Solutions of the Bethe Ansatz Equations as Spectral Determinants

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Excursions in Integrability

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References

The talk is based on three recent papers with R. Conti and A. Raimondo:

- R. Conti and D.M., *On solutions of the Bethe Ansatz for the Quantum KdV model*. arXiv 2022
- R. Conti and D.M., *Counting Monster Potentials*. JHEP 2021
- D.M. and Andrea Raimondo, *Opers for higher states of quantum KdV models*, Commm. Math. Phys, 2020.

A family of anharmonic oscillators

$$-\Psi''(x) + \left(x^{2\alpha} + \frac{\ell(\ell+1)}{x^2} - E \right) \Psi(x) = 0, \alpha > 1, \ell \geq 0, E \in \mathbb{C}.$$

E is said an eigenvalue if $\exists \Psi \neq 0$ such that

$$\lim_{x \to 0^+} \Psi(x) = \lim_{x \to +\infty} \Psi(x) = 0.$$

The **spectrum is discrete, simple and positive**, $E_n(\ell), n \in \mathbb{N}$:

$$E_n(\ell) \sim \left(\frac{2\Gamma\left(\frac{2\alpha+1}{2\alpha}\right)}{\sqrt{\pi}\Gamma\left(\frac{3\alpha+1}{2\alpha}\right)} \right)^{\frac{2\alpha}{\alpha+1}} (4n + 2\ell + 3)^{\frac{2\alpha}{\alpha+1}}, \; n \to +\infty.$$

Spectral determinant $D_\ell(E)$ is an entire function of order $\frac{1+\alpha}{2\alpha}$.

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The Dorey-Tateo discovery

- Dorey and Tateo, J.Phys A, (1998) noticed that $D_\ell(E)$ satisfies the following countable collection of identities:

$$e^{-i\pi\frac{4\ell+2}{\alpha+1}}\frac{D_\ell\left(e^{-\frac{2\pi i}{\alpha+1}}E_n\right)}{D_\ell\left(e^{\frac{2\pi i}{\alpha+1}}E_n\right)}=-1,\ \forall n\geq 0$$

- These are the Bethe Ansatz Equations (BAE) of an Integrable Quantum Field Theory known Quantum KdV model! (CFT with $c<1\approx 6$ Vertex model with $-1<\Delta<1$)
- The spectral determinant $D_\ell(E)$ should correspond to the ground state of the model.

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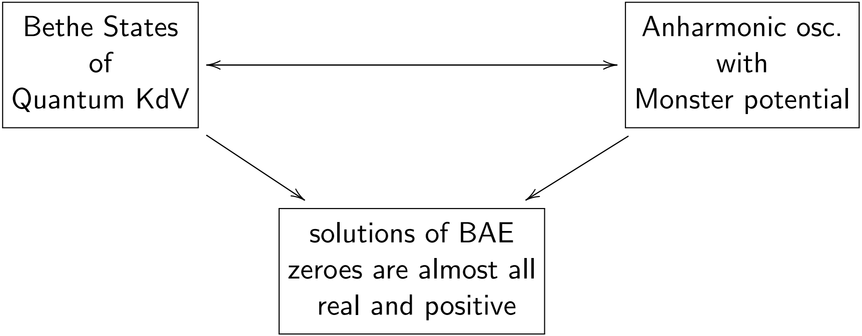
The ODE/IM Conjecture for Quantum KdV



Topological classification of solutions

- Problem: Classify solutions of the BAE, $Q(E)$, whose zeros are **all** real, positive and are asymptotics to $E_n(\ell)$ as $n \to +\infty$.
- Use as "topological index" the sequence of root numbers.

Roots and Root-Numbers

Let $Q(E)$ be a solution and $\{x_k\}$ be the increasing sequence of those positive real numbers such that

$$e^{-i\pi\frac{4l+2}{\alpha+1}}\frac{Q\left(e^{-i\frac{2\pi}{\alpha+1}}x_k\right)}{Q\left(e^{i\frac{2\pi}{\alpha+1}}x_k\right)}=-1.$$

We say that $k \in \mathbb{Z}$ is a root-number if $Q(x_k)=0$.
Root-numbers $\{k_n\}_{n\in\mathbb{N}}$ form an increasing sequence of integers.

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Fixing Ambiguities

- Numbering ambiguity: $x_k \to x_{k+m_1}$ with $m_1 \in \mathbb{Z}$
  Fix the numbering by imposing: $k_n = n$ for n large enough .

- Phase/Momentum ambiguity

$$e^{-i\pi\frac{4l+2}{\alpha+1}} = e^{-4ip}, \; p \to p + \frac{m_2}{2}$$

Fix the momentum by imposing: $2p - \frac{1}{2} \leq k_{min} < 2p + \frac{1}{2}$, $k_{min} = -\min_k\{x_k \geq 0\}$

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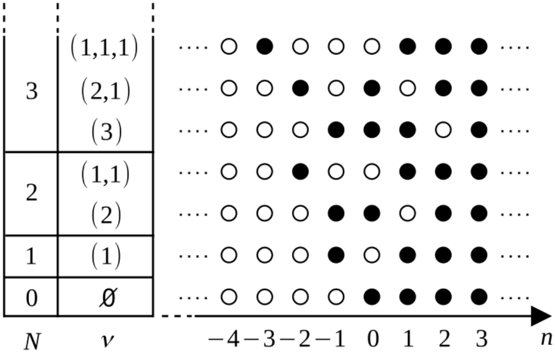
Roots and integer partitions

Root-numbers are sequences that stabilizes: $k_n = n, \text{ if } n \gg 0$.

$$\Downarrow$$

Root-numbers sequences are classified by integer partitions $\{k_n^\lambda\}$.



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1. Let $N \in \mathbb{N}$ and $2p \geq N + \frac{1}{2}$. For every $\lambda \vdash N$, the BAE admit a unique (normalised) solution $Q_p^{\lambda}(E)$ whose sequence of root-numbers coincide with $\{k_n^{\lambda}\}_{n \in \mathbb{N}}$.
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Our results. 1. Well-posedness of BAE

(1) Theorem, M. - Conti 2022

Fix $\alpha > 1$, $(N, \lambda \vdash N)$. If p is sufficiently large:
The BAE admit a unique solution $Q_p^\lambda(E)$ whose sequence of root-numbers coincide with $\{k_n^\lambda\}_{n \in \mathbb{N}}$.

$+$ Uniform asymptotics of roots/holes positions.

Earlier results:

- Well-posedness for $\alpha > 1$, $p = \frac{1}{2\alpha+2}$ and $\lambda = \emptyset$ by A. Avila in Comm. Math. Phys. (2004) - after Voros.
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Idea beyond the proof

- Introducing the counting function,

$$z(x) = -2p + \frac{1}{2\pi i} \log \frac{Q\left(e^{-i\frac{2\pi}{\alpha+1}} x\right)}{Q\left(e^{i\frac{2\pi}{\alpha+1}} x\right)}, x \geq 0,$$

- The BAE becomes (cfr. Spohn's talk)

$$z(x_{k_n}) = k_n + \frac{1}{2}, \; n \in \mathbb{N}$$

- Transform the logarithmic BAE into a Free-Boundary Nonlinear Integral Equation (known as Destri-De Vega).
- Do mathematics!

Destri-De Vega Integral Equation

Given $\lambda \vdash N$, call $H = -k_0$ (k_0 is the lowest root number).
The unknown is a tuple $(\omega, h_1, \ldots, h_H, z)$

- $[\omega, +\infty[$, $\omega > 0$, is the integration interval.
- $h_1 < \cdots < h_H$ are the holes greater than the lowest root.
- $z : C^1([\omega, \infty[)$, strictly monotone, $z(x) \sim x^{\frac{1+\alpha}{2\alpha}}, x \to +\infty$.

The Destri-De Vega (DDV) equation is

$$\begin{cases} 1.\ z(x) = -2p + \int_\omega^\infty K_\alpha(x/y)\left\lceil z(y) - \frac{1}{2}\right\rceil \frac{dy}{y} + H\, F_\alpha\left(\frac{x}{\omega}\right) - \sum_{k=1}^H F_\alpha\left(\frac{x}{h_k}\right), \\ \qquad K_\alpha(x) := \frac{\sin\left(\frac{2\pi}{1+\alpha}\right)}{\pi} \frac{x}{1+x^2-2x\cos\left(\frac{2\pi}{1+\alpha}\right)} = xF'_\alpha(x) \\ 2.\ \left\lceil z(\omega) - \frac{1}{2}\right\rceil = -H \\ 3.\ z(h_k) = \sigma(k) + \frac{1}{2},\ k = 1 \ldots N,\ \sigma(k) = \text{hole number of } h_k \end{cases}$$

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Linearisation Vs WKB (large ℓ ODE/IM)

$$l_{\omega,p}(x) = -2p + \int_{\omega}^{\infty} K_\alpha(x/y) l_{\omega,p}(y) \frac{dy}{y}, \; l_{\omega,p}(x) \sim x^{\frac{\alpha+1}{2\alpha}}, x \to \infty.$$

It is a Wiener-Hopf equation, solutions can be expressed via

$$\tau(\xi) = \frac{1}{2\pi i} \int_{\delta-i\infty}^{\delta+i\infty} \frac{\alpha^{\frac{\alpha s}{1+\alpha}}}{2\sqrt{\pi}(1+\alpha)^{s-1}} \frac{\Gamma\left(-\frac{1}{2}-\frac{\alpha s}{1+\alpha}\right)\Gamma\left(1-\frac{s}{1+\alpha}\right)}{s^2\,\Gamma(-s)} \xi^{-s} ds, \quad \xi = x/\omega.$$

We discovered a (much more useful) formula in terms of a WKB integral

$$\tau(\xi) = \frac{1}{\pi} \int_{u_-}^{u_+} \sqrt{u^2\xi - u^{2\alpha+2} - \ell(\ell+1)} \frac{du}{u}, \; \sqrt{\cdots}_{|u=u_\pm} = 0.$$

This is a first hint of the ODE/IM correspondence.

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Perturbation/Analytical challenges

We need to analyse integrals like

$$A_p[f,\varepsilon] = \int_1^\infty K_\alpha\left(\frac{x}{y}\right)\langle pf(y)+\varepsilon(y)\rangle \frac{dy}{y},\ \langle z\rangle = z - \left\lceil z - \frac{1}{2}\right\rceil$$
$$B_p[f,\varepsilon] = \int_1^\infty K_\alpha\left(\frac{x}{y}\right)\left\lceil pf(y)+\varepsilon(y) - \frac{1}{2}\right\rceil \frac{dy}{y}$$

As an example, we showed that if $f \sim x^{\frac{\alpha+1}{2\alpha}}$ and $\varepsilon, \tilde{\varepsilon}$ are bounded (+ some further hypotheses), then

$$\left| \|B_p[f,\varepsilon] - B_p[f,\tilde{\varepsilon}]\|_\infty - \frac{\alpha+1}{2\alpha}\|\varepsilon-\tilde{\varepsilon}\|_\infty \right| \lesssim_f \frac{\|\varepsilon-\tilde{\varepsilon}\|_\infty}{p}$$

$\Longrightarrow$ contractiveness of the perturbation operator $B_p[I,\cdot]$ when p is large.

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Monster potentials

Monster potentials, BLZ (2003)

1. Let P be a monic polynomial of degree N. The spectral determinant $D_\ell^P(E)$ w.r.t the potential

$$V^P = x^{2\alpha} + \frac{\ell(\ell+1)}{x^2} - 2\frac{d^2}{dx^2}\log P(x^{2\alpha+2})$$

satisfies the BAE if the monodromy about the additional poles is trivial for every E.

2. Assuming that the roots of P are distinct, the trivial monodromy is equivalent to the BLZ system

$$\sum_{j\neq k}\frac{z_k\left(z_k^2+(3+\alpha)(1+2\alpha)z_kz_j+\alpha(1+2\alpha)z_j^2\right)}{(z_k-z_j)^3}-\frac{\alpha z_k}{4(1+\alpha)}+\Delta(\ell,\alpha)=0\ , \quad k=1,\dots,N.$$

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Wronskian of Hermite polynomials

Rational extensions of the harmonic oscillator

- A rational extension of degree N is a potential

$$V^U(t) = t^2 - 2\frac{d^2}{dt^2}\ln U(t),$$

where U a polynomial of degree N such that all monodromies of $\psi''(t) = (V^U(t) - E)\psi$ are trivial for every E.

- Oblomkov's theorem (1999)

$$U \propto U^\lambda := Wr[H_{\lambda_1+j-1}, \ldots, H_{\lambda_j}], \text{ for a } \lambda := (\lambda_1, \ldots, \lambda_j) \vdash N.$$

(2) (Conditional) Theorem, M. - Conti 2021/2022

- Assume there exists a sequence P_ℓ of monster potentials with $\ell \to \infty$, then – up to subsequences –

$$z_k = \frac{\ell^2}{\alpha} + \frac{(2\alpha+2)^{\frac{3}{4}}}{\alpha} v_k^\lambda \ell^{\frac{3}{2}} + O\big(\ell\big),\ k = 1, \dots, N$$

where v_k^λ are the roots of U^λ.

- (If a monster potential with a such an asymptotics exists and) $D_\ell^\lambda(E)$ is the corresponding spectral determinant, then

$$D_\ell^\lambda(E) = Q_p^\lambda\big(E/\eta\big),\ p = \frac{2\ell+1}{\alpha+1} \text{ and } \eta = \left(\frac{2\sqrt{\pi}\,\Gamma\left(\frac{3}{2}+\frac{1}{2\alpha}\right)}{\Gamma\left(1+\frac{1}{2\alpha}\right)}\right)^{\frac{2\alpha}{1+\alpha}}.$$

Large momentum limit of Monster Potentials

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An unproven identity

Let $\lambda \vdash N$, assume U^λ has N distinct zeroes (see conjecture by Felder-Hemery-Veselov 2010). Consider the Jacobian

$$J^\lambda_{ij}(\underline{t}) = \delta_{ij}\left(1 + \sum_{l\neq j}\frac{6}{(v_i^\lambda - v_j^\lambda)^4}\right) - (1-\delta_{ij})\frac{6}{(v_i^\lambda - v_j^\lambda)^4}, \; i,j=1,\dots,N.$$

The eigenvalues of J^λ are the square numbers $\mu_k = \left(\rho_k^\lambda\right)^2$ computed from the Tableau as follows:

Example: $\lambda = (3,2,2,1,1)$ yields $\underline{\rho}^\lambda = \{1,1,1,2,2,4,4,5,7\}$.

5	4	3	2	1
6	5	4		
7				

$\longrightarrow$

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4		

$\longrightarrow$

1

$\longrightarrow \emptyset.$

$\lambda=(N)$ stated/proven in Ahmed, Bruschi, Calogero, Olshanetsky, and Perelomov ('79).

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Every solution of the BAE of every integrable quantum field theory is the **spectral determinant of a linear differential operator**.
$\rightarrow$ Bethe Roots are eigenvalues of a (possibly self-adjoint) differential operator (cf. Hilbert-Pólya Conjecture).

Ongoing work: M - Raimondo after Feigin-Frenkel and M -R- Valeri

$\widehat{\mathfrak{g}}$ an affine Kac-Moody Lie-algebra and ${}^L\widehat{\mathfrak{g}}$ the Langlands dual,

$$\left\{ \text{Bethe states of } \widehat{\mathfrak{g}} - \text{quantum KdV} \right\} \longleftarrow\!\!-\!-\!-\!\!\longrightarrow \left\{ {}^L\widehat{\mathfrak{g}} - \text{opers on } \mathbb{C}^* \right\}.$$

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