

(the Riemann - Hilbert version)

Key ingredient # 1 - the RH problem of Baik, Deift, Johansson

- $Y(z) \in H(\mathbb{C} \setminus C)$ $\quad 2 \times 2$

- $$Y_+(z) = Y_-(z) \begin{pmatrix} 1 & z^{-n}\varphi(z) \\ 0 & 1 \end{pmatrix} \qquad (21.1)$$
  $z \in C$

- $$Y(z) = (I + O(z^{-1}))\, z^{n\sigma_3} \qquad z \to \infty$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad Y(z) \equiv Y(z; n)$$

Connection to TD:

$$h_n \equiv D_{n+1}/D_n = Y_{12}(0)$$

$$\log D_n[\varphi] = -\int_0^1 \int_C R(z,z;\gamma)\,dz\,d\gamma$$

$$R(z,z';\gamma) = \frac{Y_{11}(z;\gamma)Y_{21}(z';\gamma) - Y_{11}(z';\gamma)Y_{21}(z;\gamma)}{z-z'}$$

$$\times\ (z')^{-n}\ \frac{\varphi(z')-1}{2\pi i}$$

$$Y(z;\gamma) \longleftarrow \varphi_\gamma(z) \equiv \gamma\varphi(z) + 1 - \gamma$$

The RHP (21.1)

$\oplus$

Nonlinear steepest descent method

$\Downarrow$

asymptotics of $Y(z;n)$:

(and $D_n[\varphi]$)

$$Y(z;n) = \left(I + O\left(\frac{e^{-\varepsilon n}}{1+|z|}\right)\right) \qquad n \to \infty$$

$$\times \begin{cases} \begin{pmatrix} \alpha(z) & 0 \\ 0 & \alpha^{-1}(z) \end{pmatrix} z^{n\sigma_3} & |z| > 1 \\ \\ \begin{pmatrix} 0 & \alpha(z) \\ -\alpha^{-1}(z) & 0 \end{pmatrix} & |z| < 1 \end{cases}$$

Connection to OPUC

$$\underline{Y}(z) = \begin{pmatrix} P_n(z) & \displaystyle\int_C \frac{P_n(\xi)\,\varphi(\xi)}{\xi - z}\,\frac{d\xi}{2\pi i\,\xi^n} \\ -\dfrac{1}{h_{n-1}}\, z^{n-1}\,\hat{P}_{n-1}(z^{-1}) & -\dfrac{1}{h_{n-1}} \displaystyle\int_C \frac{\hat{P}_{n-1}(\xi^{-1})\,\varphi(\xi)}{\xi - z}\,\frac{d\xi}{2\pi i\,\xi} \end{pmatrix}$$

(24.1)

$$\int_C P_n(z)\,\hat{P}_k(z^{-1})\,\varphi(z)\,\frac{dz}{2\pi i\,z} = h_n\,\delta_{nk}$$

$$\left(\quad h_n = D_{n+1}/D_n \quad \right)$$

(24.1) + determinantal formulae for $P_n(z)$ and $\hat{P}_n(z)$

$$\Downarrow$$

Key ingredient # 2:

$$D^{B}_{n+1}\left[\varphi;\, a+\frac{b}{z-c}\right] \qquad (25.1)$$

$$= \quad a\,D_n[\varphi] - b\,c^{-n-1}\,Y_{11}(c)\,D_n[\varphi]$$

if $|c|>1$

$$= \quad a\,D_n[\varphi] \qquad \text{if } 0<|c|<1$$

(24.1) ⊕ determinantal formulae

for $P_n(z)$ ⊕ N. Witte

$$\Downarrow$$

Key ingredient # 3.

$$D^{B}_{n+1}\left[\varphi; \left(a+\frac{b}{z-c}\right)\varphi\right]$$

(26.1)

$$= D_{n+1}[\varphi]\left(a-\frac{b}{c}\right)+\frac{b}{c}\, D_n[\varphi]\, Y_{12}(c)$$

if $c \neq 0$

$$= a\, D_{n+1}[\varphi] - b\, D_n[\tilde{\varphi}] \lim_{z\to\infty} \frac{\tilde{Y}_{11}(z)-z^{n}}{z^{n-1}}$$

if $c=0$

$$\left(\tilde{\varphi}(z) = \varphi(z^{-1}), \quad \tilde{Y}(z) \equiv Y(z) \longleftrightarrow \tilde{\varphi}(z) \right)$$

(25.1), (26.1)

$\oplus$

known asymptotics for

$D_n[\varphi]$ and $Y(z)$

$$\Downarrow$$

Theorem 1.

Remark about $\langle \sigma_{00} \sigma_{mn} \rangle$

$m \neq n, n-1, 0$.

- $T > T_c$: $\langle \sigma_{00} \sigma_{mn} \rangle \sim C(\theta, T) \dfrac{e^{-k g(T)}}{k^{1/2}}$

$$k = \sqrt{m^2 + n^2}, \quad \tan\theta = m/n$$

(M. Fisher, 1962)

- Determinant (Fredholm) representations of Cheng, Wu (1967)

- Form factor series representations by Wu, McCoy, Tracy and Barouch (1976)

- $\langle \sigma_{00} \sigma_{mn} \rangle \sim (1 - x^{-2})^{1/4}$ $\quad T < T_c$

$$\times \left[1 + O\left(e^{-2m\theta_1 - 2\ell \theta_2} \right) \right]$$

(Cheng, Wu)

- B. M. McCoy, The Romance of the Ising Model 2011 arXiv: 1111.7006

- P. Deift, A. Its, I. Krasovky
  Toeplitz matrices and Toeplitz determinants under the impetus of the Ising model. Some history and some recent results
  arXiv: 1207.4990.

- B. M. McCoy, T. T. Wu, The Two-Dimensional Ising Model, Harvard Univ. Prees, 1973

Rigorous asymptotic analysis of $\langle \sigma_{00} \sigma_{mn} \rangle$ is still a challenge.

- $< \sigma_{00} \sigma_{mn} >$ – $\underline{PVI}$

$$\frac{\sinh \frac{2J_h}{K_B T}}{\sinh \frac{2J_v}{K_B T}} = -\, x^{-1}$$

S. Boukraa, J.-M. Maillard, B. M. McCoy

2020

Appendix 1.

Derivation of (26.1)

$$D^{\beta}_{n+1}[\varphi;\Psi], \quad \Psi = \frac{1}{z-c}\varphi, \quad c \neq 0$$

We have

$$\Psi_k = \int_C z^{-k}\frac{1}{z-c}\varphi(z)\frac{dz}{2\pi i z}$$

$$= \int_C z^{-k}\frac{1}{z(z-c)}\varphi(z)\frac{dz}{2\pi i}$$

$$= -\frac{1}{c}\int_C z^{-k}\varphi(z)\frac{dz}{2\pi i z} + \frac{1}{c}\int_C \frac{z^{-k}\varphi(z)}{z-c}\frac{dz}{2\pi i}$$

$$= -\frac{1}{c}\varphi_k + \frac{1}{c}\int_C \frac{z^{-k}\varphi(z)}{z-c}\frac{dz}{2\pi i}$$

There fore :

$$D^B_{n+1}[\varphi; I]$$

$$= \frac{1}{c}\det\begin{pmatrix} \varphi_0 & \cdots & \varphi_{-n} \\ \varphi_1 & \cdots & \varphi_{-n+1} \\ \vdots & & \vdots \\ \varphi_{n-1} & & \varphi_{-1} \\ -\varphi_n + \int_C \frac{z^{-n}\varphi(z)}{z-c}\frac{dz}{2\pi i} & \cdots & -\varphi_0 + \int_C \frac{\varphi(z)}{z-c}\frac{dz}{2\pi i} \end{pmatrix}$$

$$= -\frac{1}{c} D_{n+1}[\varphi]$$

$$+ \frac{1}{c} \det \begin{pmatrix} \varphi_0 & \varphi_{-1} & \dots & \varphi_{-n} \\ \varphi_1 & \varphi_0 & \dots & \varphi_{-n+1} \\ \vdots & \vdots & & \vdots \\ \varphi_{n-1} & \varphi_{n-2} & \dots & \varphi_{-1} \\ \int\limits_C \frac{z^{-n}\varphi(z)}{z-c} \frac{dz}{2\pi i} & \int\limits_C \frac{z^{-n+1}\varphi(z)}{z-c} \frac{dz}{2\pi i} & \dots & \int\limits_C \frac{\varphi(z)}{z-c} \frac{dz}{2\pi i} \end{pmatrix}$$

$$= -\frac{1}{c} D_{n+1}[\varphi]$$

$$+ \frac{1}{c} \int\limits_C \det \begin{pmatrix} \varphi_0 & \varphi_{-1} & \dots & \varphi_{-n} \\ \varphi_1 & \varphi_0 & \dots & \varphi_{-n+1} \\ \vdots & \vdots & & \vdots \\ \varphi_{n-1} & \varphi_{n-2} & \dots & \varphi_{-1} \\ z^{-n} & z^{-n+1} & \dots & 1 \end{pmatrix} \frac{\varphi(z)}{z-c} \frac{dz}{2\pi i}$$

Remember now a classical formula:

$$P_n(z) = \frac{1}{D_n[\varphi]} \det \begin{pmatrix} \varphi_0 & \varphi_{-1} & \dots & \varphi_{-n} \\ \varphi_1 & \varphi_0 & \dots & \varphi_{-n+1} \\ \vdots & \vdots & & \vdots \\ \varphi_{n-1} & \varphi_{n-2} & \dots & \varphi_{-1} \\ 1 & z & \dots & z^n \end{pmatrix}$$

Hence, the last integral on page 3 is just

$$\frac{1}{c} D_n[\varphi] \int_C \frac{z^{-n} P_n(z) \varphi(z)}{z-c} \frac{dz}{2\pi i}$$

$$= \frac{1}{c} D_n[\varphi] Y_{12}(c). \quad \text{Q.E.D.}$$

Appendix 2.

Derivation of (25.1)

$$D^{B}_{n+1} \; [\varphi; q] \qquad q = \frac{1}{z-c} \; , \; c \neq 0 .$$

We have

$$q_k = \int_C z^{-k} \frac{1}{z-c} \frac{dz}{2\pi i z}$$

$$= \begin{cases} 0 & 0 < |c| < 1 \\ -c^{-k-1} & |c| > 1 \end{cases}$$

$k = 0, 1, \ldots n$

Therefore,

$$D^{B}_{n+1}[\varphi; q] = 0 \quad \text{if } 0<|c|<1$$

while for $|c|>1$ we have

$$Y_{11}(c) = P_n(c)$$

$$= \frac{1}{D_n[\varphi]} \det \begin{pmatrix} \varphi_0 & \varphi_{-1} & \cdots & \varphi_{-n} \\ \varphi_1 & \varphi_0 & \cdots & \varphi_{-n+1} \\ \vdots & \vdots & & \vdots \\ \varphi_{n-1} & \varphi_{n-2} & \cdots & \varphi_{-1} \\ 1 & c & & c^n \end{pmatrix}$$

$$\Rightarrow -c^{-n-1} D_n[\varphi] Y_{11}(c) = \det \begin{pmatrix} \varphi_0 & \varphi_{-1} & \cdots & \varphi_{-n} \\ \varphi_1 & \varphi_0 & \cdots & \varphi_{-n+1} \\ \vdots & \vdots & & \vdots \\ \varphi_{n-1} & \varphi_{n-2} & \cdots & \varphi_{-1} \\ -c^{-n-1} & -c^{-n} & \cdots & -c \end{pmatrix}$$

$$= D^{B}_{n+1}[\varphi; q] \qquad \text{Q.E.D.}$$

Appendix 3.

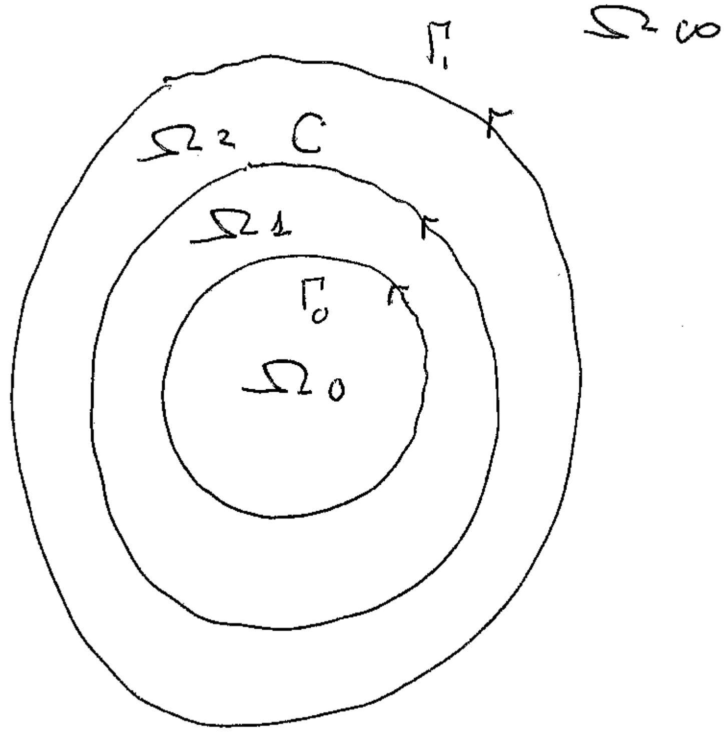
Asymptotics of $Y(z;n)$

$$Y(z) \longmapsto T(z) = \begin{cases} Y(z)\, z^{-n\sigma_3} & |z|>1 \\ Y(z) & |z|<1 \end{cases}$$

- $$T_+(z) = T_-(z) \begin{pmatrix} z^n & \varphi(z) \\ 0 & z^{-n} \end{pmatrix} \quad z \in C$$

- $$T(\infty) = I$$

$$\begin{pmatrix} z^n & \varphi(z) \\ 0 & z^{-n} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z^{-n}\varphi^{-1} & 1 \end{pmatrix} \begin{pmatrix} 0 & \varphi \\ -\varphi^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z^{n}\varphi^{-1} & 1 \end{pmatrix}$$

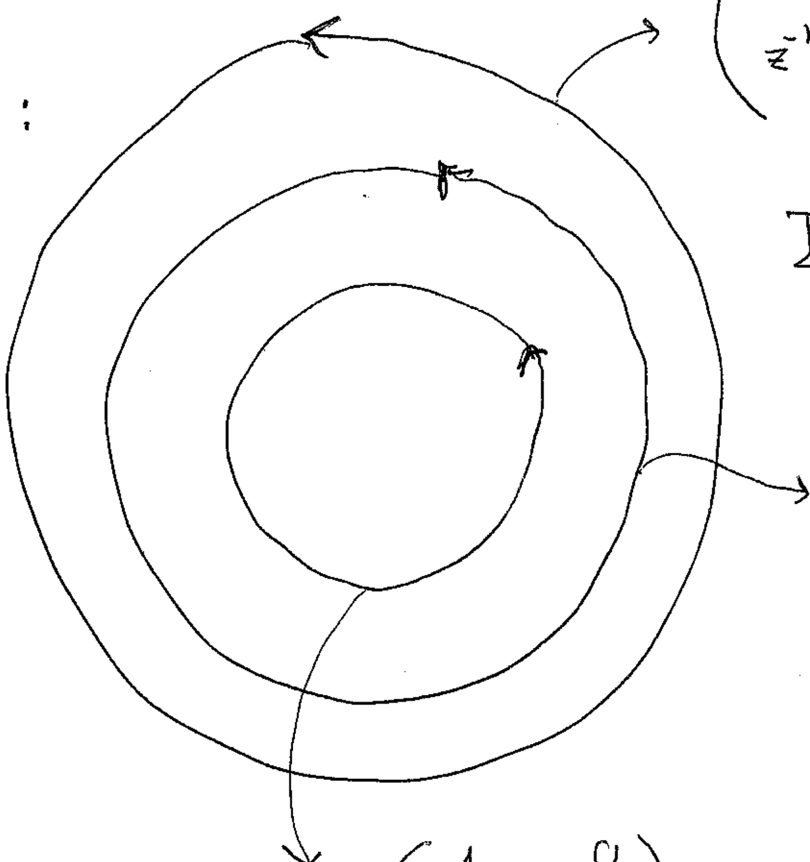


$$T(z) \longmapsto S(z) = T(z) \begin{cases} \begin{pmatrix} 1 & 0 \\ z^{-n}\varphi^{-1} & 1 \end{pmatrix} & \text{in } \Omega_2 \\ \begin{pmatrix} 1 & 0 \\ -z^{n}\varphi^{-1} & 1 \end{pmatrix} & \text{in } \Omega_1 \\ I & \text{in } \Omega_0 \cup \Omega_\infty \end{cases}$$

- $$S_+(z) = S_-(z)\, G_S(z)$$

$$z \in \Gamma_S$$

$G_S(z)$:



$$\begin{pmatrix} 1 & 0 \\ z^{-n}\varphi^{-1} & 1 \end{pmatrix} = I + O(e^{-\varepsilon n})$$

$$\begin{pmatrix} 0 & \varphi(z) \\ -\varphi^{-1}(z) & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ z^{n}\varphi^{-1} & 1 \end{pmatrix} = I + O(e^{-\varepsilon n})$$

- $$S(\infty) = I$$

$$\Downarrow$$

$$S(z) = \left(I + O\left(\frac{e^{-\varepsilon n}}{1+|z|} \right) \right) P^{(\infty)}(z)$$

- $P^{(\infty)}(z) \in H(\mathbb{C} \setminus C)$
- $P_+^{(\infty)}(z) = P_-^{(\infty)}(z) \begin{pmatrix} 0 & \varphi(z) \\ -\varphi^{-1}(z) & 0 \end{pmatrix} \quad z \in C$
- $P^{(\infty)}(\infty) = I$

Observe:

$$\begin{pmatrix} 0 & \varphi(z) \\ -\varphi^{-1}(z) & 0 \end{pmatrix} = \begin{pmatrix} \alpha_-^{-1}(z) & 0 \\ 0 & \alpha_-(z) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \alpha_+^{-1}(z) & 0 \\ 0 & \alpha_+(z) \end{pmatrix}$$

$$\alpha(z) = \exp\left[\frac{1}{2\pi i} \int_C \frac{\log \varphi(\xi)}{\xi - z} d\xi \right]$$

$$\left(\alpha_+(z) / \alpha_-(z) = \varphi(z) \; ! \right)$$

$$\Downarrow$$

$$P^{(\infty)}(z) = \begin{cases} \begin{pmatrix} \alpha(z) & 0 \\ 0 & \alpha^{-1}(z) \end{pmatrix} & |z| > 1 \\ \begin{pmatrix} 0 & \alpha(z) \\ -\alpha^{-1}(z) & 0 \end{pmatrix} & |z| < 1 \end{cases}$$

asymptotics for $Y(z; n)$ follows.