The complex elliptic Ginibre ensemble at weak non-Hermiticity

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In collaboration with

Figure 1: Alex Little.

and based on the forthcoming arXiv:220?.?????

What's the problem?

Consider the Gaussian Unitary Ensemble (GUE), i.e. matrices

$$\mathbf{X} = \frac{1}{2}(\mathbf{Y} + \mathbf{Y}^{\dagger}) \in \mathbb{C}^{n \times n}: \quad Y_{jk} \overset{\text{iid}}{\sim} N\left(0, \frac{1}{\sqrt{2}}\right) + \mathrm{i} N\left(0, \frac{1}{\sqrt{2}}\right)$$

as in (Porter 1965). Equivalently think of a log-gas system $\{x_j\}_{j=1}^n \subset \mathbb{R}$ with joint pdf for the particles' locations equal to (Mehta 1967)

$$p_n(x_1, \ldots, x_n) = \frac{1}{Z_n} \prod_{1 \leq j < k \leq n} |x_k - x_j|^2 \exp\left(-\sum_{j=1}^n x_j^2\right).$$

Question: How do the particles $\{x_j\}_{j=1}^n$ behave for large n?

The particles $\{x_j\}_{j=1}^n$ form a *DPP* on $\mathbb{R}$ (Dyson 1970),

$$R_k(x_1,\ldots,x_n) := \frac{n!}{(n-k)!}\int_{\mathbb{R}^{n-k}} p_n(x_1,\ldots,x_n)\prod_{j=k+1}^n \mathrm{d}x_j = \det\left[K_n(x_i,x_j)\right]_{i,j=1}^k$$

with correlation kernel

$$K_n(x,y) = \frac{\mathrm{e}^{-\frac{1}{2}(x^2+y^2)}}{\sqrt{\pi}}\sum_{k=0}^{n-1}\frac{1}{2^k k!}H_k(x)H_k(y),\quad H_n(z) = \frac{n!}{2\pi\mathrm{i}}\oint \mathrm{e}^{2zt-t^2}\frac{\mathrm{d}t}{t^{n+1}}.$$

Now analyze R_k asymptotically in different scaling regimes:

(A) The global eigenvalue regime: define the ESD

$$\mu_{\mathbf{X}}(s)=\frac{1}{n}\#\{1\leq j\leq n,\quad x_j\leq s\},\quad s\in\mathbb{R},$$

then, as $n\to\infty$, the random measure $\mu_{\mathbf{X}/\sqrt{n}}$ converges almost surely to the Wigner semi-circular distribution (Wigner 1955)

$$\rho(x)=\frac{1}{\pi}\sqrt{(2-x^2)_+}\,\mathrm{d}x \tag{1}$$

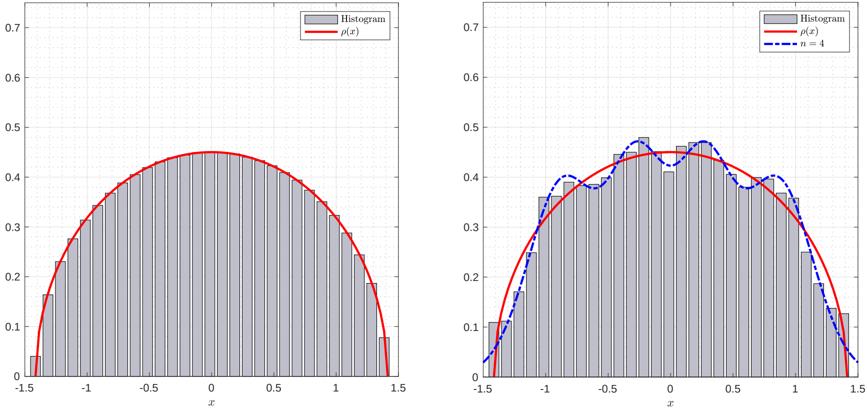

Figure 2: Wigner's law for one (rescaled) 2000×2000 GUE matrix on the left, plotted is the rescaled histogram of the 2000 eigenvalues and the semicircular density $\rho(x)$. On the right we compare Wigner's law to the exact eigenvalue density for $n = 4$ and the associated eigenvalue histogram (sampled 4000 times).

(B) The local eigenvalue regime: We shall zoom in on $x_0 = \sqrt{2n}$ only (Bowick, Brézin 1991, Forrester 1993, Nagao, Wadati 1993),

$$\frac{1}{\sqrt{2}n^{\frac{1}{6}}} K_n\left(\sqrt{2n} + \frac{x}{\sqrt{2}n^{\frac{1}{6}}}, \sqrt{2n} + \frac{y}{\sqrt{2}n^{\frac{1}{6}}}\right) \to K_{\mathrm{Ai}}(x,y), \tag{2}$$

as $n \to \infty$ uniformly in $x, y \in \mathbb{R}$ chosen from compact subsets, with

$$K_{\mathrm{Ai}}(x,y) = \int_0^\infty \mathrm{Ai}(x+z)\mathrm{Ai}(z+y)\,\mathrm{d}z,$$

which yields a trace class operator on $L^2(t,\infty)$.

In turn, the largest eigenvalue in the GUE obeys

$$\max_{i=1,\ldots,n} \lambda_i(\mathbf{X}) \Rightarrow \sqrt{2n} + \frac{1}{\sqrt{2}n^{\frac{1}{6}}} F_2, \qquad n \to \infty,$$

where the cdf of F_2 equals (Forrester 1993)

$$\mathrm{Prob}(F_2 \leq t) = \det(I - K_{\mathrm{Ai}} \restriction_{L^2(t,\infty)}),$$

which famously connects to Painlevé special function theory (Tracy, Widom 1994).

Universality

Wigner's law (1) is a universal limiting law (Arnold 1967, ...) and so is the soft edge law (2) (Soshnikov 1999). Both laws holds true for centered and scaled Hermitian *Wigner matrices* $\mathbf{X} = (X_{jk})_{j,k=1}^n$ with $\mathbb{E}|X_{jk}|^2 < \infty$ where $X_{jk}, j < k$ are iid complex variables and X_{jj} iid real variables independent of the upper triangular ones ($\oplus$ decay).

What's the problem? continued

Consider the Complex Ginibre ensemble (GinUE), i.e. matrices

$$\mathbf{X} = \mathbf{Y} \in \mathbb{C}^{n\times n}: \qquad Y_{jk} \overset{\text{iid}}{\sim} N\left(0, \frac{1}{\sqrt{2}}\right) + \mathrm{i}N\left(0, \frac{1}{\sqrt{2}}\right)$$

as in (Ginibre 1965). Equivalently think of a log-gas system $\{z_j\}_{j=1}^n \subset \mathbb{C}$ with joint pdf for the particles' locations equal to (Ginibre 1965)

$$p_n(z_1, \ldots, z_n) = \frac{1}{Z_n} \prod_{1\leq j<k\leq n} |z_k - z_j|^2 \exp\left(-\sum_{j=1}^n |z_j|^2\right).$$

Question: How do the particles $\{z_j\}_{j=1}^n$ behave for large n?

The particles $\{z_j\}_{j=1}^n$ form a *DPP* on $\mathbb{C} \simeq \mathbb{R}^2$ (Mehta 1967),

$$R_k(z_1,\ldots,z_n) := \frac{n!}{(n-k)!}\int_{\mathbb{C}^{n-k}} p_n(z_1,\ldots,z_n)\prod_{j=k+1}^{n}\mathrm{d}^2 z_j = \det\big[K_n(z_i,z_j)\big]_{i,j=1}^{k}$$

with correlation kernel

$$K_n(z,w) = \frac{\mathrm{e}^{-\frac{1}{2}(|z|^2+|w|^2)}}{\pi}\sum_{k=0}^{n-1}\frac{1}{k!}(zw^*)^k.$$

Now analyze R_k asymptotically in different scaling regimes:

(A) The global eigenvalue regime: define the ESD

$$\mu_{\mathbf{X}}(s,t)=\frac{1}{n}\#\{1\leq j\leq n,\ \ \Re z_j\leq s,\ \ \Im z_j\leq t\},\qquad s,t\in\mathbb{R}$$

then, as $n\rightarrow\infty$, the random measure $\mu_{\mathbf{X}/\sqrt{n}}$ converges almost surely to the uniform distribution on the unit disk (Ginibre 1965)

$$\rho(z)=\frac{1}{\pi}\chi_{|z|<1}(z)\mathrm{d}^2z \tag{3}$$

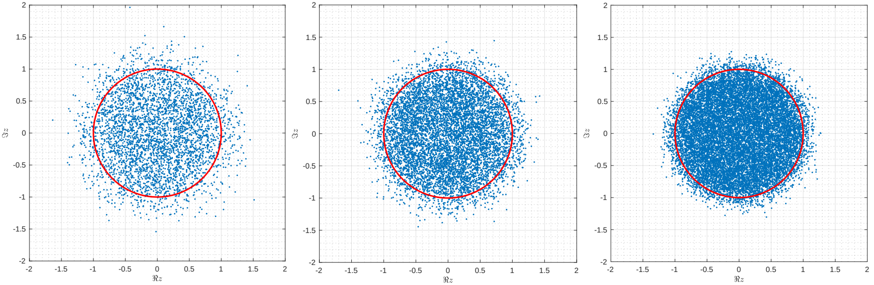

Figure 3: The circular law for 1000 complex (rescaled) Ginibre matrices of varying dimensions $n \times n$ in comparison with the unit circle boundary. We plot $n = 4, 8, 16$ from left to right.

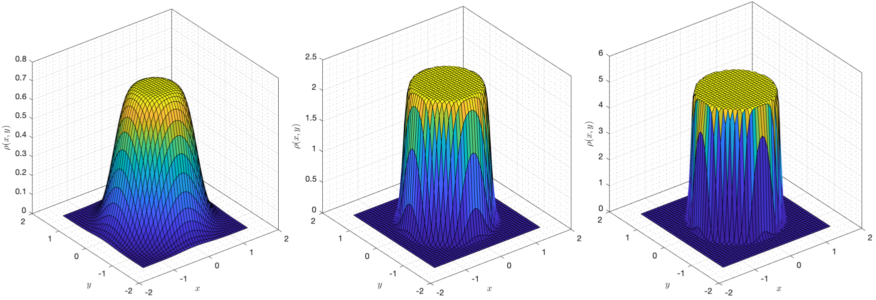

Figure 4: Rescaled eigenvalue density for $\mathbf{X} \in \mathrm{GinUE}$ with $n = 5, 50, 250$ from left to right. The larger n, the better its approach to the uniform density on $x^2 + y^2 \leq 1$.

(B) The local eigenvalue regime: We shall zoom in on $|z_0| = \sqrt{n}$ only (Ginibre 1965, Mehta 1967)

$$\frac{1}{\sqrt{n}} K_n\left(z_0 + \frac{z}{\sqrt{n}}, z_0 + \frac{w}{\sqrt{n}}\right) \to K_{\mathrm{e}}(z, w) \tag{4}$$

as $n \to \infty$ uniformly in $z, w \in \mathbb{C}$ chosen from compact subsets, with

$$K_{\mathrm{e}}(z, w) = \frac{1}{2\pi} \mathrm{erfc}\left(\sqrt{2}\Big(\mathrm{e}^{\mathrm{i}\theta} w^* + \mathrm{e}^{-\mathrm{i}\theta} z\Big)\right) \mathrm{e}^{-\frac{1}{2}(|z|^2 + |w|^2) + zw^*}$$

where $\theta = \arg z_0$.

In turn the spectral radius in the GinUE obeys

$$\max_{i=1,\ldots,n}|z_i(\mathbf{X})| \Rightarrow \sqrt{n}+\sqrt{\frac{\gamma_n}{4}}+\frac{G}{\sqrt{4\gamma_n}},\qquad n\to\infty,$$

where $\gamma_n=\ln(n/2\pi)-2\ln\ln n$ and the cdf of G equals (Rider 2003)

$$\mathrm{Prob}(G\leq t)=\mathrm{e}^{-\mathrm{e}^{-t}},$$

so no Painlevé transcendents are floating about.

Universality

The circular law (3) is a universal limiting law (Girko 1985, ...) and so is the edge law (4) (Cipolloni, Erdős, Schröder 2021). Both laws holds true for centered and scaled matrices $\mathbf{X}=(X_{jk})_{j,k=1}^n$ with iid complex entries so that $\mathbb{E}|X_{jk}|^2<\infty$ ($\oplus$ decay).

Here is the problem: the in between

Consider the Complex Elliptic Ginibre ensemble (eGinUE), i.e. matrices

$$\mathbf{X} = \sqrt{\frac{1+\tau}{2}}\,\mathbf{X}_1 + \mathrm{i}\sqrt{\frac{1-\tau}{2}}\,\mathbf{X}_2 \in \mathbb{C}^{n\times n}: \quad \mathbf{X}_1, \mathbf{X}_2 \in \text{GUE independent}$$

as in (Girko 1986). Here, $0 \leq \tau \leq 1$. Equivalently think of a log-gas system $\{z_j\}_{j=1}^n \subset \mathbb{C}$ with joint pdf equal to (Ginibre 1965)

$$p_n^\tau(z_1,\ldots,z_n) = \frac{1}{Z_n^\tau}\prod_{1\leq j<k\leq n}|z_k - z_j|^2 \exp\left(-\frac{1}{1-\tau^2}\sum_{j=1}^n\left(|z_j|^2 - \tau\Re z_j^2\right)\right).$$

Question: How do the particles $\{z_j\}_{j=1}^n$ behave for large n?

The particles $\{z_j\}_{j=1}^n$ from a DPP on $\mathbb{C} \simeq \mathbb{R}^2$ (Di Francesco,... 1994),

$$R_k^\tau(z_1,\ldots,z_n) := \frac{n!}{(n-k)!}\int_{\mathbb{C}^{n-k}} p_n^\tau(z_1,\ldots,z_n)\prod_{j=k+1}^n \mathrm{d}^2 z_j = \det\left[K_n^\tau(z_i,z_j)\right]_{i,j=1}^k$$

with correlation kernel

$$K_n^\tau(z,w) = \frac{\mathrm{e}^{-\frac{1}{2(1-\tau^2)}(|z|^2-\tau\Re z^2+|w|^2-\tau\Re w^2)}}{\pi\sqrt{1-\tau^2}}\sum_{k=0}^{n-1}\frac{\tau^k}{2^k k!}H_k\left(\frac{z}{\sqrt{2\tau}}\right)H_k\left(\frac{w^*}{\sqrt{2\tau}}\right).$$

Now analyze R_k asymptotically in different scaling regimes:

(A) The global eigenvalue regime: define the ESD

$$\mu_{\mathbf{X}}(s,t)=\frac{1}{n}\#\{1\leq j\leq n,\ \ \Re z_j\leq s,\ \ \Im z_j\leq t\},\qquad s,t\in\mathbb{R}$$

then, as $n\to\infty$, the random measure $\mu_{\mathbf{X}/\sqrt{n}}$ converges almost surely to the uniform distribution on the ellipse

$$E_\tau:=\big\{z\in\mathbb{C}:\ (\Re z)^2/(1+\tau)^2+(\Im z)^2/(1-\tau)^2<1\big\},$$

(Crisanti, Sommers, Sompolinsky, Stein 1988)

$$\rho(z)=\frac{1}{\pi(1-\tau^2)}\,\chi_{E_\tau}(z)\mathrm{d}^2z$$

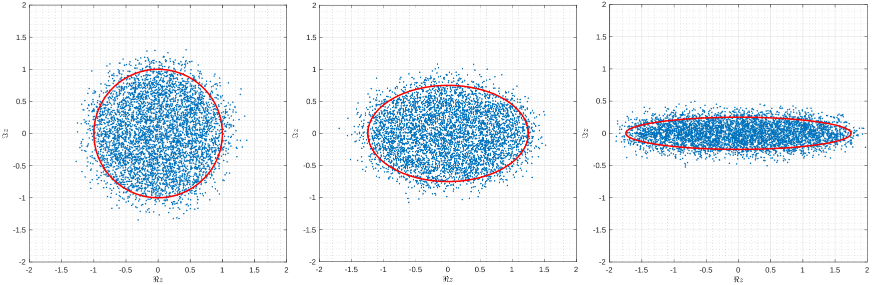

Figure 5: The elliptic law for 500 complex (rescaled) elliptic Ginibre matrices of dimension 10×10 in comparison with the ellipse boundary. We plot $\tau = 0, 0.25$, 0.75 from left to right.

(B) The local eigenvalue regime: One can look at

$$n \to \infty: \quad 1 - \tau > 0 \ \text{ uniformly in } n \qquad \text{strong non-Hermiticity}$$

as done in (Forrester, Jankovici 1996). Or, more interestingly, one can look at

$$n \to \infty: \quad \tau \uparrow 1 \qquad \text{weak non-Hermiticity}$$

as first investigated by (Fyodorov 1997). To this end, set

$$\sigma_n := n^{\alpha}\sqrt{1-\tau_n} > 0, \qquad (\tau_n)_{n=1}^{\infty} \subset [0,1),$$

which will allow us to interpolate between GUE and GinUE statistics.

We shall zoom in on the rightmost particle of the process $\{z_j\}_{j=1}^n$ $\equiv \{(x_j, y_j)\}_{j=1}^n \subset \mathbb{R}^2$ (Bender 2009). Centering and scaling,

$$x_j \mapsto \tilde{x}_j = \frac{x_j - c_n}{a_n}, \qquad y_j \mapsto \tilde{y}_j = \frac{y_j}{b_n}, \quad \alpha = \frac{1}{6},$$

accordingly, the eigenvalue process $P_n^{\tau_n} = \{(\tilde{x}_j, \tilde{y}_j)\}_{j=1}^n$

(i) converges weakly to a Poisson process on $\mathbb{R}^2$ when $\sigma_n \to \infty$,

(ii) converges weakly to the interpolating Airy process on $\mathbb{R}^2$ when $\sigma_n \to \sigma \in [0, \infty)$.

The Poisson process is determined by the correlation kernel

$$K_{\mathrm{p}}(z_1, z_2) = \delta_{z_1 z_2} \frac{1}{\sqrt{\pi}} \mathrm{e}^{-x_1 - y_1^2}, \quad z_k = (x_k, y_k) \in \mathbb{R}^2$$

and the interpolating Airy process by the correlation kernel

$$K_{\mathrm{Ai}}^{\sigma}(z_1,z_2)=\frac{1}{\sigma\sqrt{\pi}}\exp\left[-\frac{1}{2\sigma^2}(y_1^2+y_2^2)+\frac{1}{2}\sigma^2(z_1+z_2^*)+\frac{1}{6}\sigma^6\right]$$
$$\times\int_0^{\infty}\mathrm{e}^{s\sigma^2}\mathrm{Ai}\left(x_1+\mathrm{i}y_1+\frac{1}{4}\sigma^4+s\right)\mathrm{Ai}\left(x_2-\mathrm{i}y_2+\frac{1}{4}\sigma^4+s\right)\mathrm{d}s,$$

where we write $z_k=(x_k,y_k)\in\mathbb{R}^2$ for shorthand. In addition

$$\max_{i=1,\ldots,n}x_j(\mathbf{X})\Rightarrow c_n+a_nB_{\sigma},\qquad \sigma_n\to\sigma\in[0,\infty)$$

where the cdf of B_σ equals (Bender 2009)

$$F(t,\sigma):=\mathrm{Prob}(B_\sigma\leq t)=\det(I-K_{\mathrm{Ai}}^{\sigma}\restriction_{L^2((t,\infty)\times\mathbb{R})}).$$

The problem and its solution

Gernot Akemann's question

What can you say about $F(t,\sigma)$? Any Painlevé transcendents floating around? What about asymptotics?

and our answer

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For all $(t,\sigma) \in \mathbb{R} \times [0,\infty)$,

$$F(t,\sigma) = \exp\left[-\int_t^\infty (s-t)\left\{\int_{-\infty}^\infty q_\sigma^2(s,\lambda)\mathrm{d}\nu_\sigma(\lambda)\right\}\mathrm{d}s\right], \quad \frac{\mathrm{d}\nu_\sigma}{\mathrm{d}\lambda} = \frac{1}{\sigma\sqrt{\pi}}\mathrm{e}^{-\lambda^2/\sigma^2}$$

where $q_\sigma(t,\lambda)$ solve the integro-differential Painlevé-II equation

$$\frac{\partial^2}{\partial t^2}q_\sigma(t,y) = \left[t + y + 2\int_{-\infty}^\infty q_\sigma^2(t,\lambda)\mathrm{d}\nu_\sigma(\lambda)\right]q_\sigma(t,y), \quad q_\sigma(t,y) \sim \mathrm{Ai}(t+y), \ t \to +\infty.$$

The above shows in particular that

$$F(t,\sigma) = \det(I - K_{\mathrm{Ai}}^{\sigma} \upharpoonright_{L^2((t,\infty)\times\mathbb{R})}) = \det(I - L_\sigma \upharpoonright_{L^2(t,\infty)}),$$

where L_σ is trace class on $L^2(t,\infty)$ with kernel

$$L_\sigma(x,y) = \int_{-\infty}^{\infty} \Phi\left(\frac{z}{\sigma}\right) \mathrm{Ai}(x+z)\mathrm{Ai}(z+y)\,\mathrm{d}z, \tag{5}$$

with $\Phi(x) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{x} \mathrm{e}^{-y^2}\,\mathrm{d}y = 1 - \frac{1}{2}\mathrm{erfc}(x)$. Note that (5) is an example of a so-called finite-temperature Airy kernel.

Some details

Why a Painlevé connection? Put $J_t := (t,\infty)\times\mathbb{R}\subset\mathbb{R}^2$.

Trace identities

We have for all $n\in\mathbb{Z}_{\geq 0}$ and $(t,\sigma)\in\mathbb{R}\times[0,\infty)$,

$$\operatorname*{tr}_{L^2(J_t)}(K_{\mathrm{Ai}}^{\sigma})^n=\operatorname*{tr}_{L^2(J_t)}K_{\sigma}^n$$

where K_σ is trace class on $L^2(J_t)$ with kernel

$$K_\sigma(z_1,z_2):=\frac{1}{\sqrt{\pi}}\mathrm{e}^{-\frac{1}{2}y_1^2}\mathsf{K}_{\mathrm{Ai}}(x_1+\sigma y_1,x_2+\sigma y_2)\mathrm{e}^{-\frac{1}{2}y_2^2}. \tag{6}$$

The point is, (6) is an additive Hankel composition kernel in the horizontal variable!

Indeed, $K_\sigma(z_1, z_2)$ is of the type

$$K_\sigma(z_1, z_2) = \int_0^\infty \phi_\sigma(x_1 + s, y_1)\phi_\sigma(s + x_2, y_2)\,\mathrm{d}s$$

where

$$\phi_\sigma(x, y) := \frac{1}{\pi^{\frac{1}{4}}}\mathrm{e}^{-\frac{1}{2}y^2}\mathrm{Ai}(x + \sigma y).$$

Thus the methods of (Krajenbrink 2021) and (Bothner 2022) are readily available in the analysis of $F(t, \sigma)$ and the integro-differential Painlevé-II equation appears quite naturally.

More results

How about (tail) asymptotics of $F(t,\sigma)$?

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For any $\epsilon \in (0,1)$, there exists $t_0 = t_0(\epsilon)$ such that

$$F(t,\sigma) = 1 - A(t,\sigma)\mathrm{e}^{-B(t,\sigma)}\big(1+o(1)\big), \tag{7}$$

for $t \geq t_0$ and $0 \leq \sigma \leq t^{\epsilon}$. Here,

$$A(t,\sigma) = \frac{1}{2\pi t^{\frac{3}{2}}}\left(\sqrt{4+\sigma^4 t^{-1}} - \sigma^2 t^{-\frac{1}{2}}\right)^{-\frac{5}{2}}\left(4+\sigma^4 t^{-1}\right)^{-\frac{1}{4}},$$

$$B(t,\sigma) = \frac{4}{3}t^{\frac{3}{2}}\left(1+\frac{\sigma^4}{4t}\right)^{\frac{3}{2}} - t\sigma^2 - \frac{1}{6}\sigma^6.$$

And beyond $0 \leq \sigma \leq t^{\epsilon}, \epsilon \in (0,1)$?

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There exist $t_0, \sigma_0 > 0$ such that

$$F(t,\sigma) = \exp\left[\sigma^{\frac{3}{2}} C\left(\frac{t}{\sigma}\right) + \frac{1}{4}\int_{\frac{t}{\sigma}}^{\infty}\left\{\frac{\mathrm{d}}{\mathrm{d}u}D(u)\right\}^2 \mathrm{d}u\right]\big(1+o(1)\big), \tag{8}$$

for $t \geq t_0$ and $\sigma \geq \sigma_0$. Here,

$$C(y) = \frac{1}{\pi}\int_0^{\infty}\sqrt{x}\ln\Phi(x+y)\,\mathrm{d}x, \qquad D(y) = \frac{1}{\pi}\int_0^{\infty}\frac{1}{\sqrt{x}}\ln\Phi(x+y)\,\mathrm{d}x.$$

Note that (7) and (8) capture the full $(t \to +\infty)$ crossover between

$$F_2(t) = 1 - \frac{1}{16\pi t^{\frac{3}{2}}}\exp\left[-\frac{4}{3}t^{\frac{3}{2}}\right]\big(1+o(1)\big); \qquad \mathrm{e}^{-\mathrm{e}^{-t}} = 1 - \mathrm{e}^{-t}\big(1+o(1)\big)$$

The left tail (uniformly for all $\sigma \in (0,\infty)$) is work in progress.

Thank you very much for your attention!!!