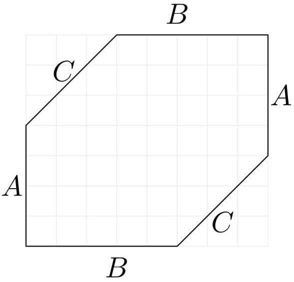
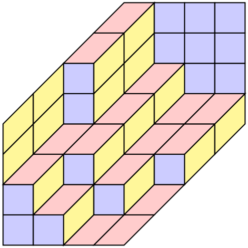
KU LEUVEN

Matrix valued orthogonality and random tilings

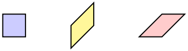
Arno Kuijlaars
KU Leuven, Belgium
Excursions in Integrability,
SISSA Trieste, Italy, 23 May 2022

1 Random tilings of a hexagon





An ABC-hexagon can be covered by three types of lozenges.



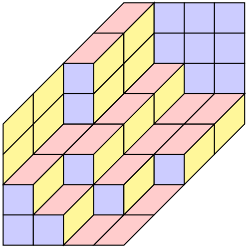
1 Weights on tiles

A weighting on tiles produces a weight on tilings $\mathcal{T}$

$$W(\mathcal{T}) = \prod_{T \in \mathcal{T}} w(T)$$

Probability of a tiling is

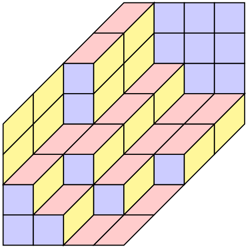
$$\textbf{Prob}(\mathcal{T}) = \frac{W(\mathcal{T})}{Z}, \quad Z = \sum_{\mathcal{T}'} W(\mathcal{T}')$$



1 Weights on tiles

A **weighting** **on tiles produces a weight on tilings** $\mathcal{T}$

$$W(\mathcal{T}) = \prod_{T \in \mathcal{T}} w(T)$$



Probability **of a tiling is**

$$\textbf{Prob}(\mathcal{T}) = \frac{W(\mathcal{T})}{Z}, \qquad Z = \sum_{\mathcal{T}'} W(\mathcal{T}')$$

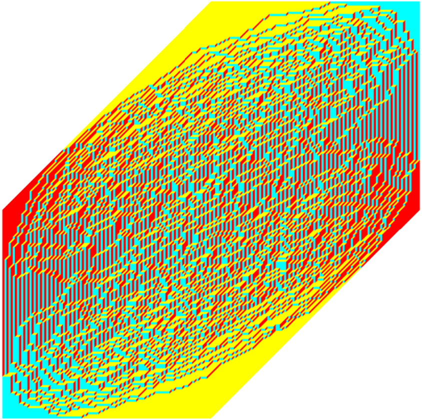
Weighting is **periodic** **with periods** $p \geq 1$ **and** $q \geq 1$ **if**

$$w_{\square}(x, y) = w_{\square}(x + p, y + q), \qquad x, y \in \mathbb{Z}$$

and similarly for $w_{\text{▱}}(x, y)$ **and** $w_{\text{▱}}(x, y)$

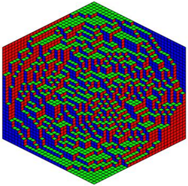
KU LEUVEN

1 Frozen and disordered regions

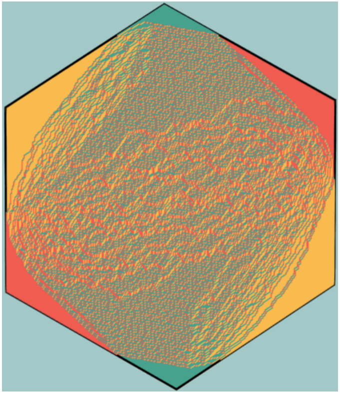


Pattern as $n \to \infty$ with **frozen regions** and **disordered regions** (a.k.a. rough phase).

Picture for $p = 1$ and $q = 2$.



1 Higher periods and smooth region

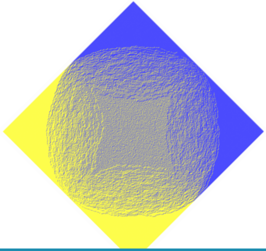


Picture for $p = 2$ and $q = 3$ due to Christophe Charlier

Correlations decay exponentially in the new smooth region
Kenyon, Okounkov, Sheffield (2006)

Analogous model: domino tilings of Aztec diamond with periodic weights

Chhita, Johansson (2016)
Berggren, Duits (2020)



KU LEUVEN

2 Determinantal point process

The positions of the lozenges in a random tiling are determinantal with a correlation kernel K

- **This means that**

$$\mathbb{P}\left[\begin{array}{l}\textbf{there is a [lozenge] or [lozenge] lozenge at each}\\ \textbf{position } (x_1,y_1),\ldots,(x_n,y_n)\end{array}\right] = \det\left[K((x_j,y_j),(x_k,y_k))\right]_{j,k=1}^{n}$$

2 Determinantal point process

The positions of the lozenges in a random tiling are determinantal with a correlation kernel K

- **This means that**

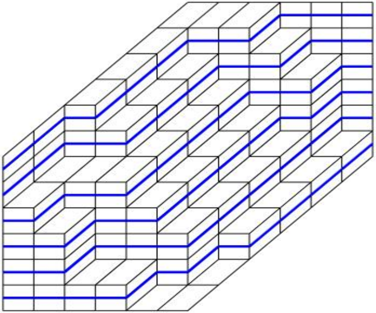
$$\mathbb{P}\left[\begin{array}{l}\text{there is a } \square \text{ or } \lozenge \text{ lozenge at each} \\ \text{position } (x_1, y_1), \ldots, (x_n, y_n)\end{array}\right]$$

$$= \det\left[K((x_j, y_j), (x_k, y_k))\right]_{j,k=1}^{n}$$

Formula for K comes from either

- **dimer model interpretation and inverse Kasteleyn matrix,** **Kenyon (1997, 2009), Chhita, Johansson (2016)**
- **or nonintersecting lattice paths and Lindström-Gessel-Viennot lemma** **Eynard, Mehta (1998)**

2 Non-intersecting paths



- **Lozenge  is horizontal step on a path,**
- **Lozenge  is a diagonal step on a path,**
- **Lozenge  is not on any path;**
  assume $w_{\text{}}$$(x, y) = 1$ without loss of generality.

2 Transition matrices

For each integer $0 \leq x < B + C$ we have a transition matrix

$$T_x(y, y') = \begin{cases} w_{\square}(x, y), & \text{if } y' = y, \\ w_{\lozenge}(x, y), & \text{if } y' = y + 1, \\ 0, & \text{otherwise with } (y, y') \in \mathbb{Z}^2. \end{cases}$$

2 Transition matrices

For each integer $0 \le x < B + C$ we have a transition matrix

$$T_x(y, y') = \begin{cases} w_{\square}(x, y), & \text{if } y' = y, \\ w_{\lozenge}(x, y), & \text{if } y' = y + 1, \\ 0, & \text{otherwise with } (y, y') \in \mathbb{Z}^2. \end{cases}$$

In case of periodic weighting

- $T_x = T_{x+p}$ **is block Toeplitz with blocks of size** $q \times q$**,**
- **The matrix symbol of** T_x **is**

$$A_x(z) = \Big[T_x(y, y')\Big]_{y,y'=0}^{q-1} + z\Big[T_x(y, y'+q)\Big]_{y,y'=0}^{q-1},$$

with $z \in \mathbb{C}$**.**

2 Double contour integral formula

Suppose $A = qN$**,** $C = qM$**,** $B + C = pL$**.**

Theorem (Duits, Kuijlaars (2021))

$K((px, qy), (px, qy))$ **is equal to the** $(0, 0)$ **entry of the matrix**

$$\frac{1}{(2\pi i)^2} \oint_\gamma \oint_\gamma A^{L-x}(w) R_N(w, z) A^x(z) \frac{w^y}{z^{y+1} w^{M+N}} dz dw$$

with $\boxed{A(z) = A_0(z) A_1(z) \cdots A_{p-1}(z)}$

- R_N **is the reproducing kernel for the matrix weight**

$$W(z) = \frac{A^L(z)}{z^{M+N}}$$

on closed contour γ **around** 0**.**

2 Full formula

Theorem

Let $0 \leq j, j' \leq p-1$ ***and*** $0 \leq k, k' \leq q-1$***. Then***

$$K((px+j, qy+k), (px'+j', qy'+k'))$$

is equal to (k, k') ***entry of***

$$\frac{1}{(2\pi i)^2} \oint_\gamma \oint_\gamma \left(\prod_{l=px'+j'}^{pL-1} A_l(w) \right) R_N(w,z) \left(\prod_{l=0}^{px+j-1} A_l(z) \right) \frac{w^{y'} dz dw}{z^{y+1} w^{M+N}} - \frac{\chi_{px+j>px'+j'}}{2\pi i} \oint_\gamma \left(\prod_{l=px'+j'}^{px+j-1} A_l(z) \right) z^{y'-y} \frac{dz}{z}$$

3 Matrix valued orthogonality

- $W(z) = \dfrac{A(z)^L}{z^{M+N}}$ **is** $q \times q$ **matrix valued function on contour** γ
- $P_n(z) = z^n I_q + \cdots$ **is monic matrix valued polynomial of degree** n

3 Matrix valued orthogonality

- $W(z) = \dfrac{A(z)^L}{z^{M+N}}$ **is** $q \times q$ **matrix valued function on contour** γ
- $P_n(z) = z^n I_q + \cdots$ **is monic matrix valued polynomial of degree** n
- P_n **is matrix valued orthogonal polynomial (MVOP) if**

$$\frac{1}{2\pi i} \oint_\gamma P_n(z) W(z) z^k dz = H_n \delta_{k,n}, \qquad k = 0, 1, \ldots, n$$

with invertible H_n

KU LEUVEN

3 Reproducing kernel

Reproducing kernel $R_N(w,z)$ is polynomial of degree $\leq N-1$ in both variables such that

$$\frac{1}{2\pi i}\oint_{\gamma} P(w)\frac{A^L(w)}{w^{M+N}} R_N(w,z)dw = P(z)$$

for matrix valued polynomial P of degree $\leq N-1$.

3 Reproducing kernel

Reproducing kernel $R_N(w,z)$ is polynomial of degree $\leq N-1$ in both variables such that

$$\frac{1}{2\pi i}\oint_\gamma P(w)\frac{A^L(w)}{w^{M+N}}R_N(w,z)dw = P(z)$$

for matrix valued polynomial P of degree $\leq N-1$.

- **If MVOP of all degrees $\leq N$ exist then**

$$R_N(w,z) = \sum_{n=0}^{N-1} P_n^T(w)H_n^{-1}P_n(z)$$

3 Riemann Hilbert problem

P_N and R_N are characterized by a matrix-valued **Riemann-Hilbert problem** of size $2q \times 2q$

- $Y : \mathbb{C} \setminus \gamma \to \mathbb{C}^{2q \times 2q}$ is analytic with jump

$$Y_+(z) = Y_-(z) \begin{pmatrix} I_q & W(z) \\ 0 & I_q \end{pmatrix}, \qquad z \in \gamma,$$

and $Y(z) = (I_{2q} + O(z^{-1})) \begin{pmatrix} z^N I_q & 0 \\ 0 & z^{-N} I_q \end{pmatrix}$ as $z \to \infty$.

Grünbaum, de la Iglesia, Martínez-Finkelshtein (2011)
Cassatella-Contra, Mañas (2012)

- Generalization of Fokas, Its, Kitaev (1992) RH problem for orthogonal polynomials

3 Solution of RH problem

- $Y : \mathbb{C} \setminus \gamma \to \mathbb{C}^{2q\times 2q}$ **is analytic with jump**

$$Y_+(z) = Y_-(z) \begin{pmatrix} I_q & W(z) \\ 0 & I_q \end{pmatrix}, \qquad z \in \gamma,$$

and $Y(z) = (I_{2q} + O(z^{-1})) \begin{pmatrix} z^N I_q & 0 \\ 0 & z^{-N} I_q \end{pmatrix}$ **as** $z \to \infty$**.**

Unique solution is

$$\boxed{Y(z) = \begin{pmatrix} P_N(z) & * \\ * & * \end{pmatrix}}$$

Reproducing kernel is

$$R_N(w,z) = \frac{1}{w-z} \begin{pmatrix} 0 & I_q \end{pmatrix} Y^{-1}(w) Y(z) \begin{pmatrix} I_q \\ 0 \end{pmatrix}$$

3 Plan for asymptotic analysis

- **Apply Deift-Zhou method of steepest descent to RH problem where $N, M, L \to \infty$. Deift, Zhou (1993)**
- **Find asymptotics for P_N and for**

$$R_N(w,z) = \frac{1}{w-z} \begin{pmatrix} 0 & I_q \end{pmatrix} Y^{-1}(w) Y(z) \begin{pmatrix} I_q \\ 0 \end{pmatrix}$$

- **Use this for asymptotic analysis of**

$$\frac{1}{(2\pi i)^2} \oint_\gamma \oint_\gamma A^{L-x}(w) R_N(w,z) A^x(z) \frac{w^y dz dw}{z^{y+1} w^{M+N}}$$

and similar double integrals

- **Identify frozen, rough and smooth regimes, and their boundary curves.**

4 Orthogonality on a Riemann surface

Case $W(z) = \dfrac{A(z)^L}{z^{M+N}}$ **on contour** γ **around** 0**.**

- **Riemann surface associated with**

$$\mathcal{R}: \quad \det(\lambda I_q - A(z)) = 0$$

It is Harnack curve of **Kenyon, Okounkov, Sheffield (2006)**

Proposition (Loose formulation ...)

Each row of P_N **corresponds to a meromorphic function on** $\mathcal{R}$ **that has orthogonality properties with respect to scalar weight**

$$\frac{\lambda^L}{z^{M+N}}$$

4 Example with $p = 3$ and $q = 2$

$$A_0(z) = A_1(z) = \begin{pmatrix} 1 & 1 \\ z & 1 \end{pmatrix}, \quad A_2(z) = \begin{pmatrix} 1 & a \\ z & 1 \end{pmatrix},$$

$$A(z) = A_0(z)A_1(z)A_2(z) = \begin{pmatrix} 3z+1 & az+a+2 \\ z^2+3z & (2a+1)z+1 \end{pmatrix}$$

- **Eigenvalues of $A(z)$ are (with $x_1 = -\frac{3}{a}$, $x_2 = -a-2$)**

$$\lambda_{1,2}(z) = (a+2)z + 1 \pm \sqrt{az(z-x_1)(z-x_2)}$$

- **Riemann surface** $\boxed{w^2 = z(z-x_1)(z-x_2)}$ **has genus one if** $a \neq 1$

4 Example with $p = 3$ and $q = 2$

$Y \mapsto X$ **of RH problem**

$$X(z) = Y(z) \begin{pmatrix} E(z) & 0 \\ 0 & E(z) \end{pmatrix}$$

where $A(z) = E(z)\Lambda(z)E(z)^{-1}$ **with** $\Lambda(z) = \begin{pmatrix} \lambda_1(z) & 0 \\ 0 & \lambda_2(z) \end{pmatrix}$

- **Jump conditions for** X

$$X_+(z) = X_-(z) \begin{pmatrix} I_2 & \frac{\Lambda(z)^L}{z^{M+N}} \\ 0 & I_2 \end{pmatrix} \quad \text{on } \gamma$$

$$X_+(z) = X_-(z) \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \quad \text{on } (-\infty, x_1] \cup [x_2, 0]$$

with $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

4 Example with $p = 2$ and $q = 3$

$$X_+(z) = X_-(z) \begin{pmatrix} I_2 & \frac{\Lambda(z)^L}{z^{M+N}} \\ 0 & I_2 \end{pmatrix} \quad \text{on } \gamma,$$

$$X_+(z) = X_-(z) \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \quad \text{on } (-\infty, x_1] \cup [x_2, 0],$$

$$X(z) = (I_4 + O(z^{-1})) \begin{pmatrix} z^N E(z) & 0 \\ 0 & z^{-N} E(z) \end{pmatrix} \qquad \text{as } z \to \infty.$$

- **Entries X_{j1}, X_{j2} give a meromorphic function f_j on $\mathcal{R}$ with a pole at infinity of order $\approx 2N$.**

4 Example with $p = 2$ and $q = 3$

$$X_+(z) = X_-(z) \begin{pmatrix} I_2 & \frac{\Lambda(z)^L}{z^{M+N}} \\ 0 & I_2 \end{pmatrix} \text{ on } \gamma,$$

$$X_+(z) = X_-(z) \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} \text{ on } (-\infty, x_1] \cup [x_2, 0],$$

$$X(z) = (I_4 + O(z^{-1})) \begin{pmatrix} z^N E(z) & 0 \\ 0 & z^{-N} E(z) \end{pmatrix} \quad \text{as } z \to \infty.$$

- Entries X_{j1}, X_{j2} give a meromorphic function f_j on $\mathcal{R}$ with a pole at infinity of order $\approx 2N$.
- Entries X_{j3}, X_{j4} give a holomorphic function ϕ_j on $\mathcal{R} \setminus (\gamma_1 \cup \gamma_2)$ with a zero at infinity of order $\approx 2N$.

4 Example with $p = 2$ and $q = 3$

Jump condition $X_+(z) = X_-(z) \begin{pmatrix} I_2 & \frac{\Lambda(z)^L}{z^{M+N}} \\ 0 & I_2 \end{pmatrix}$ **implies**

$$\phi_{j,+} = \phi_{j,-} + f_j \frac{\lambda^L}{z^{M+N}}, \qquad z \in \gamma_1 \cup \gamma_2.$$

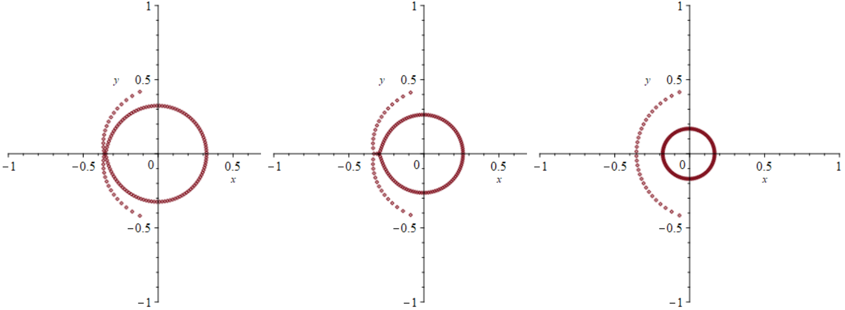
- **This leads to orthogonality**

$$\oint_{\gamma_1 \cup \gamma_2} f_j \frac{\lambda^L}{z^{M+N}} \omega = 0$$

for large class of holomorphic differentials ω **on** $\mathcal{R} \setminus \{\infty\}$ **with a pole at infinity of order at most** $\approx 2N$**.**

- **Where are the zeros of** f_j**?**

5 Pictures of zeros



Zeros tend to accumulate along certain contours.

- **Plots are for zeros of** $\det P_n$.

5 Zeros

If $p = q = 2$ then the Riemann surface has genus zero and the MVOP becomes scalar orthogonality in the complex plane.

Groot, Kuijlaars (2021)

5 Zeros

If $p = q = 2$ then the Riemann surface has genus zero and the MVOP becomes scalar orthogonality in the complex plane.

Groot, Kuijlaars (2021)

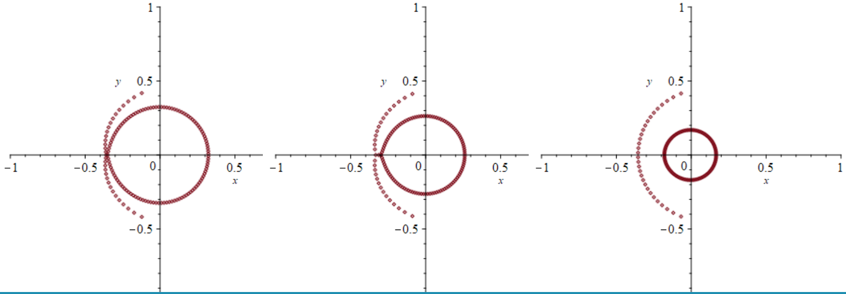
- **Limiting behavior of zeros can be found using notions of logarithmic potential theory and equilibrium measures in external fields**
- **The contour γ is not fixed; the right contour needs to have a symmetry property and is called an S-curve.**
- **The S-curve is a trajectory of a quadratic differential.**

Martínez-Finkelshtein, Rakhmanov (2011)

5 Higher genus case

- **In higher genus case, we need potential theory with bipolar Green's kernel that is adapted to the Riemann surface.**
- **We need the analogues of equilibrium measures, S-curves, and quadratic differentials, ...**

Bertola, Groot, Kuijlaars (coming soon)



Thank you for your attention.

but one more thing...

Thank you for your attention.

but one more thing...

Advertisement

- **At KU Leuven, Belgium there is opening for**

Research professor in Mathematical Analysis, Probability Theory and Random Matrices

at junior or senior level, with reduced teaching for 10 years.

- **Starting date October 1, 2023.**
https://www.kuleuven.be/personeel/jobsite/jobs/60115431