Marking and conditioning of determinantal point processes

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Fredholm determinant

The Fredholm determinant of a trace-class operator $\mathbb{K}$ on $L^2(\mathbb{R},\mu)$ with integral kernel K is given by

$$\det(1-\mathbb{K}) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_{\mathbb{R}^n} \det\left(K(x_i,x_j)\right)_{i,j=1}^n dx_j.$$

Integrable kernels

A kernel is k-integrable (in the sense of Its-Izergin-Korepin-Slavnov '93) if it is of the form

$$K(x,y) = \frac{\sum_{j=1}^k f_j(x)h_j(y)}{x-y} \qquad \text{with} \sum_{j=1}^k f_j(x)h_j(x) = 0.$$

IIKS method (Its-Izergin-Korepin-Slavnov '93, Deift-Its-Zhou '97, Bertola-Cafasso '12)

The IIKS method characterizes Fredholm determinants of integrable operators in terms of a $k \times k$ Riemann-Hilbert problem:

- ✓ by Jacobi's identity,

$$\partial_s \log\det(1 - \mathrm{M}_{\theta_s}\mathbb{K}) = -\mathrm{Tr}\left(\partial_s \mathrm{M}_{\theta_s}\mathbb{K}(1 - \mathrm{M}_{\theta_s}\mathbb{K})^{-1}\right),$$

- ✓ the kernel of the resolvent operator

$$(1 - \mathrm{M}_{\theta_s}\mathbb{K})^{-1} - 1 = \mathrm{M}_{\theta_s}\mathbb{K}\,(1 - \mathrm{M}_{\theta_s}\mathbb{K})^{-1}$$

  is characterized by a $k \times k$ Riemann-Hilbert problem,

- ✓ for a suitable s-dependence, this gives explicit identities for $\partial_s \log\det(1 - \mathrm{M}_{\theta_s}\mathbb{K})$ in terms of the Riemann-Hilbert solution.

Riemann-Hilbert problem

The kernel $R(x,y)$ of the resolvent operator $\mathrm{M}_{\theta_s}\mathbb{K}(1-\mathrm{M}_{\theta_s}\mathbb{K})^{-1}$ is also k-integrable,

$$R(x,y) = \frac{\theta_s(x)\sum_{j=1}^{k} F_j(x)H_j(y)}{x-y}, \quad F = Yf, \quad H = Y^{-T}h,$$

where Y solves the Riemann-Hilbert problem:

- ✓ $Y : \mathbb{C}\setminus\mathbb{R} \to \mathbb{C}^{k\times k}$ is analytic,
- ✓ $Y_+(z) = Y_-(z)\left(I_k - 2\pi i\theta_s(z)f(z)h(z)^T\right)$ for $z\in\mathbb{R}$,
- ✓ as $z\to\infty$, $Y(z)\to I_k$.

(Care must be taken for the precise sense of the boundary values and the asymptotic condition!)

Model Riemann-Hilbert problems

IIKS method and Riemann-Hilbert characterization are in particular effective if the kernel K itself is also characterized by a model Riemann-Hilbert problem.

Dressing procedure then allows to transform the IIKS Riemann-Hilbert problem to a problem suitable for asymptotic analysis and for deriving integrable differential equations.

Determinantal point processes

Many of the modern applications of the IIKS method are connected to kernels K of determinantal point processes.

Is there a point process interpretation of the method and of the Riemann-Hilbert problem?

Determinantal point processes on $\mathbb{R}$

A DPP on $\mathbb{R}$ is a random point process on $\mathbb{R}$ such that

1. Correlation functions are determinants:

$$\rho_m(x_1, \dots, x_m) = \det\left(K(x_i, x_j)\right)_{i,j=1}^{m}.$$

2. Average multiplicative statistics are Fredholm determinants:

$$\mathbb{E}\prod(1-\phi(x_i)) = \det(1 - \mathrm{M}_\phi \mathbb{K}).$$

   In particular, the gap probability

$$\mathbb{P}\left(\text{no points in } A\right) = \det\left(1 - 1_A \mathbb{K}\right).$$

(See e.g. Macchi '75, Soshnikov '00, Lyons '03, Shirai-Takahashi '03, Johansson '06, Hough-Krishnapur-Peres-Virag '06, Borodin '11.)

Orthogonal Polynomial Ensembles

N points with symmetric joint probability distribution

$$\frac{1}{Z_N}\prod_{1\leq i<j\leq N}(x_j-x_i)^2\prod_{j=1}^N w(x_j)dx_j.$$

Correlation kernel expressed in terms of orthonormal polynomials with respect to w:

$$K_N(x,y)=\sqrt{w(x)w(y)}\sum_{j=0}^{N-1}p_j(x)p_j(y)=c_N\sqrt{w(x)w(y)}\,\frac{p_N(x)p_{N-1}(y)-p_N(y)p_{N-1}(x)}{x-y}.$$

Multiplicative statistics are ratios of Hankel determinants:

$$\det(1-\mathrm{M}_\phi\mathbb{K})=\frac{H_N((1-\phi)w)}{H_N(w)}.$$

DPPs induced by orthogonal projections

Correlation kernel K for which the associated integral operator

$$\mathbb{K}f(x) = \int K(x,y)f(y)d\mu(y)$$

is an orthogonal projection (of finite or infinite rank).

Particular cases: OPEs (finite rank) and scaling limits of OPEs (infinite rank):

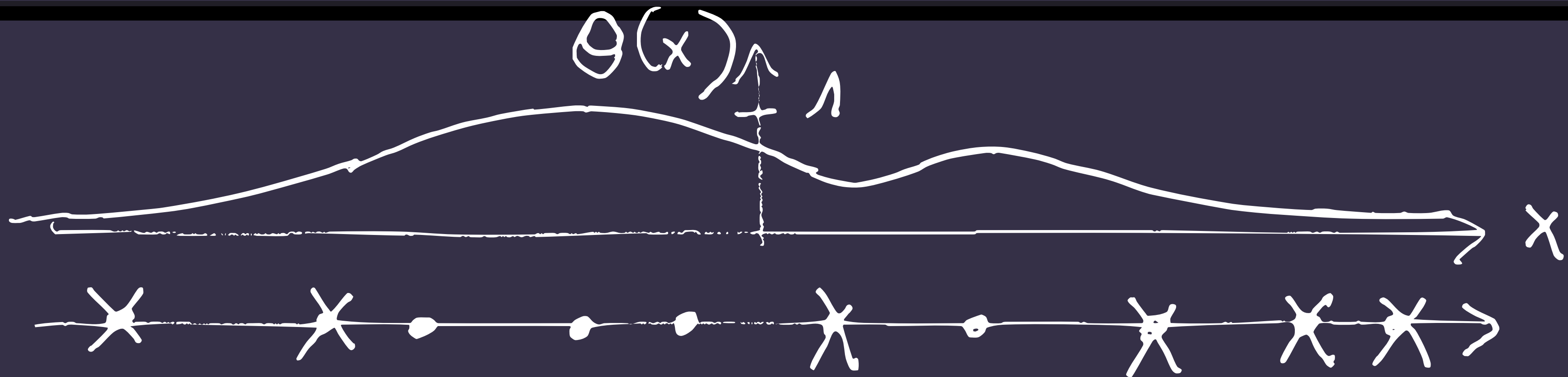
$$K^{\sin}(x,y) = \frac{\sin \pi(x-y)}{\pi(x-y)}, \; K^{\mathrm{Ai}}(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y},$$

Bessel kernels, Painlevé kernels, ...

General property (Soshnikov '00): Total number of points in a random configuration is a.s. equal to $\mathbf{rank}(\mathbb{K})$.

Thinning

Given a point process $\mathbb{P}$ on $\mathbb{R}$ and $\theta : \mathbb{R} \to [0,1]$, we define θ-thinning of $\mathbb{P}$ by removing each point x in the configuration independently with probability $1 - \theta(x)$.



Property

If $\mathbb{P}$ is the DPP with kernel $K(x,y)$ and reference measure μ, then so is its θ-thinning, but now with reference measure

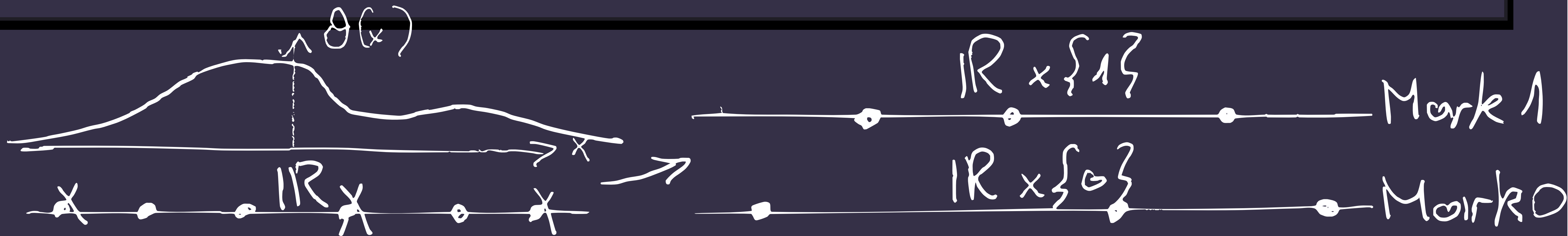
$$(1-\theta)(x)d\mu(x).$$

Thinning preserves integrable kernel DPPs, but not orthogonal projection DPPs.

Thinning, marking and conditioning of DPPs

Marked point process

Given a point process $\mathbb{P}$ on $\mathbb{R}$ and $\theta : \mathbb{R} \to [0,1]$, we define a marked point process $\mathbb{P}^\theta$ on $\mathbb{R} \times \{0,1\}$ by assigning to each point x independently mark 1 with probability $\theta(x)$ and mark 0 with probability $1 - \theta(x)$.



Property

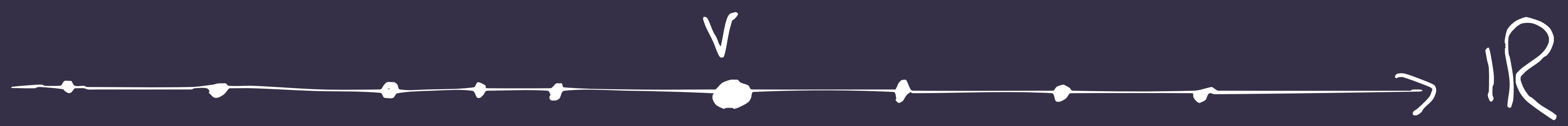
If $\mathbb{P}$ is the DPP with kernel $K(x,y)$ and reference measure μ, then so is $\mathbb{P}^\theta$, but with reference measure

$$d\mu^\theta(x,b) = \theta(x)d\mu(x)d\delta_{\{b=1\}} + (1-\theta)(x)d\mu(x)d\delta_{\{b=0\}} \text{ on } \mathbb{R} \times \{0,1\}.$$

Marking preserves integrable kernel DPPs and orthogonal projection DPPs.

Conditioning on presence of a point

Given a point process $\mathbb{P}$ on $\mathbb{R}$, the local reduced Palm measure of a point v $\mathbb{P}_v$ represents the point process $\mathbb{P}$ conditioned on v being a point in the configuration, and then removing v.



Property (Shirai-Takahashi '03)

If $\mathbb{P}$ is the DPP with kernel $K(x,y)$ and reference measure μ, then $\mathbb{P}_v$ is the DPP with kernel

$$K_v(x,y) = \frac{1}{K(v,v)} \det \begin{pmatrix} K(x,y) & K(x,v) \\ K(v,y) & K(v,v) \end{pmatrix}.$$

Palm transformation preserves OPEs, integrable kernel DPPs and orthogonal projection DPPs.

Conditioning on absence of points

Given a point process $\mathbb{P}$ on $\mathbb{R}$, define the conditional ensemble $\mathbb{P}_A$ by conditioning $\mathbb{P}$ on configurations without points in $A \subset \mathbb{R}$.

Property

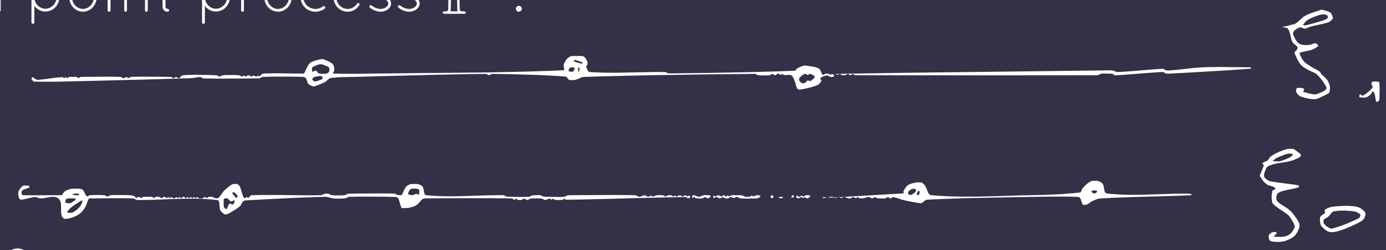
If $\mathbb{P}$ is the DPP with kernel $K(x,y)$ and reference measure μ and if $\det(1 - 1_A \mathbb{K}) \neq 0$, then $\mathbb{P}_A$ is the DPP with kernel of the operator

$$1_{A^c}\mathbb{K}(1 - 1_A \mathbb{K})^{-1}.$$

Conditioning on a gap preserves OPEs, integrable kernel DPPs, but not orthogonal projection DPPs.

Conditioning on marked point process

Given a configuration ξ and a marking function $\theta : \mathbb{R} \to [0,1]$, we write ξ_j for the configuration of mark j points in the marked point process $\mathbb{P}^\theta$.



The probability to have no mark 1 particles is

$$\mathbb{P}^\theta(\xi_1 = \emptyset) = \det(1 - \mathrm{M}_\theta \mathbb{K}).$$

If this is non-zero, we can define the conditional ensemble $\mathbb{P}^\theta_{|\emptyset}$ (on $\mathbb{R}$) by conditioning $\mathbb{P}^\theta$ on the event $\xi_1 = \emptyset$.

Conditional ensemble (cf. Bufetov '12, Bufetov-Qiu-Shamov '21)

If $\mathbb{P}$ is the DPP with kernel K of the operator $\mathbb{K}$, then $\mathbb{P}^\theta_{|\emptyset}$ is the DPP with kernel of the operator

$$\mathrm{M}_{1-\theta}\mathbb{K}(1 - \mathrm{M}_\theta \mathbb{K})^{-1} \text{ on } L^2(\mathbb{R}, d\mu), \text{ or}$$

$$\mathbb{K}(1 - \mathrm{M}_\theta \mathbb{K})^{-1} \text{ on } L^2(\mathbb{R}, (1-\theta)d\mu).$$

We want to define a conditional ensemble $\mathbb{P}^{\theta}_{|\mathbf{v}}$ which represents the conditioning of $\mathbb{P}$ on particles at $v_1, \ldots, v_k$, and on no other mark 1 particles.

Conditional ensemble

Let $\mathbb{P}$ be a DPP with kernel K of the operator $\mathbb{K}$, and suppose that $\mathbb{P}^{\theta}(\#\xi_1 = k) > 0$. For $\mathbb{P}^{\theta}$-a.e. k-point mark 1 configuration $\mathbf{v} = \{v_1, \ldots, v_k\}$,
$\mathbb{P}^{\theta}_{|\mathbf{v}}$ is the DPP with kernel of the operator

$$\mathbb{K}_{\mathbf{v}}(1 - \mathrm{M}_{\theta}\mathbb{K}_{\mathbf{v}})^{-1} \text{ on } L^2(\mathbb{R}, (1-\theta)d\mu).$$

Compare this to the Poisson process $\mathbb{P}$ with intensity ρ on $(\mathbb{R}, d\mu)$: then $\mathbb{P}^{\theta}_{|\emptyset}$ is the Poisson point process with intensity ρ on $(\mathbb{R}, (1-\theta)d\mu)$, which is the same as the unconditioned distribution of mark 0 points.

Conditional ensembles of OPEs

If $\mathbb{P}$ is an N-point OPE with density $\frac{1}{Z_n}\prod_{1\le i<j\le N}(x_j-x_i)^2\ \prod_{j=1}^N w(x_j)dx_j$, then $\mathbb{P}^\theta_{|\mathbf{v}}$ is an $n=(N-k)$-point OPE with density

$$\frac{1}{Z_n'}\prod_{1\le i<j\le n}(x_j-x_i)^2\prod_{j=1}^n\left(\prod_{\ell=1}^k(x_j-v_\ell)^2\right)(1-\theta)(x_j)w(x_j)dx_j.$$

For $w(x)=e^{-Nx^2}$, $k=0$, $1-\theta(x)=e^{-NW(x)}$ for $W\ge 0$, $\mathbb{P}^\theta_{|\mathbf{v}}$ is the OPE with confining potential x^2 replaced by $x^2+W(x)$. So any unitary invariant ensemble with confining potential $\ge x^2$ is a conditional ensemble of the GUE.

Theorem (C-Glesner '21)

The IIKS Riemann-Hilbert problem

- ✓ $Y : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{k\times k}$ is analytic,
- ✓ $Y_+(z) = Y_-(z)\left(I_k - 2\pi i\theta(z) f(z) h(z)^T\right)$ for $z \in \mathbb{R}$,
- ✓ as $z \to \infty$, $Y(z) \to I_k$,

characterizes the kernel of the conditional ensemble $K^{\theta}_{|\emptyset}$:

$$K^{\theta}_{|\emptyset} = \frac{\sum_{j=1}^{k} F_j(x) H_j(y)}{x-y}, \quad F = Yf, \quad H = Y^{-T}h.$$

A similar characterization holds for $\mathbb{P}^{\theta}_{|\mathbf{v}}$, but then the kernel is connected to a Darboux-Schlesinger transformation of the Riemann-Hilbert problem and to Janossy densities (see Sofia Tarricone's talk).

Point process interpretation of the IIKS method

We can re-write Jacobi's identity as

$$\partial_s \log \mathbb{E} \prod_j (1-\theta_s(x_j)) = -\mathbb{E}^{\theta_s}_{|\emptyset} \sum_j \frac{\partial_s \theta_s(x_j)}{1-\theta_s(x_j)},$$

relating a average multiplicative statistic of the DPP to an average additive statistic in the conditional ensemble.

The IIKS RH problem does not only characterize logarithmic derivatives of average multiplicative statistics, but it characterizes also the kernel of the conditional ensemble $\mathbb{P}^{\theta_s}_{|\emptyset}$.

Asymptotic analysis and g-functions

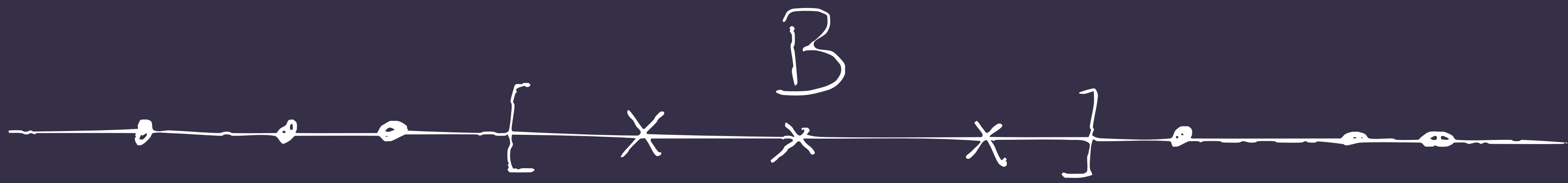
This point process interpretation of the IIKS method is not just a different viewpoint, it also helps to analyze the Riemann-Hilbert problem asymptotically.

A crucial step in the Deift-Zhou steepest descent method is to construct a suitable g-function, related to an equilibrium measure. This equilibrium measure should be the limiting one-point function in the conditional ensemble $\mathbb{P}^{\theta}_{|\emptyset}$. Constructing the g-function purely on analytic grounds is often hard, the point process interpretation helps to guess the form of the equilibrium measure.

Cf. pushed Coulomb gas interpretation to describe tails of $\det(1 - \mathrm{M}_\theta \mathbb{K}^{\mathrm{Ai}})$ and of the KPZ equation (Corwin-Ghosal-Krajenbrink-Le Doussal-Tsai '18), and asymptotic analysis of the associated Riemann-Hilbert problem (Cafasso-C, Cafasso-C-Ruzza, Charlier-C-Ruzza '19-'21).

Definition (Ghosh '16, Ghosh-Peres '17)

A point process is number rigid if for any bounded set $B \subset \mathbb{R}$, the configuration of points outside B almost surely determines the number of points in B.



Properties

DPPs induced by finite rank projections are trivially number rigid, since number of points is a.s. equal to the rank of the projection.

DPPs can only be number rigid if they are induced by a projection operator (Ghosh-Krishnapur '15).

What about DPPs induced by infinite rank projections?

Theorem (Bufetov '16)

DPPs induced by orthogonal projections with sufficiently regular $\mathbf{2}$-integrable kernels

$$K(x,y) = \frac{f_1(x)g_1(y) + f_2(x)g_2(y)}{x-y}$$

are number rigid (e.g., Sine, Airy, Bessel).

Question

Under which conditions does the mark $\mathbf{1}$ configuration in the marked point process $\mathbb{P}^{\theta}$ determine the number of points with mark $\mathbf{0}$?
(For $\theta = \mathbf{1}_{B^c}$, this is the same as asking under which condition the point process is number rigid.)

Definition

A point process $\mathbb{P}$ is marking rigid if for any measurable $\theta : \mathbb{R} \to [0,1]$ and for $\mathbb{P}_1^\theta$-a.e. mark 1 configuration $\mathbf{v}$, there exists $\ell_{\mathbf{v}} \in \mathbb{N} \cup \{0, \infty\}$ such that

$$\mathbb{P}^\theta_{|\mathbf{v}}(\#\xi_0 = \ell_{\mathbf{v}}) = 1.$$

(Note that marking rigidity implies number rigidity, by setting $\theta = 1_{B^c}$.)

Theorem (C-Glesner '21)

DPPs induced by orthogonal projections with sufficiently regular 2-integrable kernels (including Sine, Airy, Bessel point processes) are marking rigid.

Conditional DPPs

The transformation $\mathbb{P} \mapsto \mathbb{P}^{\theta}_{|\mathbf{v}}$ is a well-behaving transformation of point processes, which preserves DPPs and important subclasses like OPEs, projection DPPs, and integrable kernel DPPs.

This transformation already appeared implicitly in:

- ✓ IIKS method to study Fredholm determinants,
- ✓ unitary invariant random matrix ensembles,
- ✓ study of number rigidity.

Why study conditional DPPs

Why study conditional ensembles?

- ✓ Natural in view of the search for error-correcting codes/spectrum completion codes,
- ✓ useful in asymptotic analysis of Fredholm determinants of the form $\det(1 - \mathbf{M}_\theta \mathbb{K})$ via the IIKS method, where it helps to guess a convenient g-function,
- ✓ allows to study refined notion of number rigidity,
- ✓ allows to give a probabilistic interpretation to Jacobi's identity for Fredholm determinants, Darboux-Schlesinger-Backlund transformations for integrable systems (work in progress C-Glesner-Ruzza-Tarricone) ...

Thank you for your attention!