Differential geometry of orbit space of extended Jacobi group A_1

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Dubrovin-Frobenius Manifolds

A Dubrovin-Frobenius structure on the manifold M is the data (M, $\bullet$, $<,>$, e, E) satisfying:

1. $\eta := <,>$ is a flat pseudo-Riemannian metric;
2. $\bullet$ is product of Frobenius algebra on $T_m M$ which depends smoothly on m;
3. e is the unity vector field for the product $\bullet$ and $\nabla e = 0$;
4. $\nabla_w c(x, y, z)$ is symmetric, where $c(x, y, z) :=< x \bullet y, z >$;
5. A linear vector field $E \in \Gamma(M)$ must be fixed on M, i.e. $\nabla \nabla E = 0$ such that:

$$L_E <,>= (2 - d) <,>,$$
$$L_E \bullet = \bullet,$$
$$L_E e = e.$$

The function $F(t), t = (t^1, t^2, .., t^n)$ is a solution of WDVV equation if its third derivatives

$$c_{\alpha\beta\gamma} = \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\gamma} \tag{1}$$

satisfies the following conditions:

1. $$\eta_{\alpha\beta} = c_{1\alpha\beta}$$

   is constant nondegenerate matrix.
2. The function

   $$c^\gamma_{\alpha\beta} = \eta^{\gamma\delta} c_{\alpha\beta\delta}$$

   is structure constant of assosciative algebra.
3. F(t) must be quasihomogeneous function

   $$F(c^{d_1} t^1, .., c^{d_n} t^n) = c^{d_F} F(t^1, .., t^n)$$

   for any nonzero c and for some numbers $d_1, ..., d_n, d_F$.

WDDV equation/Dubrovin Frobenius manifold correspondence

Theorem (Dubrovin 1992)

There is a one to one correspondence between a Dubrovin-Frobenius manifold and solutions of WDVV equation.

Main applications of Dubrovin Frobenius manifold theory

1. Gromov Witten theory,
2. Singularity theory,
3. Hamiltonian theory of integrable hierarchies.

Intersection form and Monodromy

The intersection form is the bilinear pairing in T^*M defined by:

$$(\omega_1, \omega_2)^* := \iota_E(\omega_1 \bullet \omega_2)$$

where $\omega_1, \omega_2 \in T^*M$ and $\bullet$ is the induced Frobenius algebra product in the cotangent space. Let us denote by g^* the intersection form.

Intersection form and Monodromy

The intersection form g^* of a Dubrovin-Frobenius manifold is a flat almost everywhere nondegenerate metric. Let us define:

$$\Sigma = \{x \in M : det(g) = 0\}$$

Hence, the linear system of differential equations,

$$g^{\alpha\epsilon}\partial_\epsilon\partial_\beta x + \Gamma^{\alpha\epsilon}_{\beta}\partial_\epsilon x = 0,$$

denoted by Gauss-Manin connection, determining g^*-flat coordinates has poles , and consequently its solutions $x_a(t^1, .., t^n)$ are multivalued, where $(t^1, .., t^n)$ are flat coordinates of η. The analytical continuation of the solutions $x_a(t^1, .., t^n)$ has monodromy corresponding to loops around Σ. This gives rise to a monodromy representation of $\pi_1(M \setminus \Sigma)$, which is called Monodromy of the Dubrovin-Frobenius manifold.

Frobenius Manifolds as Orbit spaces

Theorem (Dubrovin Conjecture, Hertling 1999)

Any irreducible semisimple polynomial Dubrovin-Frobenius manifold with positive invariant degrees is isomorphic to the orbit space of a finite Coxeter group.

Main Point

Differential geometry of the orbit spaces of reflection groups and of their **extensions** $\mapsto$ Dubrovin-Frobenius manifolds.

Example: W is Extended affine Weyl Group [Dubrovin, Zhang 1998] and for Jacobi groups [Bertola 1999].

Examples of Orbit spaces

Example 1:
For $\mathbb{C}/A_1$:

1. Group action: $v_0 \mapsto -v_0$;
2. Invariant metric: $ds^2 = dv_0^2$
3. Invariant functions: $\mathbb{C}[v_0^2]$,
4. WDVV solution: $F(t_1) = \frac{t_1^3}{6}$.

Example 2:
For $\mathbb{C}^2/\tilde{A}_1$:

1. Group action: $(v_0, v_2) \mapsto (-v_0 + m_0, v_2 + m_2)$;
2. Invariant metric: $ds^2 = dv_0^2 - dv_2^2$
3. Invariant functions: $\mathbb{C}[e^{2\pi i v_2}\cos(2\pi i v_0), e^{2\pi i v_2}]$,
4. WDVV solution: $F(t_1, t_2) = \frac{t_1^2 t^2}{2} + e^{t^2}$.

Examples of Orbit spaces

Example 3:
For $\mathbb{C} \oplus \mathbb{C} \oplus \mathbb{H}/\mathcal{J}(A_1)$:

1. Group action:
2. $(\phi, v_0, \tau) \mapsto (\phi, -v_0, \tau)$
3. $(\phi, v_0, \tau) \mapsto (\phi - nv_0 - \frac{n^2\tau}{2}, v_0 + m + n\tau, \tau)$;
4. $(\phi, v_0, \tau) \mapsto (\phi - \frac{cv_0^2}{c\tau+d}, \frac{v_0}{c\tau+d}, \frac{a\tau+b}{c\tau+d})$;
5. Invariant metric: $ds^2 = dv_0^2 + d\phi d\tau$
6. Invariant functions: $M_\bullet[\varphi_0, \varphi_2]$,
7. WDVV solution: $F(t_1, t_2, \tau) = \frac{t_1^2\tau}{2} + \frac{t_1 t_2^2}{2} - \frac{i\pi t_2^4}{48} E_2(\tau)$.

where $(m, n) \in \mathbb{Z}^2$, and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, and

$$
\begin{aligned}
\varphi_2 &= e^{2\pi i\phi}\left(\frac{\theta_1(v_0, \tau)}{\theta_1'(0, \tau)}\right)^2, \\
\varphi_0 &= \varphi_2 \wp(v_0, \tau).
\end{aligned} \tag{2}
$$

Hurwitz space as Frobenius manifold

The Hurwitz space $H_{g,n_0,n_1,..,n_m}$ is the moduli space of curves C_g of genus g endowed with $N = m + 1 + n_0 + ..n_m$ branched covering $\lambda : C_g \mapsto \mathbb{C}P^1$ with $m + 1$ branching points over ∞ in $\mathbb{C}P^1$ of branching degree $n_j + 1$, $j = 0, .., m$.
The set of branch points $\{\lambda_1, ..., \lambda_n\}$ gives coordinates on the Hurwitz space $\hat{H}_{g;n_0,...,n_m}$.
To build a frobenius structure on $\hat{H}_{g;n_0,...,n_m}$ take $\partial_i := \frac{\partial}{\partial \lambda_i}$,

1. the multiplication as $\partial_i \bullet \partial_j = \delta_{ij}\partial_i$,
2. $e = \sum \partial_i$,
3. $E = \sum \lambda^i \partial_i$,
4. $\eta = \sum res_{P_i} \frac{\phi^2}{d\lambda}(d\lambda^i)^2$,

where ϕ are the primary differential.

Examples of Hurwitz spaces

Example 1:
For $H_{0,1}$:

1. $\lambda(p, v_0) = p^2 - v_0^2$;
2. $H_{0,1} \cong \mathbb{C}/A_1$.

Example 2:
For $H_{0,0,0}$:

1. $\lambda(p, a, b) = p + \frac{a}{p-b}$;
2. $H_{0,0,0} \cong \mathbb{C}^2/\tilde{A}_1$,

Example 3:
For $H_{1,1}$:

1. $\lambda(v, v_0, \phi, \tau) = e^{2\pi i \phi} \frac{\theta_1(v-v_0|\tau)\theta_1(v+v_0|\tau)}{\theta_1^2(v|\tau)}$;
2. $H_{1,1} \cong \mathbb{C}^3/\mathcal{J}(A_1)$,

Problem Setting

$H_{1,1} \cong \mathbb{C}^3/\mathcal{J}(A_1)$
Example of Orbit space of Jacobi Group

$H_{0,0,0} \cong \mathbb{C}^2/\tilde{A}_1$
Example of Orbit space of Extended Affine Weyl Group

Mixed of Extended Affine Weyl Group + Jacobi Group?

$$H_{1,0,0} \cong \mathbb{C}^4/W$$

Results

$$\begin{array}{ccc} H_{0,0,0} \cong \mathbb{C}^2/\tilde{A}_1 & \longleftarrow & H_{0,1} \cong \mathbb{C}/A_1 \\ \downarrow & & \downarrow \\ H_{1,0,0} \cong \mathbb{C}^4/\mathcal{J}(\tilde{A}_1) & \longleftarrow & H_{1,1} \cong \mathbb{C}^3/\mathcal{J}(A_1) \end{array}$$

1. $H_{0,1}$, g=0, 1 double pole.
2. $H_{0,0,0}$,g=0, 2 simple pole.
3. $H_{1,1}$, g=1, 1 double pole.
4. $H_{1,0,0}$, g=1, 2 simple pole.

Action of $\mathcal{J}(\tilde{A}_1)$

For $(\mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{H})/\mathcal{J}(\tilde{A}_1)$
$\mathcal{J}(\tilde{A}_1) \curvearrowright \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{H} \ni (\phi, v_0, v_2, \tau)$

$$\begin{aligned} w(\phi, v_0, v_2, \tau) &= (\phi, -v_0, v_2, \tau) \\ t(\phi, v_0, v_2, \tau) &= (\phi - 2 < n, v > + < n, n > \tau, v + m + n\tau, \tau) \\ \gamma(\phi, v_0, v_2, \tau) &= (\phi - \frac{c < v, v >}{c\tau + d}, \frac{v}{c\tau + d}, \frac{a\tau + b}{c\tau + d}) \end{aligned} \tag{3}$$

where $v = (v_0, v_2), m, n \in \mathbb{Z}^2$, and

$$< (v_0, v_2), (v_0, v_2) > = v_0^2 - v_2^2 \tag{4}$$

Jacobi forms of $\mathcal{J}(\tilde{A}_1)$

The weak $\tilde{A}_1$ -invariant Jacobi forms of weight k, order l, and index m are functions on $\Omega = \mathbb{C} \oplus \mathbb{C}^2 \oplus \mathbb{H} \ni (\phi, v_0, v_2, \tau) = (\phi, v, \tau)$ which satisfy

$$\begin{aligned}
&\varphi(w(\phi, v, \tau)) = \varphi(\phi, v, \tau), \quad A_1 \text{ invariant condition} \\
&\varphi(t(\phi, v, \tau)) = \varphi(\phi, v, \tau) \\
&\varphi(\gamma(\phi, v, \tau)) = (c\tau + d)^{-k}\varphi(\phi, v, \tau) \\
&E\varphi(\phi, v, \tau) := -\frac{1}{2\pi i}\frac{\partial}{\partial \phi}\varphi(\phi, v_0, v_2, \tau) = m\varphi(\phi, v_0, v_2, \tau)
\end{aligned} \tag{5}$$

Chevalley theorem, and generating function of the invariants

Hurwitz space/ Orbit space correspondence

$$[(\phi, v_0, v_2, \tau)] \leftrightarrow e^{2\pi i\phi}\frac{\theta_1(v-v_0|\tau)\theta_1(v+v_0|\tau)}{\theta_1(v-v_2|\tau)\theta_1(v+v_2|\tau)} \\ = \varphi_0 + \varphi_1[\zeta(v-v_2|\tau) - \zeta(v+v_2|\tau) + 2\zeta(v_2|\tau)] \tag{6}$$

Theorem 1

The trigraded algebra of Jacobi forms $J_{\bullet,\bullet,\bullet}^{\mathcal{J}(\tilde{A}_1)} = \bigoplus_{k,l,m} J_{k,l,m}^{\tilde{A}_1}$ *is freely generated by 2 fundamental Jacobi forms* $(\varphi_0^{\tilde{A}_1}, \varphi_1^{\tilde{A}_1})$ *over the graded ring* $E_{\bullet,\bullet}$

$$J_{\bullet,\bullet,\bullet}^{\mathcal{J}(\tilde{A}_1)} = E_{\bullet,\bullet}[\varphi_0^{\tilde{A}_1}, \varphi_1^{\tilde{A}_1}] \tag{7}$$

where $E_{\bullet,\bullet} := J_{\bullet,\bullet,0}$

Dubrovin Frobenius structure on the Orbit space of $\mathcal{J}(\tilde{A}_1)$

The natural candidate to be the intersection form of $\mathcal{J}(\tilde{A}_1)$ is:

$$ds^2 = 2dv_0^2 - 2dv_2^2 + 2d\phi d\tau \tag{8}$$

Lemma 2

The metric

$$ds^2 = 2dv_0^2 - 2dv_2^2 + 2d\phi d\tau \tag{9}$$

is invariant under the action of A_1, and translations. Moreover, the $SL_2(\mathbb{Z})$ transformations determine a conformal transformation of the metric ds^2, i.e:

$$2dv_0^2 - 2dv_2^2 + 2d\phi d\tau \mapsto \frac{2dv_0^2 - 2dv_2^2 + 2d\phi d\tau}{(c\tau + d)^2} \tag{10}$$

Dubrovin Frobenius structure on the Orbit space of $\mathcal{J}(\tilde{A}_1)$

For $H_{1,0,0}$:

1. $ds^2 = 2dv_0^2 - 2dv_2^2 + 2d\phi d\tau$
2. $e = \frac{\partial}{\partial \varphi_0}$;
3. $E = \varphi_0 \frac{\partial}{\partial \varphi_0} + \varphi_1 \frac{\partial}{\partial \varphi_1}$;
4. $L_e g^* = \eta^*$
5. $(t^1, t^2, t^3, t^4) = (\varphi_0 + 2\varphi_1 \frac{\theta_1'(v_2|\tau)}{\theta_1(v_2|\tau)}, \varphi_1, v_2, \tau)$
6. $F^{\alpha\beta} = \eta^{\alpha\mu}\eta^{\beta\lambda}\frac{\partial^2 F}{\partial t^\mu \partial t^\lambda} = \frac{g^{\alpha\beta}}{deg(g^{\alpha\beta})}$
7. $F(t^1, t^2, t^3, t^4) = \frac{i}{4\pi}(t^1)^2 t^4 - 2t^1 t^2 t^3 - (t^2)^2 \log(t^2 \frac{\theta_1'(0,t^4)}{\theta_1(2t^3,t^4)})$

Generalization

$$\begin{array}{ccc} H_{0,n-1,0} \cong \mathbb{C}^{n+1}/\tilde{A}_n & \longleftarrow & H_{0,n} \cong \mathbb{C}^n/A_n \\ \downarrow & & \downarrow \\ H_{1,n-1,0} \cong \mathbb{C}^{n+3}/\mathcal{J}(\tilde{A}_n) & \longleftarrow & H_{1,n} \cong \mathbb{C}^{n+2}/\mathcal{J}(A_n) \end{array}$$

1. $H_{0,n}$, g=0, 1 pole of order n;
2. $H_{0,n-1,0}$,g=0, 1 simple pole, 1 pole of order n-1;
3. $H_{1,n}$, g=1, 1 pole of order n;
4. $H_{1,n-1,0}$, g=1, 1 simple pole, 1 pole of order n-1.

Action of $\mathcal{J}(\tilde{A}_n)$

I will consider the A_n in the following extended space

$$L^{\tilde{A}_n} = \{(z_0, z_1, .., z_n, z_{n+1}) \in \mathbb{Z}^{n+2} : \sum_{i=0}^{n} v_i = 0\}.$$

The action of A_n on $L^{\tilde{A}_n}$ is given by

$$w(z_0, z_1, z_2, .., z_{n-1}, z_n, z_{n+1}) = (z_{i_0}, z_{i_1}, z_{i_2}, .., z_{i_{n-1}}, z_{i_n}, z_{n+1})$$

permutations in the first $n+1$ variables. Let the quadratic form $<, >$ given by

$$< v, v >= = v^T \begin{pmatrix} 2 & 1 & 1 & ...1 & 1 & 0 \\ 1 & 2 & 1 & ....1 & 1 & 0 \\ 1 & 1 & 2 & ....1 & 1 & 0 \\ . & . & . & ... & . & 0 \\ . & . & . & ... & . & 0 \\ 1 & 1 & 1 & ...2 & 1 & 0 \\ 1 & 1 & 1 & .... & 2 & 0 \\ 0 & 0 & 0 & .... & 0 & -(n+1) \end{pmatrix} v$$

Action of $\mathcal{J}(\tilde{A}_n)$

Consider the action $\mathcal{J}(\tilde{A}_n) \curvearrowright \Omega = \mathbb{C} \oplus \mathbb{C}^{n+1} \oplus \mathbb{H}$

Definition 3 (Jacobi group of $\tilde{A}_n$)

The "Jacobi group of $\tilde{A}_n$" is represented on the Tits cone $\Omega = \mathbb{C} \oplus \mathbb{C}^{n+1} \oplus \mathbb{H}$ by the definition of the action $w \in A_n$, $t = (\lambda, \mu) \in (\mathbb{Z} + \tau\mathbb{Z})^{n+1}$, $\gamma \in SL_2(\mathbb{Z})$ as :

1. $w(\phi, v, \tau) = (\phi, wv, \tau)$
2. $t(\phi, v, \tau) = (\phi - < \mu, v > -\frac{1}{2} < \mu, \mu > \tau, v + \lambda + \tau\mu, \tau)$
3. $\gamma(\phi, v, \tau) = (\phi - \frac{c}{2(c\tau+d)} < v, v > \tau, \frac{v}{c\tau+d}, \frac{a\tau+b}{c\tau+d})$

Jacobi forms of $\tilde{A}_n$

The weak $\tilde{A}_n$ -invariant Jacobi forms of weight k, order l, and index m are functions on $\Omega = \mathbb{C} \oplus \mathbb{C}^{n+2} \oplus \mathbb{H} \ni (\phi, v', v_{n+1}, \tau) = (\phi, v, \tau)$ which satisfy

$$\begin{aligned}
&\varphi(w(\phi, v, \tau)) = \varphi(\phi, v, \tau), \quad A_n \text{ invariant condition} \\
&\varphi(t(\phi, v, \tau)) = \varphi(\phi, v, \tau) \\
&\varphi(\gamma(\phi, v, \tau)) = (c\tau + d)^{-k} \varphi(\phi, v, \tau) \\
&E\varphi(\phi, v, \tau) := -\frac{1}{2\pi i}\frac{\partial}{\partial \phi}\varphi(\phi, v, \tau) = m\varphi(\phi, v, \tau)
\end{aligned} \tag{11}$$

Chevalley theorem

Theorem 4

The trigraded algebra of Jacobi forms $J^{\mathcal{J}(\tilde{A}_n)}_{\bullet,\bullet,\bullet} = \bigoplus_{k,l,m} J^{\tilde{A}_n}_{k,l,m}$ *is freely generated by* $n+1$ *fundamental Jacobi forms* $(\varphi_0^{\tilde{A}_n}, \varphi_1^{\tilde{A}_n},, \varphi_2^{\tilde{A}_n}, .., , \varphi_n^{\tilde{A}_n})$ *over the graded ring* $E_{\bullet,\bullet}$

$$J^{\mathcal{J}(\tilde{A}_n)}_{\bullet,\bullet,\bullet} = E_{\bullet,\bullet}[\varphi_0^{\tilde{A}_n}, \varphi_1^{\tilde{A}_n},, \varphi_2^{\tilde{A}_n}, .., , \varphi_n^{\tilde{A}_n}] \tag{12}$$

Theorem 5

The functions $(\varphi_0^{\tilde{A}_n}, \varphi_1^{\tilde{A}_n}, .., \varphi_n^{\tilde{A}_n})$ *obtained by the formula*

$$\begin{aligned}\lambda^{\tilde{A}_n} &= e^{2\pi i \phi_1} \frac{\prod_{i=0}^{n} \theta_1(z - v_i + v_{n+1}, \tau)}{\theta_1^n(z,\tau)\theta_1(z + (n+1)v_{n+1})} \\ &= \varphi_n^{\tilde{A}_n} \wp^{n-2}(z,\tau) + \varphi_{n-1}^{\tilde{A}_n} \wp^{n-3}(z,\tau) + ... + \varphi_2^{\tilde{A}_n} \wp(z,\tau) \\ &+ \varphi_1^{\tilde{A}_n}[\zeta(z,\tau) - \zeta(z + (n+1)v_{n+1}, \tau) + \varphi_0^{\tilde{A}_n}\end{aligned} \tag{13}$$

Work in progress

Using the orbifold charts of $\Omega/\mathcal{J}(\tilde{A}_n)$, it is possible to prove that there is an unique bilinear form that transforms as a modular form of weight 2 under the action of $SL_2(\mathbb{Z})$,i.e under $\tau \mapsto \frac{a\tau+b}{c\tau+d}$, $ds^2 \mapsto \frac{ds^2}{(c\tau+d)^2}$. This bilinear form is:

$$ds^2 = ds^2_{\tilde{A}_n} + 2d\tilde{\phi}d\tau \tag{14}$$

1. The unit vector field and Euler vector field are given in terms of the invariants. Indeed:

$$e = \frac{\partial}{\partial \varphi_0} \tag{15}$$

$$E = \varphi_0 \frac{\partial}{\partial \varphi_0} + \varphi_1 \frac{\partial}{\partial \varphi_1} + \varphi_2 \frac{\partial}{\partial \varphi_2} + .. + \varphi_n \frac{\partial}{\partial \varphi_n} \tag{16}$$

2. The last step is just to prove that $(\Omega/\mathcal{J}(\tilde{A}_n), g, L_e g, e, E)$ has a flat pencil strcuture, and therefore, a Frobenius structure. To prove it, note that $(\Omega/\mathcal{J}(\tilde{A}_n), g, e, E)$ is isomorphic to $(H_{1,n-1,0}, g, e, E)$, therefore, $(\Omega/\mathcal{J}(\tilde{A}_n), g, L_e g, e, E)$ has a flat pencil structure because $(H_{1,n-1,0}, g, L_e g, e, E)$ has it.

Thank you!

Formulas for g and η

$$< \partial_a, \partial_b >= - \sum_{|\lambda|<\infty} res_{d\lambda=0} \frac{\partial_a(\lambda(p)dp)\partial_b(\lambda(p)dp)}{d\lambda(p)} \tag{17}$$

$$(\partial_a, \partial_b) = - \sum_{|\lambda|<\infty} res_{d\lambda=0} \frac{\partial_a(Log\lambda(p)dp)\partial_b(Log\lambda(p)dp)}{dLog\lambda(p)} \tag{18}$$

$$c(\partial_a, \partial_b, \partial_c) = - \sum_{|\lambda|<\infty} res_{d\lambda=0} \frac{\partial_a(\lambda(p)dp)\partial_b(\lambda(p)dp)\partial_c(\lambda(p)dp)}{d\lambda(p)} \tag{19}$$

Flat coordinates of η on Hurwitz space

Theorem (Dubrovin 1992)

The corresponding flat coordinates $t_A, A = 1, ..., N$ consist of the five parts:

1. $t^{i;\alpha} = res_{\infty_i} \lambda^{\frac{-1}{n_i+1}} p d\lambda \quad i=0,...m, \quad \alpha = 1, ..., n_i;$
2. $p^i = v.p \int_{\infty 0}^{\infty_i} dp \quad i=0,...m;$
3. $q^i = res_{\infty i} \lambda dp \quad i=0,...m;$
4. $\tau^i = \int_{bi} dp \quad i=1,...g;$
5. $s^i = \int_{ai} \lambda dp \quad i=1,...g.$

Formulas

$$\wp(z,\omega,\omega') = \frac{1}{z^2} + \sum_{m^2+n^2\neq 0} \frac{1}{(z+2m\omega+2n\omega')^2} + \frac{1}{(2m\omega+2n\omega')^2} \tag{20}$$

$$\frac{d\zeta}{dz} = -\wp \tag{21}$$

$$\frac{dLog\sigma}{dz} = \zeta \tag{22}$$

$$\eta = \zeta(\omega,\omega,\omega') \tag{23}$$

$$\Theta_1(v|\tau) = 2\sum_{n=0}^{\infty}(-1)^n exp(i\pi(n+\frac{1}{2})^2\tau)sin((2n+1)\pi v) \tag{24}$$

$$\sigma(z,\omega,\omega') = 2\omega\frac{\Theta_1(\frac{z}{2\omega}|\tau)}{\Theta_1'(0|\tau)}exp(\frac{\eta z^2}{2\omega}) \tag{25}$$