Generalized Gibbs Ensembles

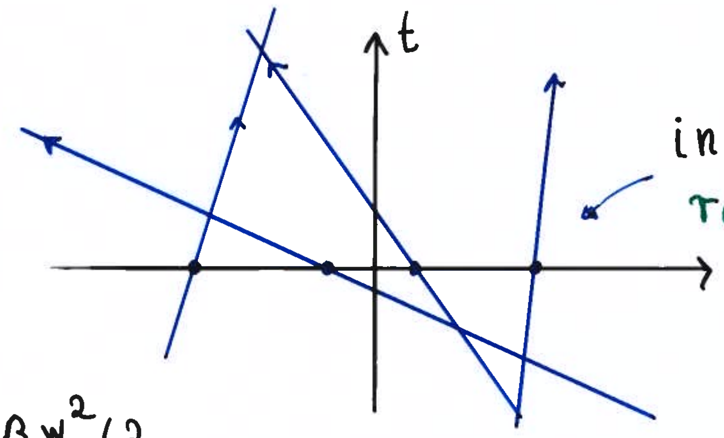
of the Calogero Fluid

GGE

Herbert Spohn

TUMünchen

1. Generalized Gibbs GGE



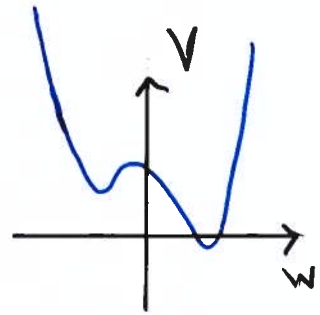
initial random

ideal gas

- thermal equilibrium Poisson
  intensity $\rho \, dx \, \frac{1}{Z} e^{-\beta w^2/2} dw$

- isolated // NO thermalization /

- long time limit GGE Poisson $\rho \, dx \, \frac{1}{Z} e^{-V(w)} dw$
  $t \to \infty$



parameter $\rho > 0$, confining potential V

Theorem (Kallenberg 1978)

space time stationary, $\rho > 0$, $\langle p_0^2 \rangle < \infty$, finite entropy / length !!

$\Rightarrow$ mixture of GGE

2. Integrable and interacting

ideal gas $\sum_j f(p_j)$ conserved, no interaction

- 1D fluids

$$H_N = \sum_{j=1}^{N} \frac{1}{2} p_j^2 + \sum_{i<j=1}^{N} V_{mec}(q_i - q_j)$$

" integrable "

$$I_1, \dots, I_N, \quad I_2 = H_N$$

$$\{ I_m, I_n \} = 0$$

meta theorem

ONLY solutions

better: N local conservation laws

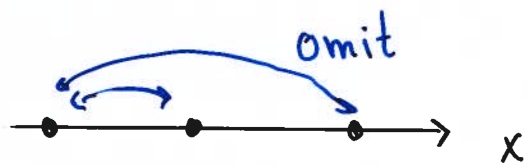
$$V_{mec} = 0, \quad V_{mec}(x) = \begin{cases} & |x| \le a \\ 0 & |x| > a \end{cases}$$ hard rods

Calogero 1975 $\quad V(x) = \dfrac{1}{\sinh^2(x)} \quad + \dots$

(Sutherland-Calogero-Moser

$$V(x) = \frac{1}{x^2}$$ high density limit

- dilute limit



$$H_{toda} = \sum_j \left(\frac{1}{2} p_j^2 + e^{-(q_{j+1} - q_j)} \right)$$

3. Calogero fluid

phase space $\mathbb{R}^N_{(+)} \times \mathbb{R}^N = T_N$

conserved fields Lax matrix $N \times N$

$$L_{ij} = \delta_{ij} p_j + i(1-\delta_{ij}) \frac{1}{\sinh(q_i - q_j)} \qquad L = L^*$$

Lax pair $L(q,p)$, $M(q,p)$

$$\frac{d}{dt} L = [L, M]$$

⇝ $L \psi_\alpha = \lambda_\alpha \psi_\alpha$ eigenvalues are conserved NON LOCAL

$$\Big(\sum_{j=1}^{N} p_j\Big)^2$$

⇝ local fields

$$Q^{[n]}(x) = \sum_{j=1}^{N} \delta(x - q_j)(L^n)_{jj}, \quad Q^{[n]} = \int dx\, Q^{[n]}(x) = \operatorname{tr} L^n$$

density total

$Q^{[0]}$ particle number, $Q^{[1]}$ momentum, $Q^{[2]}$ energy, $Q^{[3]}$?, ...

4. GGE on ring

periodize on $[0,\ell]$

$f : \mathbb{R} \to \mathbb{R}$

$$f_\ell(x) = \sum_{m \in \mathbb{Z}} f(x + m\ell)$$

$\leadsto$ H_ℓ L_ℓ

still integrable V_{mec} double periodic Weierstrass

$\leadsto$ GGE . $Q_\ell^{[n]}$

$$e^{-\sum_{n=0}^{\infty} \mu_n Q_\ell^{[n]}} = e^{-\operatorname{tr} V(L_\ell)}$$

relative to $\frac{1}{N!} d^N q \, d^N p$

on $[0,\ell]^N \times \mathbb{R}^N$

parameters volume ℓ, number N

- confining potential V fixed

spacetime stationary

canonical

goal infinite volume $\frac{\ell}{N} = \nu$ fixed

free energy / length $\lim_{\ell\to\infty} -\frac{1}{\nu} \log Z_N(\underset{w}{\nu}, \underset{m}{V}) = F(\nu, V)$

physically more relevant

density of states DOS of Lax $\rho_{Q,N}(w) = \frac{1}{N}\sum_{j=1}^{N} \delta(w - \lambda_j)$ eigenvalues

expect

LLN $\lim_{N\to\infty} \rho_{Q,N} = \rho_Q$ a.s.

- also CLT relates to average currents

NO IDEA

$V_{mec}(x) = \frac{1}{x^2}$

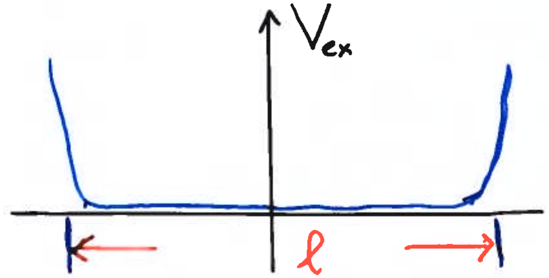
P. Choquard 2000

$\left(\frac{1}{x^2}\right)_\ell = \frac{1}{\sin^2(x/\ell)}$

5. External potential

$$T_N^{\blacktriangle} = \mathbb{W}_N \times \mathbb{R}^N$$

$$\{ x_1 < \dots < x_N \}$$



$$Z_N(\nu, V) = \int_{T_N^{\blacktriangle}} d^N q \, d^N p \; e^{-\operatorname{tr} V(L)} \; e^{-\sum_{j=1}^{N} V_{ex}(q_j)}$$

Rujsenaars 1988 - 1995

scattering coordinates

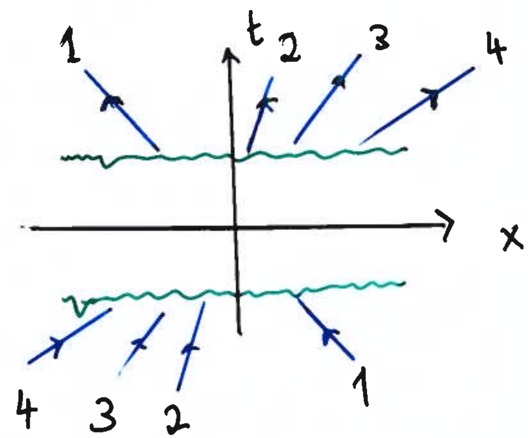
$$\lim_{t \to \infty} p_j(t) = \lambda_j \, , \quad \lim_{t \to \infty} q_j(t) - \lambda_j t = \phi_j \in \mathbb{R}$$

$\lambda \in \mathbb{W}_N$ eigenvalues of L

$\Phi : (\lambda, \phi) \mapsto (q, p)$ one-to-one, canonical

generic

Result (R 1988) algebraic construction of Φ



$$Z_N(\nu, V) = \int_{T_N^{\blacktriangle}} d^N \lambda \, d^N \phi \; e^{-\sum_{j=1}^{N} V(\lambda_j)} \; e^{-\sum_{j=1}^{N} V_{ex}(q_j(\lambda, \phi))}$$

?

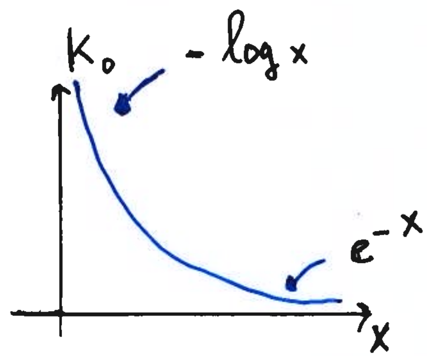
- special choice $V_{ex}(x) = e^{-\ell/2} \cosh x$

$$\sum_{j=1}^{N} V_{ex}(q_j) = \sum_{j=1}^{N} e^{-\ell/2} Y_j \cosh \phi_j$$

$$Y_j = \prod_{\substack{m=1 \\ m\neq j}}^{N} \left(1 + \frac{1}{(\lambda_m - \lambda_j)^2} \right)^{1/2}$$

confining!

modified Bessel



$\rightsquigarrow$

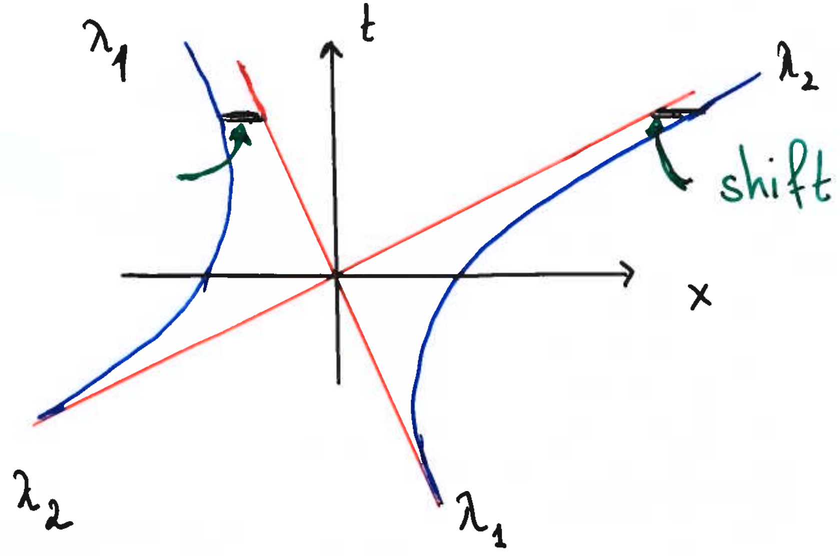
$$Z_N(\nu, V) = \frac{1}{N!} \int_{\mathbb{R}^N} d^N\lambda \prod_{j=1}^{N} e^{-V(\lambda_j)} \prod_{j=1}^{N} 2 K_0 (2 e^{-\ell/2} Y_j)$$

// mean-field //

- Calogero 2-particle scattering shift

$$\phi_{ca}(w) = -\log\left(1 + \frac{1}{w^2}\right)$$

$$w = \lambda_1 - \lambda_2$$



6. free energy functional (1 - particle)

$$\rho \geq 0 \quad , \quad \int_{\mathbb{R}} dw \, \rho(w) = \frac{1}{\nu}$$

$$\mathcal{F}(\rho) = \int_{\mathbb{R}} dw \, \rho(w) \Big(V(w) - 1 + \log \rho(w) - \log \big(1 + \int_{\mathbb{R}} dw' \rho(w') \phi_{ca}(w - w') \big) \Big)$$

minimizer ρ^* (unique)

$$\mathcal{F}(\rho^*) = F(\nu, V) = \lim_{N \to \infty} - \frac{1}{N} \log Z_N(\nu, V) \qquad \Big\| \quad \nu = \frac{\ell}{N}$$

Lax DOS $\rho_Q = \nu \rho^*$

TBA formalism (Yang, Yang 1969) Lagrange multiplier μ minimizer $\rho_p(\mu)$

$$T \psi(w) = \int dw' \phi_{ca}(w - w') \psi(w')$$

$$\rho_p(\mu) = \frac{1}{1 - \rho_n T} \rho_n$$

TBA equation $\varepsilon = V - 1 - \mu - T e^{-\varepsilon}$

quasi-energy $\rho_n = e^{-\varepsilon}$

adjust $\mu \rightsquigarrow \rho^*$

7. Bethe equations

- Lieb-Liniger δ-Bose gas

phase shift θ_{ll}, scattering shift $\phi_{ll} = \theta'_{ll}$ $\left(\frac{2c}{w^2+c^2} \right)$

$$\underbrace{2\pi I_j}_{\text{input}} = \nu N k_j + \sum_{i=1}^{N} \theta_{ll}(k_j - k_i)$$

output $(k_1, \dots, k_N)$

$I_1 < \dots < I_N$

I_j integer

counting measure

weight $e^{-\sum_{j=1}^{N} V(k_j)}$

- DOS $\rho_{Q,N}(w) = \frac{1}{N} \sum_{j=1}^{N} \delta(w - k_j)$

$$\lim_{N\to\infty} \rho_{Q,N} = \rho_Q$$

Dorlas, Lewis, Pulé 1989

$V(w) = \frac{1}{2}\beta w^2 - \mu\beta$

thermal

- Calogero fluid

$$y_j = \nu N \lambda_j + \sum_{i=1}^{N} \theta_{ca}(\lambda_j - \lambda_i)$$

$y \in \mathbb{W}_N$

weight $e^{-\sum_{j=1}^{N} V(\lambda_j)}$

DOS

Lebesgue

agrees with previous result for $N \to \infty$

asymptotic BA

Outlook

HS 2020, Guionnet, Memin 2021

- Toda lattice $\sum_{j=1}^{N} \frac{1}{2} p_j^2 + \sum_{j=1}^{N-1} e^{-(q_{j+1}-q_j)}$ $\quad \phi_{to}(w) = \log w^2$ $\quad w \to \infty$ of $-\log(1+\frac{1}{w^2})$

  ‖ same method ‖

- Calogero-Moser $V_{mec}(x) = \frac{1}{x^2}$ | classical + quantum on ring

  TBA exact

‖ $\phi_{CM} = 0$ ‖ $\rightsquigarrow$

$$\partial_t \rho(x,t;w) + \partial_x(w\, \rho(x,t;w)) = 0$$ non-interacting

$\Uparrow$ local DOS

Calogero fluid

‖ generalized hydrodynamics ‖

$$\partial_t \rho(x,t;w) + \partial_x\big(\underbrace{v^{\mathrm{eff}}(x,t;w)}\, \rho(x,t;w)\big) = 0$$ interacting

‖ local nonlinear functional of ρ ‖