The Maxwell-Bloch System in the Sharp-Line Limit

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Joint work with Sitai Li (Xiamen)

Maxwell equation: $q_z = -\int_{\mathbb{R}} Pg(\lambda)\,\mathrm{d}\lambda$.

Describes the evolution of an optical pulse $q(t,z) \in \mathbb{C}$ along an active-medium fiber with spatial coordinate z. The optical field is driven by the medium polarization $P(t,z;\lambda) \in \mathbb{C}$ with frequency detuning λ.

Bloch subsystem: $P_t - 2\mathrm{i}\lambda P = -2qD$ and $D_t = 2\mathrm{Re}(q^* P)$.

Describes the (retarded, $t = t_{\mathrm{lab}} - z/c$) time variation of the medium polarization and population inversion $D(t,z;\lambda) \in \mathbb{R}$. These are driven by the optical field.

$g(\lambda)$ is a probability density describing the distribution of atoms with different frequency detunings λ, for instance if the atoms are in a gas, there is a Doppler shift in their frequencies due to relative motion.

To study the injection of an optical pulse into the end of a half-line optical medium ($z \geq 0$) that is prepared in some asymptotic state in the distant past ($t \to -\infty$) we take the Maxwell-Bloch system

$$\frac{\partial q}{\partial z}(t,z) = -\int_{\mathbb{R}} P(t,z;\lambda)g(\lambda)\,\mathrm{d}\lambda$$

$$\frac{\partial P}{\partial t}(t,z;\lambda) - 2\mathrm{i}\lambda P(t,z;\lambda) = -2q(t,z)D(t,z;\lambda)$$

$$\frac{\partial D}{\partial t}(t,z) = 2\mathrm{Re}(q(t,z)^* P(t,z;\lambda))$$

with a (mathematical) initial condition

$$q(t,0) = q_0(t) \quad \text{(the incident pulse)}$$

and (mathematical) boundary conditions

$$\lim_{t\to-\infty} q(t,z) = 0, \quad D_- := \lim_{t\to-\infty} D(t,z;\lambda) = \pm 1, \quad P_- := \lim_{t\to-\infty} P(t,z;\lambda) = 0.$$

If the atoms are in a crystal instead of a gas, minimal Doppler shift $\implies g(\lambda) = \delta_0$. Then it is only necessary to track the polarization $P(t,z;\lambda)$ and population inversion $D(t,z;\lambda)$ for detuning $\lambda = 0$, so with $P(t,z) := P(t,z;0)$ and $D(t,z) := D(t,z;0)$ we obtain the *sharp-line limit*:

$$q_z = -P, \quad q(t,0) = q_0(t), \quad \lim_{t\to-\infty} q(t,z) = 0$$
$$P_t = -2qD, \quad \lim_{t\to-\infty} P(t,z) = 0$$
$$D_t = 2\mathrm{Re}(q^* P), \quad \lim_{t\to-\infty} D(t,z) = D_- = \pm 1.$$

The Bloch subsystem and BCs imply that $|P(t,z)|^2 + D(t,z)^2 = 1$. If furthermore $q_0(t) \in \mathbb{R}$, then $q(t,z)$ and $P(t,z)$ are both real, and

$$P = \sin(\Theta), \quad D = \cos(\Theta), \quad q = -\frac{1}{2}\Theta_t \implies \Theta_{tz} = 2\sin(\Theta)$$

the sine-Gordon equation in characteristic/light-cone coordinates.

The sine-Gordon equation has been studied by Cheng-Venakides-Zhou and Chen-Liu-Lu in the long-time limit for the (non-characteristic) Cauchy problem in *laboratory coordinates*:

$$\Theta_{\tau\tau} - \Theta_{\chi\chi} + \sin(\Theta) = 0, \quad \Theta(\chi, 0) = F(\chi), \quad \Theta_{\tau}(\chi, 0) = G(\chi).$$

Some important observations:

- For this problem, the reflection coefficient $r(\lambda)$ comes from the Faddeev-Takhtajan scattering problem which automatically yields $r(0) = 0$. But for the characteristic Cauchy problem r comes instead from the Zakharov-Shabat problem, and $r(0) \neq 0$ in general.
- For this hyperbolic problem, the solution is asymptotically confined to the light cone $|\chi/\tau| < 1$, and $r(0) = 0$ implies that $\Theta \to 0$ as $|\chi/\tau| \to 1$. For Maxwell-Bloch we may expect something different if $r(0) \neq 0$ in the sine-Gordon reduction, or if the reduction is not possible…

We study the characteristic Cauchy problem for the sharp-line Maxwell-Bloch system near the light cone edge: $z/t \to 0$ as $t \to +\infty$:

- We find a *boundary-layer phenomenon*: the pulse undergoes a sudden transition upon entering the medium. The transition is described by a specific 1-parameter family of solutions of the Painlevé-III equation recently seen in large-amplitude limits for the focusing nonlinear Schrödinger equation:
  - Nongeneric focusing of waves in the semiclassical limit (Suleimanov, Buckingham-Jenkins-M).
  - Near-field high-order limits of iterated Bäcklund transformations (rogue waves of infinite order: Bilman-Ling-M; high-order solitons: Bilman-Buckingham; general backgrounds: Bilman-M).
- Further implications: the optical pulse fails to be in $L^1(\mathbb{R})$ for all $z > 0$ even if it has compact support at $z = 0$; most pulses, but not all, switch the medium into its ground state as $t \to +\infty$.

Similar results without full justification were reported by Gabitov-Zakharov-Mikhaĭlov. Fokas-Menyuk gave a more rigorous analysis of a similar problem, with different results.

For $z \geq 0$ and $t \geq 0$, set $x = \sqrt{2tz}$ (similarity variable). Try

$$q(t,z) = t^{-1}y(X), \quad P(t,z) = 2X^{-1}s(X), \quad D(t,z) = 1 - 2X^{-1}U(X), \quad X = x.$$

Then the sharp-line Maxwell-Bloch equations for real q and P imply

$$\begin{aligned} y'(X) &= -2s(X) \\ Xs'(X) &= s(X) - 2Xy(X) + 4y(X)U(X) \\ XU'(X) &= U(X) - 4y(X)s(X). \end{aligned}$$

A related system replaces the third ODE with

$$XU'(X) = U(X) - 4Xs(X)^{-1}y(X)U(X) + 4s(X)^{-1}y(X)U(X)^2.$$

The modified system implies that $u(X) := -s(X)^{-1}y(X)$ satisfies

$$u''(X) = \frac{u'(X)^2}{u(X)} - \frac{u'(X)}{X} + \frac{4}{X} + 4u(X)^3 - \frac{4}{u(X)}, \quad \text{(Painlevé-III)}$$

and it has a first integral $J := s(X)^{-2}U(X)(U(X) - X)$. When $J = -1$ we recover the original ODE for $U(X)$.

Causality

Our Cauchy problem is characteristic (data given on the light cone).

Definition (Causal solutions)

A solution of the Cauchy problem for a given incident pulse $q_0(t)$ is called *causal* if $q(z,t) = 0$ holds for all $t < 0$ and $z \geq 0$.

Obviously a causal solution can only be generated from an incident pulse $q_0(t)$ vanishing identically for $t < 0$. From the boundary conditions at $t = -\infty$, the Bloch subsystem implies that for causal solutions, $D(z,t) = D_-$ and $P(z,t) = 0$ for all $t < 0$ and $z \geq 0$.

Theorem

If $q_0(t) = 0$ for all $t < 0$, there can exist at most one causal solution of the Maxwell-Bloch Cauchy problem.

Generally, there exist multiple non-causal solutions for the same Cauchy data. Note that causality is fundamentally connected with the reflection coefficient; reflectionless solutions (solitons) are non-causal.

Lax pair

Jost solutions for $z = 0$

The Lax pair for the Maxwell-Bloch system reads

$$\boldsymbol{\phi}_t = (\mathrm{i}\lambda\sigma_3 + \mathbf{Q})\boldsymbol{\phi}, \quad \mathbf{Q} := \begin{pmatrix} 0 & q(t,z) \\ -q(t,z)^* & 0 \end{pmatrix}$$

$$\boldsymbol{\phi}_z = \frac{1}{2\mathrm{i}\lambda}\boldsymbol{\rho}\boldsymbol{\phi}, \quad \boldsymbol{\rho} := \begin{pmatrix} D(t,z) & P(t,z) \\ P(t,z)^* & -D(t,z) \end{pmatrix}.$$

Thus, the spectral problem that can be analyzed when $z = 0$ is the nonselfadjoint Zakharov-Shabat equation. The inverse-scattering transform should be based on that problem, with the z-equation supplying (mathematical) time-evolution of scattering data.

Taking $z = 0$ and $q(t,0) = q_0(t) \in L^1(\mathbb{R})$ with support on $t \geq 0$ (for causality), Jost matrices are defined for $\lambda \in \mathbb{R}$ by the asymptotic behavior $\boldsymbol{\phi}_{\pm}(t;\lambda) = \mathrm{e}^{\mathrm{i}\lambda t\sigma_3} + o(1)$ as $t \to \pm\infty$.

The scattering matrix is defined by $\mathbf{S}(\lambda) := \boldsymbol{\phi}_-(t;\lambda)^{-1}\boldsymbol{\phi}_+(t;\lambda)$ and is independent of t. The assumption that $q_0(t) = 0$ for $t < 0$ means that $\boldsymbol{\phi}_-(t;\lambda) = \mathrm{e}^{\mathrm{i}\lambda t\sigma_3}$ holds exactly for all $t \leq 0$, so $\mathbf{S}(\lambda) = \boldsymbol{\phi}_+(0;\lambda)$. The *reflection coefficient* $r(\lambda)$ is defined by

$$r(\lambda) := \frac{S_{21}(\lambda)}{S_{11}(\lambda)} = \frac{\phi_{+,21}(0;\lambda)}{\phi_{+,11}(0;\lambda)}.$$

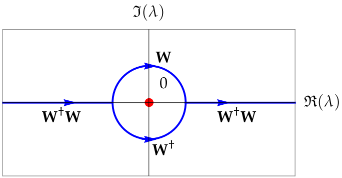
Lemma

Suppose that $q_0(t) \in \mathscr{S}(\mathbb{R})$*, that* $q_0(t) = 0$ *for* $t < 0$*, and that* $S_{11}(\lambda) \neq 0$ *for* $\mathrm{Im}(\lambda) \geq 0$*. Then* $r(\lambda) \in \mathscr{S}(\mathbb{R})$ *admits analytic continuation to* $\mathrm{Im}(\lambda) > 0$.

Evolution of the reflection coefficient in $z > 0$ is difficult to justify; nonetheless we can formulate a Riemann-Hilbert problem that produces the unique causal solution of the Cauchy problem.

Riemann-Hilbert problem

Let $\Sigma_{\mathbf{M}}$ be the contour shown and take $r(\lambda) \in \mathscr{S}(\mathbb{R})$ analytic for $\mathrm{Im}(\lambda) > 0$. For given $D_- := \pm 1$ and coordinates $(t, z) \in \mathbb{R}^2$, seek $\mathbf{M}(\lambda) = \mathbf{M}(\lambda; t, z)$, 2×2, analytic for $\lambda \in \mathbb{C} \setminus \Sigma_{\mathrm{M}}$ with $\mathbf{M} \to \mathbb{I}$ as $\lambda \to \infty$, and with the indicated jumps, where



$$\mathbf{W}(\lambda) := \begin{pmatrix} 1 & 0 \\ r(\lambda)\mathrm{e}^{-2i\theta(\lambda)} & 1 \end{pmatrix}, \quad \theta(\lambda) := \lambda t - \frac{D_- z}{2\lambda},$$

and where $\mathbf{W}^{\dagger}(\lambda) := \mathbf{W}(\lambda^*)^{\dagger}$. A dressing argument proves:

Theorem

Let $q_0(t) \in \mathscr{S}(\mathbb{R})$ with $q_0(t) = 0$ for $t < 0$ generate no discrete eigenvalues or spectral singularities and have reflection coefficient $r(\lambda)$. Then the RHP is uniquely solvable for all $(t, z) \in \mathbb{R}^2$ and the unique causal solution to the Maxwell-Bloch Cauchy problem is

$$q(t, z) = -2i \lim_{\lambda \to \infty} \lambda M_{12}(\lambda; t, z) \quad \textit{and} \quad \boldsymbol{\rho}(t, z) = D_- \mathbf{M}(0; t, z) \sigma_3 \mathbf{M}(0; t, z)^{-1}.$$

Key quantities obtained from the reflection coefficient

Denote $r_0^{(m)} := r^{(m)}(0)$, write r_0 for $r_0^{(0)}$, and let M be the index m of the first nonzero $r_0^{(m)}$. Also, set

$$\aleph := \frac{1}{\pi}\int_{\mathbb{R}} \ln(1+|r(\lambda)|^2)\frac{\mathrm{d}\lambda}{\lambda}; \quad \aleph_M := \arg(r_0^{(M)}) + \aleph.$$

At $z = 0$, $\boldsymbol{\rho}(t,0)$ can be expressed in terms of the Jost matrices as

$$\boldsymbol{\rho}(t,0) = \begin{pmatrix} D(t,0) & P(t,0) \\ P(t,0)^* & -D(t,0) \end{pmatrix} = D_- \boldsymbol{\phi}_-(t;0)\sigma_3\boldsymbol{\phi}_-(t;0)^{-1}, \quad t > 0.$$

This satisfies the enforced boundary condition $\boldsymbol{\rho}(t,0) \to D_-\sigma_3$ as $t \to -\infty$, and using the scattering matrix and a trace identity,

$$\lim_{t\to+\infty} P(t,0) = -D_- \frac{2|r_0|\mathrm{e}^{-\mathrm{i}\aleph_0}}{1+|r_0|^2} \quad \text{and} \quad \lim_{t\to+\infty} D(t,0) = D_- \frac{1-|r_0|^2}{1+|r_0|^2}.$$

Physically, one expects $D(t,z) \to -1$ as $t \to +\infty$. Obviously not true at $z = 0$ unless $r_0 = 0$ and $D_- = -1$.

Selection of self-similar solutions

For each given $\omega \in \mathbb{C}$, there is a unique odd analytic solution $u = u(X;\omega) = -u(-X;\omega)$ of Painlevé-III ($\alpha = 0$, $\beta = \gamma = -\delta = 4$)

$$u''(X) = \frac{u'(X)^2}{u(X)} - \frac{u'(X)}{X} + \frac{4}{X} + 4u(X)^3 - \frac{4}{u(X)}, \quad u'''(X) = \omega.$$

Taking $\omega \in (-3,3)$ and enforcing the consistent constraint $J = -1$ gives a solution $(y(X), s(X), U(X))$ of the self-similar Maxwell-Bloch system with Taylor expansions

$$y(X;\omega) = \frac{1}{2}\sqrt{1 - \frac{\omega^2}{9}}X^2 - \frac{\omega}{12}\sqrt{1 - \frac{\omega^2}{9}}X^4 + \mathcal{O}(X^6)$$

$$s(X;\omega) = -\frac{1}{2}\sqrt{1 - \frac{\omega^2}{9}}X + \frac{\omega}{6}\sqrt{1 - \frac{\omega^2}{9}}X^3 + \mathcal{O}(X^5)$$

$$U(X;\omega) = \left(\frac{\omega}{6} + \frac{1}{2}\right)X + \mathcal{O}(X^3), \quad X \to 0.$$

These functions are exactly the ones that describe infinite-order solitons and rogue waves in the focusing NLS at time $t = 0$ (Bilman-Buckingham and Bilman-Ling-M).

Selection of self-similar solutions

The functions $y(X;\omega)$, $s(X;\omega)$, and $U(X;\omega)$ are analytic on the real and imaginary axes of the X-plane.

Definition (Particular self-similar solutions of Maxwell-Bloch)

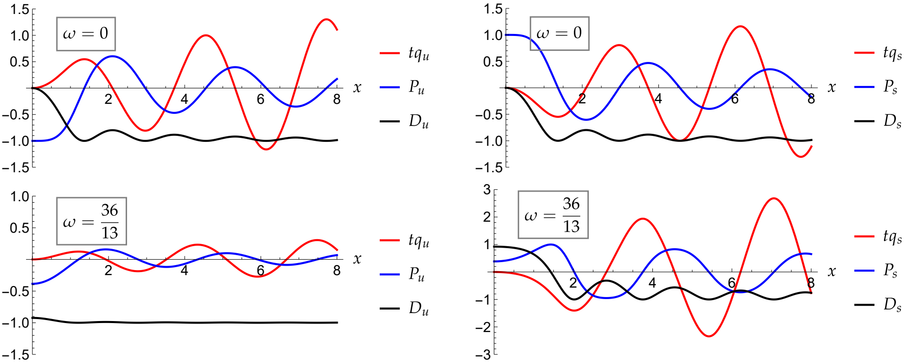
Given $\omega \in (-3,3)$ and $\xi = \mathrm{e}^{\mathrm{i}\kappa}$, $\kappa \in \mathbb{R}$, with $x = \sqrt{2tz} \geq 0$, two real-valued self-similar solutions of the Maxwell-Bloch system are

$$
\begin{aligned}
q(t,z) &= q_{\mathrm{u}}(t,z;\omega,\xi) := t^{-1}\xi y(x;\omega) \\
P(t,z) &= P_{\mathrm{u}}(t,z;\omega,\xi) := 2\xi x^{-1} s(x;\omega) \\
D(t,z) &= D_{\mathrm{u}}(t,x;\omega) := 1 - 2x^{-1}U(x;\omega)
\end{aligned}
$$

and

$$
\begin{aligned}
q(t,z) &= q_{\mathrm{s}}(t,z;\omega,\xi) := t^{-1}\xi y(-\mathrm{i}x;\omega) \\
P(t,z) &= P_{\mathrm{s}}(t,z;\omega,\xi) := -2\mathrm{i}\xi x^{-1} s(-\mathrm{i}x;\omega) \\
D(t,z) &= D_{\mathrm{s}}(t,x;\omega) := -1 + 2\mathrm{i}x^{-1}U(-\mathrm{i}x;\omega).
\end{aligned}
$$

Plots of the particular self-similar solutions: $\xi = 1$

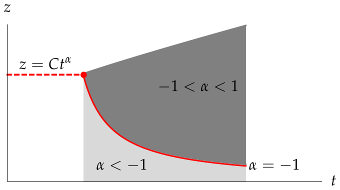


Asymptotic regimes within the light cone

Definition (Asymptotic regimes)

Let $C > 0$ be a fixed constant. The three asymptotic regimes within the light cone $z \geq 0$, $t \geq 0$ are defined as follows.

- The *medium-edge regime* corresponds to $t \to +\infty$ with $z = Ct^{\alpha}$ and $\alpha < -1$.
- The *transition regime* corresponds to $t \to +\infty$ with $z = Ct^{\alpha}$ and $\alpha = -1$.
- The *medium-bulk regime* corresponds to $t \to +\infty$ with $z = Ct^{\alpha}$ and $|\alpha| < 1$.



Main theorem

Theorem (Global asymptotics — generic case)

Suppose that the incident pulse satisfies $q_0(t) \in \mathscr{S}(\mathbb{R})$ and $q_0(t) = 0$ for $t < 0$, and q_0 generates no discrete eigenvalues or spectral singularities. If $M = 0$ (i.e., $r_0 = r(0) \neq 0$), then with

$$\omega := 3\frac{|r_0|^2 - 1}{|r_0|^2 + 1} \in (-3, 3),$$

as $t \to +\infty$ with $z \geq 0$ and $z = o(t)$, the causal solution of the Maxwell-Bloch Cauchy problem

$$\begin{aligned} q(t,z) &= q_{\mathrm{m}}(t,z;\omega,\mathrm{e}^{-\mathrm{i}\aleph_0}) + \mathcal{O}(z/t) + \mathcal{O}(t^{-\infty}) \\ P(t,z) &= P_{\mathrm{m}}(t,z;\omega,\mathrm{e}^{-\mathrm{i}\aleph_0}) + \mathcal{O}((z/t)^{1/2}) + \mathcal{O}(t^{-\infty}) \\ D(t,z) &= D_{\mathrm{m}}(t,z;\omega) + \mathcal{O}((z/t)^{1/2}) + \mathcal{O}(t^{-\infty}), \end{aligned}$$

where $\mathrm{m} = \mathrm{s}$ for propagation in an initially-stable medium ($D_- = -1$) and $\mathrm{m} = \mathrm{u}$ for propagation in an initially-unstable medium ($D_- = 1$).

The following is proved by Taylor expansion of the functions $y(X;\omega)$, $s(X;\omega)$, and $U(X;\omega)$.

Corollary

Under the same assumptions on $q_0(t)$, as $t \to +\infty$ with $z = Ct^\alpha$ and $\alpha < -1$,

$$q(t,z) = D_- \frac{2|r_0|e^{-i\aleph_0}}{1+|r_0|^2} z + \mathcal{O}(t^{2\alpha+1}) + \mathcal{O}(t^{\alpha-1})$$

$$P(t,z) = -D_- \frac{2|r_0|e^{-i\aleph_0}}{1+|r_0|^2} + \mathcal{O}(t^{\alpha+1}) + \mathcal{O}(t^{\frac{1}{2}(\alpha-1)})$$

$$D(t,z) = D_- \frac{1-|r_0|^2}{1+|r_0|^2} + \mathcal{O}(t^{\alpha+1}) + \mathcal{O}(t^{\frac{1}{2}(\alpha-1)}),$$

regardless of whether $D_- = -1$ or $D_- = 1$.

This corollary shows that the solution in the medium-edge regime is very close to the exact solution for $z = 0$.

Corollaries

Medium-bulk asymptotics — generic case.

Corollary

Under the same assumptions on $q_0(t)$, *let*

$$\varepsilon := \frac{1}{2\pi}\ln(1+|r_0|^{-2D_-}) > 0,\ A := \sqrt{\frac{2}{\pi}}\frac{|\Gamma(1+\mathrm{i}\varepsilon)|}{|r_0|^{\frac{1}{2}D_-}(1+|r_0|^{2D_-})^{\frac{1}{4}}} > 0$$

and for $x > 0$ *define* $\varphi(x) := 2x - \varepsilon\ln(8x) - \frac{1}{4}\pi + \arg(\Gamma(1+\mathrm{i}\varepsilon))$. *Then as* $t \to +\infty$ *with* $z = Ct^{\alpha}$ *and* $\alpha \in (-1,1)$, *in both cases* $D_- = \pm 1$,

$$q(t,z) = D_-\mathrm{e}^{-\mathrm{i}\aleph_0}\frac{1}{t}\left(\frac{tz}{2}\right)^{\frac{1}{4}} A\sin(\varphi(\sqrt{2tz})) + \mathcal{O}(t^{-\frac{1}{4}(\alpha+5)}) + \mathcal{O}(t^{\alpha-1})$$

$$P(t,z) = -D_-\mathrm{e}^{-\mathrm{i}\aleph_0}\left(\frac{tz}{2}\right)^{-\frac{1}{4}} A\cos(\varphi(\sqrt{2tz})) + \mathcal{O}(t^{-\frac{3}{4}(\alpha+1)}) + \mathcal{O}(t^{\frac{1}{2}(\alpha-1)})$$

$$D(t,z) = -1 + \frac{1}{2}\left(\frac{tz}{2}\right)^{-\frac{1}{2}} A^2\cos^2(\varphi(\sqrt{2tz})) + \mathcal{O}(t^{-(\alpha+1)}) + \mathcal{O}(t^{\frac{1}{2}(\alpha-1)}).$$

Corollaries

Lack of absolute integrability of the optical pulse.

The previous result allows for $\alpha = 0$. It shows that, unlike the situation near the edge of the medium $z = 0$, *for each fixed* $z > 0$*, the active medium decays as* $t \to +\infty$ *to the stable pure state* ($P = 0$ *and* $D = -1$) *regardless of whether the initial state was stable* ($D_- = -1$) *or unstable* ($D_- = 1$). The decay is quite slow however:

Corollary

Under the same assumptions on $q_0(t)$*, for every* $z > 0$ *the optical pulse function* $t \mapsto q(t,z)$ *does not lie in* $L^1(\mathbb{R})$*.*

This is important to observe, because $q(\cdot,z) \in L^1(\mathbb{R})$ is the fundamental assumption of scattering theory for the Zakharov-Shabat system. Jost solutions are not guaranteed to exist for all $\lambda \in \mathbb{R}$ as soon as $z > 0$.

However, this is not an obstruction to using the Riemann-Hilbert problem to capture the unique causal solution because existence and uniqueness are proved by independent means.

Using an elementary symmetry $\mathcal{S} : (q(t,z), P(t,z), D(t,z)) \mapsto (\mathcal{S}q(t,z), \mathcal{S}P(t,z), \mathcal{S}D(t,z)) := (q(T-t,z), P(T-t,z), -D(T-t,z))$ we can use the $t \to +\infty$ decay of causal solutions to the stable pure state to prove the following.

Corollary

There exist incident pulses $q_0(t)$ satisfying the same assumptions as above for which the Maxwell-Bloch Cauchy problem for an initially-unstable medium ($D_- = 1$) has (other) solutions that are not causal and that decay to both stable and unstable pure states as $t \to +\infty$.

This proves rigorously that *without the imposition of causality, the Cauchy problem on an initially-unstable medium is ill-posed.*

The nongeneric case — initially-stable medium

Theorem

Suppose that the incident pulse satisfies $q_0(t) \in \mathscr{S}(\mathbb{R})$ and $q_0(t) = 0$ for $t < 0$, and q_0 generates no discrete eigenvalues or spectral singularities. If $r_0 = r(0) = 0$, so that the index M of the first nonzero value $r_0^{(m)}$ for $m = M$ is positive, then the causal solution of the Maxwell-Bloch Cauchy problem on an initially-stable medium ($D_- = -1$) satisfies

$$q(t,z) = -2\frac{\mathrm{i}^M}{M!}|r_0^{(M)}|\mathrm{e}^{-\mathrm{i}\aleph_M}\left(\frac{z}{2t}\right)^{\frac{1}{2}(M+1)} J_{M+1}(2\sqrt{2tz}) + \mathcal{O}((z/t)^{\frac{1}{2}(M+2)})$$

$$P(t,z) = 2\frac{\mathrm{i}^M}{M!}|r_0^{(M)}|\mathrm{e}^{-\mathrm{i}\aleph_M}\left(\frac{z}{2t}\right)^{\frac{1}{2}M} J_{M}(2\sqrt{2tz}) + \mathcal{O}((z/t)^{\frac{1}{2}(M+1)})$$

$$D(t,z) = -1 + 2\frac{|r_0^{(M)}|^2}{(M!)^2}\left(\frac{z}{2t}\right)^{M} J_{M}(2\sqrt{2tz})^2 + \mathcal{O}((z/t)^{\frac{1}{2}(M+1)}),$$

as $t \to +\infty$ with $z \geq 0$ and $z = o(t)$.

The nongeneric case — initially-stable medium

This result admits corollaries obtained by restriction to the medium-edge regime with the help of

$$J_n(2x) = \frac{x^n}{n!}(1 + \mathcal{O}(x^2)), \quad x \to 0$$

and by restriction to the medium-bulk regime with the help of

$$J_n(2x) = \frac{1}{\sqrt{\pi x}}\left(\cos\left(2x - \frac{1}{2}n\pi - \frac{1}{4}\pi\right) + \mathcal{O}(x^{-1})\right), \quad x \to +\infty.$$

In the latter case, we can obtain asymptotics as $t \to +\infty$ with $z > 0$ fixed and obtain that $q(t,z) = \mathcal{O}(t^{-1-\frac{1}{2}M})$, so as $M = 1, 2, 3, \ldots$, absolute integrability of $t \mapsto q(t,z)$ is recovered. Also $D(t,z) \to -1$ as $t \to +\infty$, so the medium returns to the stable state in this limit, even for a nongeneric incident pulse.

The nongeneric case — initially-unstable medium

Theorem

Suppose that the incident pulse satisfies $q_0(t) \in \mathscr{S}(\mathbb{R})$ and $q_0(t) = 0$ for $t < 0$, and q_0 generates no discrete eigenvalues or spectral singularities. If $r_0 = r(0) = 0$, so that the index M of the first nonzero value $r_0^{(m)}$ for $m = M$ is positive, then the causal solution of the Maxwell-Bloch Cauchy problem on an initially-unstable medium ($D_- = 1$) satisfies

$$q(t,z) = 2\frac{\mathrm{i}(-1)^{M+1}}{M!}|r_0^{(M)}|\mathrm{e}^{-\mathrm{i}\aleph_M}\left(\frac{z}{2t}\right)^{\frac{1}{2}(M+1)} J_{M+1}(2\mathrm{i}\sqrt{2tz}) + \mathcal{O}(t^{-\frac{1}{2}(M+2)(1-\alpha)})$$

$$P(t,z) = 2\frac{(-1)^{M+1}}{M!}|r_0^{(M)}|\mathrm{e}^{-\mathrm{i}\aleph_M}\left(\frac{z}{2t}\right)^{\frac{1}{2}M} J_{M}(2\mathrm{i}\sqrt{2tz}) + \mathcal{O}(t^{-\frac{1}{2}(M+1)(1-\alpha)})$$

$$D(t,z) = 1 + 2(-1)^{M+1}\frac{|r_0^{(M)}|^2}{(M!)^2}\left(\frac{z}{2t}\right)^{M} J_{M}(2\mathrm{i}\sqrt{2tz})^2 + \mathcal{O}(t^{-\frac{1}{2}(M+1)(1-\alpha)}),$$

as $t \to +\infty$ with $z = Ct^{\alpha}$ and $\alpha \le -1$ (medium-edge and transition regimes only).

The nongeneric case — initially-unstable medium

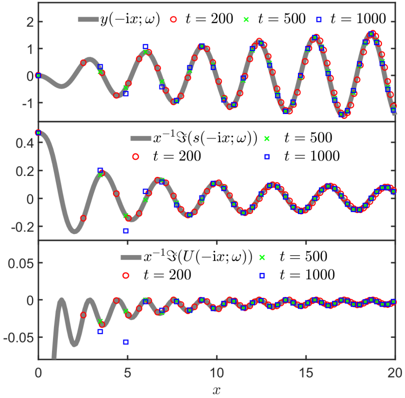
Because this result for a nongeneric pulse incident on an initially-unstable medium is not (provably) valid in the medium-bulk regime that includes $t \to +\infty$ with $z > 0$ fixed, *we cannot tell whether a nongeneric pulse can trigger the decay of the medium to the stable pure state as* $t \to +\infty$.

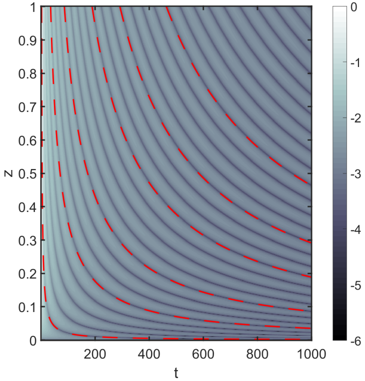
Indeed, the simplest example of a nongeneric incident pulse is $q_0(t) \equiv 0$, for which the unique causal solution is exactly

$$q(t,z) \equiv 0, \quad P(t,z) \equiv 0, \quad D(t,z) \equiv 1.$$

On the other hand, we know that a generic pulse results in $D(t,z) \to -1$ as $t \to +\infty$ for each $z > 0$.

Here we take $q_0(t) = 0.5e^{it}e^{-(1/t+1/(3.5-t))/10}\chi_{[0,3.5]}(t)$ and consider propagation with $D_- = -1$.



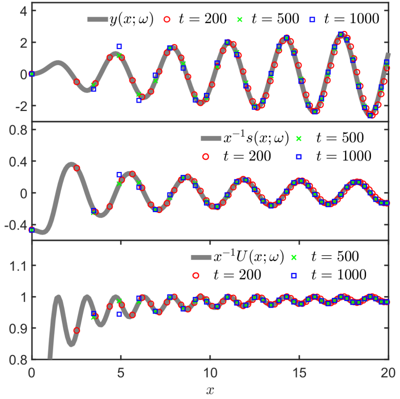


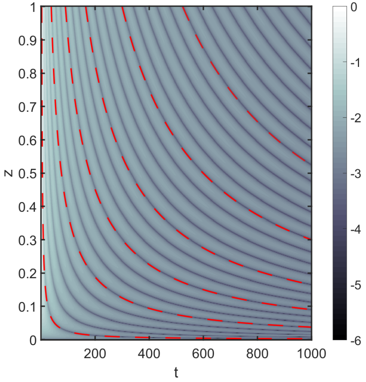
Here $r_0 = -0.50723 - 0.47903i$, $\omega = -1.03564$, and $\aleph = 1.26584$.

Numerical experiments

A generic pulse incident on an initially-unstable medium

Here we take $q_0(t) = 0.5e^{it}e^{-(1/t+1/(3.5-t))/10}\chi_{[0,3.5]}(t)$ and consider propagation with $D_- = 1$.



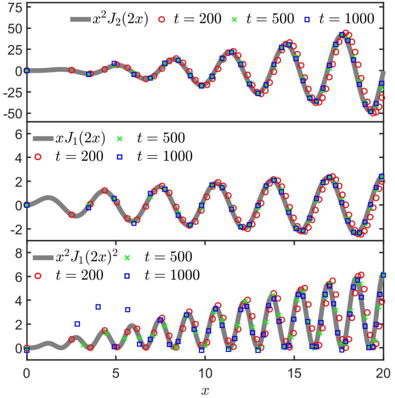


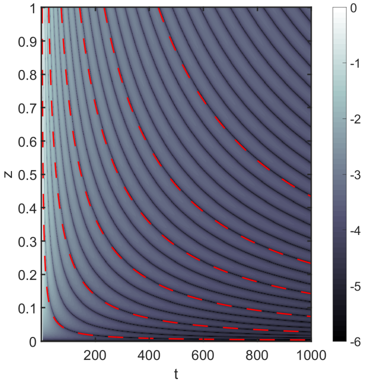
Here $r_0 = -0.50723 - 0.47903i$, $\omega = -1.03564$, and $\aleph = 1.26584$.

Numerical experiments

A nongeneric pulse incident on an initially-stable medium

Here we take $q_0(t) = 0.5e^{-(1/t+1/(6-t))/10} \tanh(t-3)\chi_{[0,6]}(t)$ and consider propagation with $D_- = -1$.



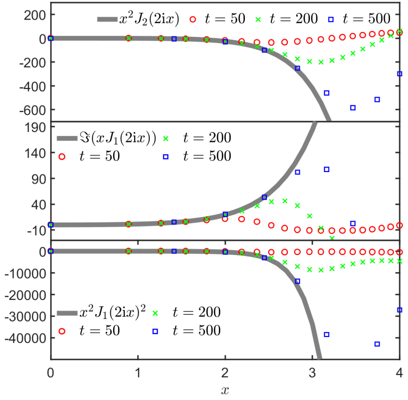


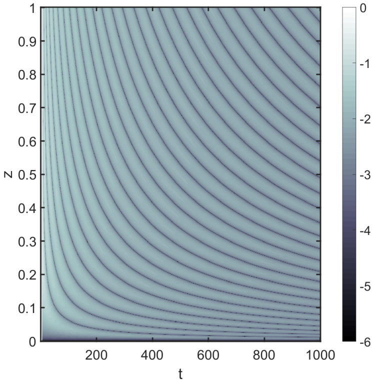
Here $M = 1$, $r_0^{(1)} = 4.26238i$ and $\aleph = 0$.

Numerical experiments

A nongeneric pulse incident on an initially-unstable medium

Here we take $q_0(t) = 0.5e^{-(1/t+1/(6-t))/10}\tanh(t-3)\chi_{[0,6]}(t)$ and consider propagation with $D_- = 1$.

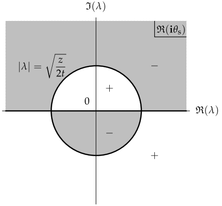




Here $M = 1$, $r_0^{(1)} = 4.26238i$ and $\aleph = 0$.

Proofs by $\overline{\partial}$ steepest-descent analysis: $D_- = -1$

For propagation in an initially-stable medium ($D_- = -1$), the phase in the Riemann-Hilbert problem is $\theta = \theta_s(\lambda; t, z) := \lambda t + z/(2\lambda)$, and the sign chart of $\mathrm{Re}(i\theta_s)$ involves the circle of (small, under our assumptions) radius $\lambda_\circ := \sqrt{z/(2t)}$.



We choose this small circle to coincide with that in the RHP. Then, we formulate an equivalent hybrid Riemann-Hilbert-$\overline{\partial}$ problem based on a non-analytic extension to the complex plane of the function $R(\lambda) := r(\lambda)/(1 + |r(\lambda)|^2)$:

$$R(\lambda) \to Q_N(u, v) := \sum_{n=0}^{N-2} \frac{(iv)^n}{n!} \frac{d^n R}{du^n}(u), \quad (u, v) \in \mathbb{R}^2, \quad \lambda = u + iv.$$

This can be done for any N, which gives rise to the error terms $\mathcal{O}(t^{-\infty})$ in our theorems. The resulting unknown has jumps on and within the small circle only, but is nonanalytic in a strip bisected by $\mathbb{R}$ (excluding the circle).

Proofs by $\overline{\partial}$ steepest-descent analysis: $D_- = -1$

We then construct a parametrix for the hybrid RH-$\overline{\partial}$ problem by:

- Neglecting the $\overline{\partial}$ part;
- Approximating the jumps on and within the circle using the Taylor expansion of $r(\lambda)$ and the fact that the radius is small.

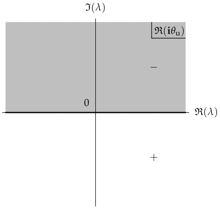
The parametrix is analytic for $|k| \neq 1$, tends to the identity as $k \to \infty$, and with counterclockwise orientation the jump matrix for $|k| = 1$ is

$$\begin{pmatrix} \Delta_M(\lambda_\circ k)^{-\frac{1}{2}} & \dfrac{\lambda_\circ^M |a_M|}{\sqrt{\Delta_M(\lambda_\circ k)}} k^M \mathrm{e}^{\mathrm{i}x(k+k^{-1})} \\ -\dfrac{\lambda_\circ^M |a_M|}{\sqrt{\Delta_M(\lambda_\circ k)}} k^M \mathrm{e}^{-\mathrm{i}x(k+k^{-1})} & \Delta_M(\lambda_\circ k)^{-\frac{1}{2}} \end{pmatrix}, \quad \begin{cases} x := \sqrt{2tz} \\ \lambda_\circ := \sqrt{z/(2t)} \\ a_M := r_0^{(M)}/M! \\ \Delta_M(\lambda) := 1 + |a_M|^2 \lambda^{2M}. \end{cases}$$

When $M = 0$, this is solved in terms of Painlevé-III. When $M > 0$, it is a small-norm problem, and its Neumann-series solution produces Bessel functions at subleading order.

Proofs by $\overline{\partial}$ steepest-descent analysis: $D_- = 1$

For propagation in an initially-unstable medium ($D_- = 1$), the phase is instead $\theta = \theta_u(\lambda; t, z) = \lambda t - z/(2\lambda)$ and the change in sign on the second term yields a different sign chart for $\mathrm{Re}(i\theta_u)$. The phase factors $e^{\pm i\theta_u}$ are now non-oscillatory for $|\lambda| = \lambda_\circ$.



We can compensate with an explicit "g-function" cut on the whole circle, the effect of which is:

- the "unstable" phase $\theta_u(\lambda; t, z)$ is replaced with the "stable" phase $\theta_s(\lambda; t, z)$;
- the phase factors are moved from the off-diagonal jump matrix elements to the diagonal.

The resulting parametrix is again solved by Painlevé-III (on a rotated axis) when $M = 0$. When $M > 0$, after restoring the unstable phase θ_u we have a small-norm problem *but only if zt is controlled*. It is enough to assume that $zt = \mathcal{O}(1)$ as $t \to +\infty$.

Thanks for your attention!

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