Trieste-SISSA May 2022

A Dubrovin-Frobenius manifold structure of NLS type on the orbit space of B_n.

Paolo Lorenzoni

Joint work with **A. Arsie, I. Mencattini and G. Moroni**

arXiv:2111.03964

UNIVERSITA' DEGLI STUDI DI MILANO BICOCCA

Plan of the talk

1. (Bi-)Flat F-manifolds and Dubrovin-Frobenius manifolds and reflection groups.

2. The case of B_2: focusing and defocusing NLS.

3. The case of B_n.

Flat F-manifolds (Manin)

Definition

A **flat F-manifold** $(M, \circ, \nabla, e)$ is a manifold equipped with a product

$$\circ : TM \times TM \to TM$$

with structure constants c^i_{jk}, a connection ∇ with Christoffel symbols Γ^i_{jk} and a distinguished vector field e s.t.

- the one parameter family of connections ∇_λ

$$\Gamma^i_{jk} - \lambda c^i_{jk}$$

  is flat and torsionless for any λ.
- e is the unit of the product.
- e is flat: $\nabla e = 0$.

For a given λ the torsion and the curvature are

$$\begin{aligned} T^{(\lambda)k}_{ij} &= \Gamma^k_{ij} - \Gamma^k_{ji} + \lambda(c^k_{ij} - c^k_{ji}) \\ R^{(\lambda)k}_{ijl} &= R^k_{ijl} + \lambda(\nabla_i c^k_{jl} - \nabla_j c^k_{il}) + \lambda^2(c^k_{im}c^m_{jl} - c^k_{jm}c^m_{il}), \end{aligned}$$

We obtain

1. the connection ∇ is torsionless,
2. the product $\circ$ is commutative,
3. the connection ∇ is flat,
4. the tensor field $\nabla_l c^k_{ij}$ is symmetric in the low indices,
5. the product $\circ$ is associative.

The above conditions imply

$$c^i_{jk} = \partial_j \partial_k F^i.$$

Dubrovin-Frobenius manifolds

To define a Dubrovin-Frobenius manifold we need

- a metric η satisfying the conditions

$$\eta_{il}c^l_{jk} = \eta_{jl}c^l_{ik}, \qquad \nabla\eta = 0.$$

  It turns out that

$$c^i_{jk} = \eta^{il}\partial_l\partial_j\partial_k F$$

  where F is a solution of WDVV equations.

- a second distinguished linear vector field, called the Euler vector field and denoted by E. It turns out that

$$\mathcal{L}_E F = d_F F + \cdots$$

Duality for flat-F manifolds and Dubrovin-Frobenius manifolds

- Duality for Dubrovin-Frobenius manifold (almost-dual structure, Dubrovin 2004): it is defined by the data $(g, *, E)$ where E is the Euler vector field,

$$g = (E\circ)\eta^{-1}, \qquad X * Y = (E\circ)^{-1} X \circ Y$$

- Duality for flat-F manifolds $(\nabla, \circ, e) \rightarrow$ bi-flat F-manifolds $(\nabla, \circ, e, \nabla^*, *, E)$ (A.Arsie and P.L. 2013): the dual structure is defined by a flat structure $(\nabla^*, *, E)$ where

$$d_\nabla(X\circ) = d_{\nabla^*}(X\circ) \qquad \text{or} \qquad d_\nabla(X*) = d_{\nabla^*}(X*).$$

A different but equivalent definition of duality appears in a paper of Konishi, Minabe and Shiraishi (2018).

Some results on flat and bi-flat F-manifolds

Flat and bi-flat F-manifolds share many properties of Dubrovin-Frobenius manifolds. Among them

- Relations with Painlevé transcendents:
  Arsie and L. (2013, 2015), L. (2014),
  Kato, Mano and Sekiguchi (2015),
  Kawakami and Mano (2019).
- Existence of associated integrable PDEs:
  Arsie, Buryak, L., Rossi (2021).
- Relations with reflection groups:...

Flat F-manifolds, Dubrovin-Frobenius manifolds and reflection groups

1. Saito's metric and Saito's flat coordinates (Saito 1979 and Saito, Yano and Sekiguchi 1980).
2. From Saito's metric to flat pencil of metrics and Dubrovin-Frobenius manifolds (Dubrovin, 1993).
3. The case of Shephard groups (Dubrovin, 2004)
4. A generalization in the case of B_n and D_n (D. Zuo, 2007)
5. A standard flat structure on the orbit space of well generated complex reflection groups (Kato, Mano and Sekiguchi, 2015).
6. An alternative construction starting from a dual "logarihtmic" flat structure (Arsie, PL 2017)
7. An alternative proof of the existence of the "standard" KMS flat structure (Konishi, Minabe and Shiraishi, 2018).

Chevalley's theorem

Let G be a Coxeter group acting on a n-dimensional euclidean space and R be the subring of invariant polynomials.

Theorem

(Chevalley) There esist n positive integers $(d_1, ..., d_n)$, called degrees of G, such that R is generated by n homogeneous invariant polynomials $(u^1, ..., u^n)$, algebraically independent s.t.

$$deg(u^i) = d_i, \qquad d_1 = 2 < d_2 \leqslant d_3 \leqslant \cdots \leqslant d_{n-1} < d_n.$$

Polynomial invariants are not uniquely defined: how to fix them?

Saito's answer: as flat coordinates of a flat metric

Let

$$g^{ij} = \frac{\partial u^i}{\partial p^l}\delta^{lm}\frac{\partial u^j}{\partial p^m} = \sum_{l=1}^{n}\frac{\partial u^i}{\partial p^l}\frac{\partial u^j}{\partial p^l}$$

be the components the inverse of the euclidean metric g in the coordinates $(u^1, ..., u^n)$.

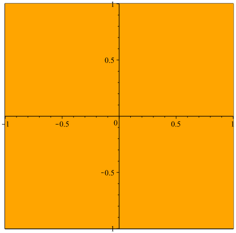
There exists a unique choice (up to rescalings $u^i \to c^i u^i$) of basic invariant polynomials $(u^1, ..., u^n)$ such that the quantities

$$\eta^{ij} = \frac{\partial g^{ij}}{\partial u^n}$$

define a constant non degenerate matrix.

The case of B_2 (group of symmetries of a square)

In this case $d_1 = 2, d_2 = 4$.



The polynomial invariants must be invariant w.r.t.

$$(p_1, p_2) \to (\pm p_1, \pm p_2),\ (\pm p_1, \mp p_2),\ (\pm p_2, \pm p_1),\ (\pm p_2, \mp p_1),$$

These transformations are generated by

$$(p_1, p_2) \to (p_2, p_1), \qquad (p_1, p_2) \to (p_1, -p_2)$$

Saito's procedure

Basic invariants are defined in terms of elementary symmetric polynomials in (p_1^2, p_2^2):

$$u_1 = \frac{1}{8}(p_1^2 + p_2^2), \qquad u_2 = p_1^2 p_2^2 + c u_1^2.$$

Rewriting the Euclidean cometric in the coordinates (u_1, u_2) we get

$$g = \begin{pmatrix} \frac{1}{2}u_1 & u_2 \\ u_2 & -2c(c+16)u_1^3 + 4(c+8)u_1 u_2 \end{pmatrix}.$$

The cometric

$$\eta = \mathcal{L}_{\frac{\partial}{\partial u_2}} g = \begin{pmatrix} 0 & 1 \\ 1 & 4(c+8)u_1 \end{pmatrix}$$

is constant if only if $c = -8$.

Saito's metric and Saito's flat coordinates for B_2

With this choice we obtain

$$\eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and the Saito's flat coordinates are

$$\begin{aligned} u_1 &= \frac{1}{8}(p_1^2 + p_2^2), \\ u_2 &= p_1^2 p_2^2 - 8u_1^2. \end{aligned}$$

Saito's result and bihamiltonian geometry

The pencil of contravariant metrics $g - \lambda\eta$ is a flat pencil. This means that:

- The pencil of contravariant metrics $g - \lambda\eta$ is flat for any λ.
- The contravariant Christoffel symbols of the pencil $\Gamma^{ij}_{(g_\lambda)k}$ coincide with the pencil of contravariant Christoffel symbols $\Gamma^{ij}_{(\eta)k}$ and $\Gamma^{ij}_{(g)k}$:

$$\Gamma^{ij}_{(g_\lambda)k} = \Gamma^{ij}_{(g)k} - \lambda\Gamma^{ij}_{(\eta)k}.$$

Flat pencils of metrics define bi-Hamiltonian structures of hydrodynamic type (Dubrovin, 1998).

Flat pencils associated with Dubrovin-Frobenius manifolds

The flat pencil $g - \lambda\eta$ associated with a Dubrovin-Frobenius manifold satisfies the following additional properties

- **Exactness**: there exists a vector field e called the **Liouville vector field** such that

$$\mathcal{L}_e g = \eta, \qquad \mathcal{L}_e \eta = 0.$$

- **Homogeneity**:

$$\mathcal{L}_E g = (d-1) g,$$

  where $E^i := g^{il}\eta_{lj}e^j$.
- **Egorov property**: locally there exists a function τ such that

$$e^i = \eta^{is}\partial_s \tau, \qquad E^i = g^{is}\partial_s \tau.$$

Flat pencils satisfying these properties are called **quasi-homogeneous**.

Regular pencil and Dubrovin-Frobenius manifolds

Definition

A quasi-homogeneous pencil is called **regular** if the operator

$$R := \nabla_{(\eta)}E - \nabla_{(g)}E$$

is invertible.

Theorem

Let $g_\lambda = g - \lambda\eta$ be a regular quasi-homogeneous flat pencil of metrics, then the data $(\circ, \eta, e, E)$, where the product $\circ$ is defined by the structure constants

$$c^j_{hk} := g^{sm}\eta_{mh}\left(\Gamma^{(\eta)l}_{sk} - \Gamma^{(g)l}_{sk}\right)(R^{-1})^j_l,$$

define a Dubrovin-Frobenius manifold.

Polynomials solutions of WDVV

Applying this theorem to Saito's flat pencil of metrics one gets a polynomial solution of WDVV equation. For instance

$$F_{B_2} = \frac{1}{2} u_1 u_2^2 + \frac{64}{15} u_1^5.$$

Dubrovin conjectured that any semisimple Dubrovin-Frobenius manifold (assuming that the Euler vector field has positive weights) and with polynomial prepotential is isomorphic to the orbit space of a finite Coxeter group. This was proved by Hertling (2002).

A remark on B_2 and a modified construction

Notice that for $c = 0$ the components of

$$g = \begin{pmatrix} \frac{1}{2}u_1 & u_2 \\ u_2 & 32u_1u_2 \end{pmatrix}.$$

depends linearly on u^i. Moreover the contravariant metric

$$\eta = \mathcal{L}_{\frac{\partial}{\partial u_1}} g = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 32u_2 \end{pmatrix}$$

is flat and $g - \lambda\eta$ is a quasi-homogeneous flat pencil.

This remark can be generalized to B_n (and also to D_n) choosing

$$e = \frac{\partial}{\partial u^k}, \qquad k = 1, ..., n-1$$

and it is the starting point of the construction of D. Zuo (2007).

Shephard groups

A Shephard group is the symmetry group of a regular complex polytope. The space of orbits of a Shephard group is endowed with the structure of a Dubrovin-Frobenius manifold (Dubrovin, 2004).

Due to the results of Orlik and Solomon (1988), the inverse of the Hessian of the basic invariant polynomial of lowest degree defines a flat (pseudo)-metric which depends linearly on the highest degree polynomial (in a coordinate system given by basic invariant polynomials).

Well generated complex reflection groups

Theorem

(Kato-Mano-Sekiguchi, 2015) The orbit space of a well-generated complex reflection group is equipped with a flat F-structure $(\nabla, \circ, e, E)$ *with* linear *Euler vector field where*

1. *The flat coordinates for* ∇ *are basic invariants* $(u_1, \ldots, u_n)$ *of the group (generalized Saito coordinates).*
2. *In the Saito flat coordinates*

$$e = \frac{\partial}{\partial u_n}, \;\; E = \sum_{i=1}^{n} \left(\frac{d_i}{d_n} \right) u_i \frac{\partial}{\partial u_i}.$$

A flat structure associated with the reflecting hyperplanes

Let $H_1, \ldots, H_M$ be the reflecting hyperplanes, α_i be a linear form defining H_i and π_i be the unitary projection onto the unitary complement of H_i.

Theorem

The data

$$\left(\nabla^* = \nabla^0 - \sum_{i=1}^{M} \frac{d\alpha_i}{\alpha_i} \otimes \tau_i\pi_i,\ * = \sum_{i=1}^{M} \frac{d\alpha_i}{\alpha_i} \otimes \sigma_i\pi_i,\ E = \sum p_k \frac{\partial}{\partial p_k}\right)$$

where ∇^0 is the standard flat connection on $\mathbb{C}^n$ and the collections of constants $(\sigma_1, \ldots, \sigma_M)$ and $(\tau_1, \ldots, \tau_M)$ are G-invariant (i.e. $\sigma_i = \sigma_j$ and $\tau_i = \tau_j$ if H_i and H_j belong to the same orbit) and satisfy

$$\sum_{i=1}^{M} \sigma_i\pi_i = \sum_{i=1}^{M} \tau_i\pi_i = Id.$$

define a flat F-structure on $\mathbb{C}^n$.

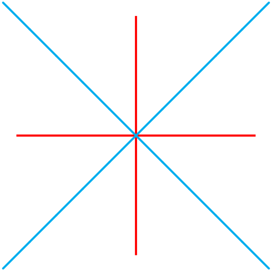
A proof the of flatness of the connection for G-invariant weights was done by Looijenga (1999). In a more general setting, different flatness conditions for this kind of connections were first studied by Kohno (1984,1989). Similar structures were studied by M. Broué, G. Malle, and R. Rouquier (1998) and by Dunkl, E.M. Opdam (2003) and W. Couwenberg, G. Heckman and E. Looijenga (2004).

In the case of Coxeter groups, these Dunk-Kohno–type connections coincide with the flat connection appearing in the theory of $\vee$-system (A,P. Veselov, 1998). Kohno's flatness conditions are equivalent to Veselov's definition of $\vee$-system (A.Arsie and P.L. 2014, M.V. Feigin and A.P. Veselov 2017).

Mirrors of reflections in the case of B_2

Idea: Identify $(\nabla^*, *, E)$ with the dual structure of a bi-flat F-manifold (A.Arsie and P.L. 2017).

Let us consider the case of B_2.



A simple example: B_2

We start from the dual structure

$$\nabla^* = \nabla^0 - \sum_{i=1}^{4} \frac{d\alpha_i}{\alpha_i} \otimes \tau_i \pi_i, \; * = \sum_{i=1}^{4} \frac{d\alpha_i}{\alpha_i} \otimes \sigma_i \pi_i, \; E = p_1 \frac{\partial}{\partial p_1} + p_2 \frac{\partial}{\partial p_2}$$

where

$$\alpha_1 = p_1, \qquad \alpha_2 = p_2, \qquad \alpha_3 = p_1 - p_2, \qquad \alpha_4 = p_1 + p_2$$

Moreover $\sigma_1 = \sigma_2$, $\sigma_3 = \sigma_4$ and $\tau_1 = \tau_2$, $\tau_3 = \tau_4$.

Results: the case $e = \frac{\partial}{\partial u_2}$

The weights are

$$\sigma_1 = \sigma_2 = \sigma_3 = \sigma_4 = \frac{1}{2} = \frac{2}{4} = \frac{\textit{order of reflection}}{\textit{number of mirrors}}.$$
$$\tau_1 = \tau_2 = -4c - 1,\ \tau_3 = \tau_4 = 4c.$$

The corresponding vector potentials are

$$F^1_{B_2} = u_1 u_2 - \frac{1}{12} u_1^3 (8c+1), \qquad F^2_{B_2} = -\frac{c}{12}(4c+1)u_1^4 + \frac{1}{2}u_2^2. \quad (1)$$

For $c = -\frac{1}{8}$ the vector potential comes from the prepotential of the Dubrovin-Frobenius manifold associated with B_2.

A conjecture for well-generated complex reflection groups

Let G be any finite well-generated irreducible (real or complex) reflection group acting on a vector space V. Then the orbit space V/G is equipped with a family of bi-flat structures (of the form described above) depending on $\mu - 1$ parameters, where μ is the number of orbits for the action of G on the reflecting hyperplanes.

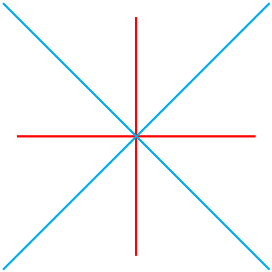
For instance in the case of B_n we have two orbits and this should imply the existence of a one parameter family, in the case of A_n there is a single orbit and this should imply the existence of a unique bi-flat structure (the Dubrovin-Saito structure of A_n).

This conjecture has been verified for Weyl groups of rank 2, 3 and 4, for the dihedral groups $I_2(m)$, for any of the exceptional well-generated complex reflection groups of rank 2 and 3 and for any of the groups $G(m, 1, 2)$ and $G(m, 1, 3)$.

Additional solutions with $e \neq \frac{\partial}{\partial u_2}$

In the case of B_2 if we remove the hypothesis $e = \frac{\partial}{\partial u_2}$ we get two additional solutions with $e = \frac{\partial}{\partial u_1}$

1. $c = 0$, $\sigma_1 = \sigma_2 = 0, \sigma_3 = \sigma_4 = 1$,
2. $c = -\frac{1}{4}$, $\sigma_1 = \sigma_2 = 1, \sigma_3 = \sigma_4 = 0$.



Defocusing and focusing NLS

The corresponding bi-flat F-manifolds are the Dubrovin-Frobenius manifolds associated with the prepotentials

$$F = \frac{1}{2}u_1^2u_2 \pm \frac{1}{2}u_2^2\left(\ln u_2 - \frac{3}{2}\right).$$

These are the prepotentials of the Dubrovin-Frobenius manifolds associated with defocusing/focusing NLS equation.

Dubrovin-Frobenius prepotentials

In the case $n = 3, 4$ only the first choice survives. For $n = 3, 4$ the Dubrovin-Frobenius prepotentials

$$
\begin{aligned}
F_{B_3} &= \frac{1}{6}u_2^3 + u_1 u_2 u_3 + \frac{1}{12}u_1^3 u_3 - \frac{3}{2}u_3^2 + u_3^2 \ln u_3, \\
F_{B_4} &= \frac{1}{108}u_1^4 u_4 + \frac{1}{6}u_1^2 u_2 u_4 - \frac{1}{72}u_2^4 + u_1 u_3 u_4 + \frac{1}{2}u_2^2 u_4 \\
&\quad + \frac{1}{2}u_2 u_3^2 - \frac{9}{4}u_4^2 + \frac{3}{2}u_4^2 \ln u_4,
\end{aligned}
$$

What happens for $n > 4$?

The intersection form in the case $n = 2, 3, 4$

For $n = 2, 3, 4$ the intersection form is

$$g_{B_2} = \begin{bmatrix} 0 & \frac{1}{p_1 p_2} \\ \frac{1}{p_1 p_2} & 0 \end{bmatrix},$$

$$g_{B_3} = \begin{bmatrix} 0 & \frac{1}{p_1 p_2} & \frac{1}{p_1 p_3} \\ \frac{1}{p_1 p_2} & 0 & \frac{1}{p_2 p_3} \\ \frac{1}{p_1 p_3} & \frac{1}{p_2 p_3} & 0 \end{bmatrix},$$

$$g_{B_4} = \begin{bmatrix} 0 & \frac{1}{p_2 p_1} & \frac{1}{p_1 p_3} & \frac{1}{p_1 p_4} \\ \frac{1}{p_1 p_2} & 0 & \frac{1}{p_2 p_3} & \frac{1}{p_2 p_4} \\ \frac{1}{p_1 p_3} & \frac{1}{p_2 p_3} & 0 & \frac{1}{p_3 p_4} \\ \frac{1}{p_1 p_4} & \frac{1}{p_2 p_4} & \frac{1}{p_3 p_4} & 0 \end{bmatrix}$$

...this suggests $\boxed{g^{ij}(p) = \frac{(1 - \delta_{ij})}{p_i p_j}}$ for arbitrary n.

Dubrovin-Saito procedure

1. Applying the Saito's procedure to the metric g one obtains a quasi-homogeneous flat pencil associated with B_n with Liouville vector field
$$e = \frac{\partial}{\partial u_{n-1}}.$$
2. This pencil **is not regular**. In Saito's flat coordinates $(t_1, ..., t_n)$ R is diagonal with $R^1_1 = 0$. It turns out that

$$\begin{aligned}
\eta_{ij} &= \delta_{i,n+1-i}, \\
c^i_{jk} &= \frac{\Gamma^{n+1-j,i}_k}{R^i_i}, \qquad i \neq 1, \\
c^1_{ij} &= c^{n+1-j}_{ni}, \qquad \forall (i,j) \neq (n,n), \\
c^1_{nn} &= \frac{(n-1)}{t_n}.
\end{aligned}$$

Conclusions and open problems

For $n = 2, 3, 4$ the dual product is defined by

$$* = \sum_{i=1}^{n^2} \frac{d\alpha_i}{\alpha_i} \otimes \sigma_i \pi_i$$

with $\sigma_i = 0$ for all the coordinate hyperplanes $p_i = 0$ (Orbit I) and $\sigma_i = 1$ for all the remaining mirrors $p_i \pm p_j = 0$ (Orbit II).
Is it true for arbitrary n?

- For $n = 2, 3, 4$ the Dubrovin-Frobenius prepotentials are related to constrained KP (Liu, Zhang and Zhou) with superpotential

$$\lambda = p^n + a_2 p^{n-2} + a_3 p^{n-3} + \cdots + a_n + \frac{a_{n+1}}{p - a_1}.$$

In particular the case $n = 2$ is related to higher genera Catalan numbers (Carlet, van de Leur, Posthuma, Shadrin).
Is it true for arbitrary n?

References

1. A. Arsie and P. Lorenzoni, *Complex reflection groups, logarithmic connections and bi-flat F-manifolds*, Lett. Math. Phys. 107, pp 1919–1961 (2017).

2. A. Arsie and P. Lorenzoni, *Bi-flat F-manifolds: a survey*, in "Integrability and Related Areas: A recognition of Emma Previato's work in algebra and geometry " (2019), Cambridge University Press: LMS Lecture Notes Series.

3. A. Arsie, P. Lorenzoni, I. Mencattini and G. Moroni, *A Dubrovin-Frobenius manifold structure of NLS type on the orbit space of* B_n, arXiv:2111.03964 (2021).