The complex elliptic Ginibre ensemble at weak non-Hermiticity

Thomas Bothner

School of Mathematics
University of Bristol

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In collaboration with

Figure 1: Alex Little.

and based on the forthcoming arXiv:2207.?????
Consider the **Gaussian Unitary Ensemble (GUE)**, i.e. matrices

\[ X = \frac{1}{2}(Y + Y^\dagger) \in \mathbb{C}^{n \times n} : \ Y_{jk} \overset{iid}{\sim} \mathcal{N} \left( 0, \frac{1}{\sqrt{2}} \right) + i\mathcal{N} \left( 0, \frac{1}{\sqrt{2}} \right) \]

as in (Porter 1965). Equivalently think of a log-gas system \( \{x_j\}_{j=1}^n \subset \mathbb{R} \) with joint pdf for the particles’ locations equal to (Mehta 1967)

\[
p_n(x_1, \ldots, x_n) = \frac{1}{Z_n} \prod_{1 \leq j < k \leq n} |x_k - x_j|^2 \exp \left( - \sum_{j=1}^n x_j^2 \right).
\]

**Question:** How do the particles \( \{x_j\}_{j=1}^n \) behave for large \( n \)?
The particles \( \{x_j\}_{j=1}^{n} \) form a DPP on \( \mathbb{R} \) (Dyson 1970),

\[
R_k(x_1, \ldots, x_n) := \frac{n!}{(n-k)!} \int_{\mathbb{R}^{n-k}} p_n(x_1, \ldots, x_n) \prod_{j=k+1}^{n} dx_j = \det \left[ K_n(x_i, x_j) \right]_{i,j=1}^{k}
\]

with correlation kernel

\[
K_n(x, y) = \frac{e^{-\frac{1}{2}(x^2+y^2)}}{\sqrt{\pi}} \sum_{k=0}^{n-1} \frac{1}{2^k k!} H_k(x) H_k(y), \quad H_n(z) = \frac{n!}{2\pi i} \oint e^{2zt-t^2} \frac{dt}{t^{n+1}}.
\]

Now analyze \( R_k \) asymptotically in different scaling regimes:
(A) The global eigenvalue regime: define the ESD

\[ \mu_X(s) = \frac{1}{n} \#\{1 \leq j \leq n, \ x_j \leq s\}, \quad s \in \mathbb{R}, \]

then, as \( n \to \infty \), the random measure \( \mu_X/\sqrt{n} \) converges almost surely to the Wigner semi-circular distribution \((\text{Wigner 1955})\)

\[ \rho(x) = \frac{1}{\pi} \sqrt{(2 - x^2)_+} \, dx \quad (1) \]
Figure 2: Wigner’s law for one (rescaled) $2000 \times 2000$ GUE matrix on the left, plotted is the rescaled histogram of the 2000 eigenvalues and the semicircular density $\rho(x)$. On the right we compare Wigner’s law to the exact eigenvalue density for $n = 4$ and the associated eigenvalue histogram (sampled 4000 times).
(B) The local eigenvalue regime: We shall zoom in on $x_0 = \sqrt{2n}$ only (Bowick, Brézin 1991, Forrester 1993, Nagao, Wadati 1993),

$$
\frac{1}{\sqrt{2n^{1/6}}} K_n \left( \sqrt{2n} + \frac{x}{\sqrt{2n^{1/6}}}, \sqrt{2n} + \frac{y}{\sqrt{2n^{1/6}}} \right) \to K_{Ai}(x, y),
$$

(2)
as $n \to \infty$ uniformly in $x, y \in \mathbb{R}$ chosen from compact subsets, with

$$
K_{Ai}(x, y) = \int_0^\infty \text{Ai}(x + z)\text{Ai}(z + y) \, dz,
$$

which yields a trace class operator on $L^2(t, \infty)$. 
In turn, the largest eigenvalue in the GUE obeys

$$\max_{i=1,\ldots,n} \lambda_i(\mathbf{X}) \Rightarrow \sqrt{2n} + \frac{1}{\sqrt{2n^6}} F_2, \quad n \to \infty,$$

where the cdf of $F_2$ equals (Forrester 1993)

$$\text{Prob}(F_2 \leq t) = \det(I - K_{\text{Ai}} |_{L^2(t,\infty)}),$$

which famously connects to Painlevé special function theory (Tracy, Widom 1994).

Universality

Wigner’s law (1) is a universal limiting law (Arnold 1967, ...) and so is the soft edge law (2) (Soshnikov 1999). Both laws holds true for centered and scaled Hermitian Wigner matrices $\mathbf{X} = (X_{jk})_{j,k=1}^n$ with $\mathbb{E}|X_{jk}|^2 < \infty$ where $X_{jk}, j < k$ are iid complex variables and $X_{jj}$ iid real variables independent of the upper triangular ones ($\oplus$ decay).
Consider the Complex Ginibre ensemble (GinUE), i.e. matrices

\[ \mathbf{X} = \mathbf{Y} \in \mathbb{C}^{n \times n} : \quad Y_{jk} \overset{\text{iid}}{\sim} \mathcal{N} \left(0, \frac{1}{\sqrt{2}}\right) + i\mathcal{N} \left(0, \frac{1}{\sqrt{2}}\right) \]

as in (Ginibre 1965). Equivalently think of a log-gas system \( \{z_j\}_{j=1}^n \subset \mathbb{C} \) with joint pdf for the particles’ locations equal to (Ginibre 1965)

\[
p_n(z_1, \ldots, z_n) = \frac{1}{Z_n} \prod_{1 \leq j < k \leq n} |z_k - z_j|^2 \exp \left(-\sum_{j=1}^{n} |z_j|^2 \right).
\]

**Question:** How do the particles \( \{z_j\}_{j=1}^n \) behave for large \( n \)?
The particles \( \{z_j\}_{j=1}^n \) form a DPP on \( \mathbb{C} \simeq \mathbb{R}^2 \) (Mehta 1967),

\[
R_k(z_1, \ldots, z_n) := \frac{n!}{(n-k)!} \int_{\mathbb{C}^{n-k}} p_n(z_1, \ldots, z_n) \prod_{j=k+1}^n \int \mathbb{C} d^2 z_j = \det [K_n(z_i, z_j)]_{i,j=1}^k
\]

with correlation kernel

\[
K_n(z, w) = \frac{e^{-\frac{1}{2}(|z|^2+|w|^2)}}{\pi} \sum_{k=0}^{n-1} \frac{1}{k!} (zw^*)^k.
\]

Now analyze \( R_k \) asymptotically in different scaling regimes:
(A) The global eigenvalue regime: define the ESD

$$\mu_X(s, t) = \frac{1}{n} \# \{1 \leq j \leq n, \Re z_j \leq s, \Im z_j \leq t \}, \quad s, t \in \mathbb{R}$$

then, as $n \to \infty$, the random measure $\mu_X/\sqrt{n}$ converges almost surely to the uniform distribution on the unit disk (Ginibre 1965)

$$\rho(z) = \frac{1}{\pi} \chi_{|z|<1}(z) d^2 z$$  \hspace{1cm} (3)
Figure 3: The circular law for 1000 complex (rescaled) Ginibre matrices of varying dimensions $n \times n$ in comparison with the unit circle boundary. We plot $n = 4, 8, 16$ from left to right.
**Figure 4:** Rescaled eigenvalue density for $X \in \text{GinUE}$ with $n = 5, 50, 250$ from left to right. The larger $n$, the better its approach to the uniform density on $x^2 + y^2 \leq 1$. 
(B) The local eigenvalue regime: We shall zoom in on $|z_0| = \sqrt{n}$ only (Ginibre 1965, Mehta 1967)

$$
\frac{1}{\sqrt{n}} K_n \left( z_0 + \frac{z}{\sqrt{n}}, z_0 + \frac{w}{\sqrt{n}} \right) \to K_e(z, w)
$$

as $n \to \infty$ uniformly in $z, w \in \mathbb{C}$ chosen from compact subsets, with

$$
K_e(z, w) = \frac{1}{2\pi} \text{erfc} \left( \sqrt{2} \left( e^{i\theta} w^* + e^{-i\theta} z \right) \right) e^{-\frac{1}{2}(|z|^2 + |w|^2 + zw^*)}
$$

where $\theta = \text{arg} \, z_0$. 

In turn the spectral radius in the GinUE obeys

$$\max_{i=1,...,n} |z_i(X)| \Rightarrow \sqrt{n} + \sqrt{\frac{\gamma_n}{4}} + \frac{G}{\sqrt{4\gamma_n}}, \quad n \to \infty,$$

where $\gamma_n = \ln(n/2\pi) - 2 \ln \ln n$ and the cdf of $G$ equals (Rider 2003)

$$\text{Prob}(G \leq t) = e^{-e^{-t}},$$

so no Painlevé transcendents are floating about.

**Universality**

The circular law (3) is a universal limiting law (Girko 1985, ...) and so is the edge law (4) (Cipolloni, Erdős, Schröder 2021). Both laws holds true for centered and scaled matrices $X = (X_{jk})_{j,k=1}^n$ with iid complex entries so that $\mathbb{E}|X_{jk}|^2 < \infty$ (⊕ decay).
Consider the Complex Elliptic Ginibre ensemble (eGinUE), i.e. matrices

\[
X = \sqrt{\frac{1 + \tau}{2}} X_1 + i \sqrt{\frac{1 - \tau}{2}} X_2 \in \mathbb{C}^{n \times n} : \ X_1, X_2 \in \text{GUE independent}
\]

as in (Girko 1986). Here, \(0 \leq \tau \leq 1\). Equivalently think of a log-gas system \(\{z_j\}_{j=1}^n \subset \mathbb{C}\) with joint pdf equal to (Ginibre 1965)

\[
p_\tau^n(z_1, \ldots, z_n) = \frac{1}{Z_\tau^n} \prod_{1 \leq j < k \leq n} |z_k - z_j|^2 \exp \left( - \frac{1}{1 - \tau^2} \sum_{j=1}^n \left( |z_j|^2 - \tau \Re z_j^2 \right) \right).
\]

**Question:** How do the particles \(\{z_j\}_{j=1}^n\) behave for large \(n\)?
The particles \( \{z_j\}_{j=1}^n \) from a DPP on \( \mathbb{C} \cong \mathbb{R}^2 \) (Di Francesco,... 1994),

\[
R_k^\tau(z_1, \ldots, z_n) := \frac{n!}{(n-k)!} \int_{\mathbb{C}^{n-k}} p_n^\tau(z_1, \ldots, z_n) \prod_{j=k+1}^n \! d^2z_j = \det [K_n^\tau(z_i, z_j)]_{i,j=1}^k
\]

with correlation kernel

\[
K_n^\tau(z, w) = \frac{e^{-\frac{1}{2(1-\tau^2)}(|z|^2 - \tau \Re z^2 + |w|^2 - \tau \Re w^2)}}{\pi \sqrt{1-\tau^2}} \sum_{k=0}^{n-1} \frac{\tau^k}{2^k k!} H_k \left( \frac{z}{\sqrt{2\tau}} \right) H_k \left( \frac{w^*}{\sqrt{2\tau}} \right).
\]

Now analyze \( R_k \) asymptotically in different scaling regimes:
(A) The global eigenvalue regime: define the ESD

\[ \mu_X(s, t) = \frac{1}{n} \# \{ 1 \leq j \leq n, \ \Re z_j \leq s, \ \Im z_j \leq t \}, \quad s, t \in \mathbb{R} \]

then, as \( n \to \infty \), the random measure \( \mu_X / \sqrt{n} \) converges almost surely to the uniform distribution on the ellipse

\[ E_\tau := \{ z \in \mathbb{C} : (\Re z)^2 / (1 + \tau)^2 + (\Im z)^2 / (1 - \tau)^2 < 1 \} \]

(Crisanti, Sommers, Sompolinsky, Stein 1988)

\[ \rho(z) = \frac{1}{\pi(1 - \tau^2)} \chi_{E_\tau}(z) d^2 z \]
Figure 5: The elliptic law for 500 complex (rescaled) elliptic Ginibre matrices of dimension $10 \times 10$ in comparison with the ellipse boundary. We plot $\tau = 0, 0.25, 0.75$ from left to right.
(B) The local eigenvalue regime: One can look at

\[ n \to \infty : \quad 1 - \tau > 0 \quad \text{uniformly in } n \quad \text{strong non-Hermiticity} \]

as done in (Forrester, Jankovici 1996). Or, more interestingly, one can look at

\[ n \to \infty : \quad \tau \uparrow 1 \quad \text{weak non-Hermiticity} \]

as first investigated by (Fyodorov 1997). To this end, set

\[ \sigma_n := n^\alpha \sqrt{1 - \tau_n} > 0, \quad (\tau_n)_{n=1}^\infty \subset [0, 1), \]

which will allow us to interpolate between GUE and GinUE statistics.
We shall zoom in on the rightmost particle of the process \( \{z_j\}_{j=1}^n \equiv \{(x_j, y_j)\}_{j=1}^n \subset \mathbb{R}^2 \) (Bender 2009). Centering and scaling,

\[
x_j \mapsto \tilde{x}_j = \frac{x_j - c_n}{a_n}, \quad y_j \mapsto \tilde{y}_j = \frac{y_j}{b_n}, \quad \alpha = \frac{1}{6},
\]

accordingly, the eigenvalue process \( P_{n}^{\tau_n} = \{(\tilde{x}_j, \tilde{y}_j)\}_{j=1}^n \)

(i) converges weakly to a Poisson process on \( \mathbb{R}^2 \) when \( \sigma_n \to \infty \),

(ii) converges weakly to the interpolating Airy process on \( \mathbb{R}^2 \) when \( \sigma_n \to \sigma \in [0, \infty) \).

The Poisson process is determined by the correlation kernel

\[
K_p(z_1, z_2) = \delta_{z_1 z_2} \frac{1}{\sqrt{\pi}} e^{-x_1 - y_1^2}, \quad z_k = (x_k, y_k) \in \mathbb{R}^2
\]
and the interpolating Airy process by the correlation kernel

\[ K_{\text{Ai}}^\sigma(z_1, z_2) = \frac{1}{\sigma\sqrt{\pi}} \exp \left[ -\frac{1}{2\sigma^2}(y_1^2 + y_2^2) + \frac{1}{2}\sigma^2(z_1 + z_2^*) + \frac{1}{6}\sigma^6 \right] \]

\[ \times \int_0^\infty e^{s^2} \text{Ai} \left( x_1 + iy_1 + \frac{1}{4}\sigma^4 + s \right) \text{Ai} \left( x_2 - iy_2 + \frac{1}{4}\sigma^4 + s \right) \, ds, \]

where we write \( z_k = (x_k, y_k) \in \mathbb{R}^2 \) for shorthand. In addition

\[ \max_{i=1, \ldots, n} x_j(X) \Rightarrow c_n + a_n B_\sigma, \quad \sigma_n \to \sigma \in [0, \infty) \]

where the cdf of \( B_\sigma \) equals (Bender 2009)

\[ F(t, \sigma) := \text{Prob}(B_\sigma \leq t) = \det(I - K_{\text{Ai}}^\sigma \mid \mathcal{L}^2((t,\infty) \times \mathbb{R})). \]
Gernot Akemann’s question

What can you say about $F(t, \sigma)$? Any Painlevé transcendents floating around? What about asymptotics?

and our answer

B-Little 2022

For all $(t, \sigma) \in \mathbb{R} \times [0, \infty)$,

$$F(t, \sigma) = \exp \left[ - \int_t^\infty (s - t) \left\{ \int_{-\infty}^\infty q_\sigma^2(s, \lambda) d\nu_\sigma(\lambda) \right\} ds \right], \quad \frac{d\nu_\sigma}{d\lambda} = \frac{1}{\sigma \sqrt{\pi}} e^{-\lambda^2/\sigma^2}$$

where $q_\sigma(t, \lambda)$ solve the integro-differential Painlevé-II equation

$$\frac{\partial^2}{\partial t^2} q_\sigma(t, y) = \left[ t + y + 2 \int_{-\infty}^\infty q_\sigma^2(t, \lambda) d\nu_\sigma(\lambda) \right] q_\sigma(t, y), \quad q_\sigma(t, y) \sim Ai(t + y), \ t \to +\infty.$$
The above shows in particular that

\[ F(t, \sigma) = \det(I - K_{\text{Ai}}^{\sigma} \upharpoonright L^2((t, \infty) \times \mathbb{R})) = \det(I - L_{\sigma} \upharpoonright L^2(t, \infty)), \]

where \( L_{\sigma} \) is trace class on \( L^2(t, \infty) \) with kernel

\[ L_{\sigma}(x, y) = \int_{-\infty}^{\infty} \Phi \left( \frac{z}{\sigma} \right) \text{Ai}(x + z) \text{Ai}(z + y) \, dz, \quad (5) \]

with \( \Phi(x) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{x} e^{-y^2} \, dy = 1 - \frac{1}{2} \text{erfc}(x) \). Note that (5) is an example of a so-called finite-temperature Airy kernel.
Some details

Why a Painlevé connection? Put $J_t := (t, \infty) \times \mathbb{R} \subset \mathbb{R}^2$.

Trace identities

We have for all $n \in \mathbb{Z}_{\geq 0}$ and $(t, \sigma) \in \mathbb{R} \times [0, \infty)$,

$$
\text{tr}_{L^2(J_t)} (K^\sigma_{\text{Ai}})^n = \text{tr}_{L^2(J_t)} K^n
$$

where $K^\sigma$ is trace class on $L^2(J_t)$ with kernel

$$
K^\sigma(z_1, z_2) := \frac{1}{\sqrt{\pi}} e^{-\frac{1}{2}y_1^2} K_{\text{Ai}}(x_1 + \sigma y_1, x_2 + \sigma y_2) e^{-\frac{1}{2}y_2^2}.
$$

The point is, (6) is an additive Hankel composition kernel in the horizontal variable!
Indeed, $K_\sigma(z_1, z_2)$ is of the type

$$K_\sigma(z_1, z_2) = \int_0^\infty \phi_\sigma(x_1 + s, y_1)\phi_\sigma(s + x_2, y_2) \, ds$$

where

$$\phi_\sigma(x, y) := \frac{1}{\pi^{1/4}} e^{-\frac{1}{2} y^2} \text{Ai}(x + \sigma y).$$

Thus the methods of (Krajenbrink 2021) and (Bothner 2022) are readily available in the analysis of $F(t, \sigma)$ and the integro-differential Painlevé-II equation appears quite naturally.
How about (tail) asymptotics of $F(t, \sigma)$?

For any $\epsilon \in (0, 1)$, there exists $t_0 = t_0(\epsilon)$ such that

$$F(t, \sigma) = 1 - A(t, \sigma) e^{-B(t, \sigma)} (1 + o(1)), \quad (7)$$

for $t \geq t_0$ and $0 \leq \sigma \leq t^\epsilon$. Here,

$$A(t, \sigma) = \frac{1}{2\pi t^{\frac{3}{2}}} \left( \sqrt{4 + \sigma^4 t^{-1}} - \sigma^2 t^{-\frac{1}{2}} \right)^{-\frac{5}{2}} (4 + \sigma^4 t^{-1})^{-\frac{1}{4}},$$

$$B(t, \sigma) = \frac{4}{3} t^{\frac{3}{2}} \left( 1 + \frac{\sigma^4}{4t} \right)^{\frac{3}{2}} - t\sigma^2 - \frac{1}{6} \sigma^6.$$
There exist $t_0, \sigma_0 > 0$ such that

$$F(t, \sigma) = \exp \left[ \sigma^2 C \left( \frac{t}{\sigma} \right) + \frac{1}{4} \int_{\frac{t}{\sigma}}^{\infty} \left\{ \frac{d}{du} D(u) \right\}^2 du \right] \left(1 + o(1)\right), \quad (8)$$

for $t \geq t_0$ and $\sigma \geq \sigma_0$. Here,

$$C(y) = \frac{1}{\pi} \int_{0}^{\infty} \sqrt{x} \ln \Phi(x + y) \, dx, \quad D(y) = \frac{1}{\pi} \int_{0}^{\infty} \frac{1}{\sqrt{x}} \ln \Phi(x + y) \, dx.$$

Note that (7) and (8) capture the full ($t \to +\infty$) crossover between

$$F_2(t) = 1 - \frac{1}{16\pi t^{3/2}} \exp \left[ -\frac{4}{3} t^{3/2} \right] \left(1 + o(1)\right); \quad e^{-e^{-t}} = 1 - e^{-t} \left(1 + o(1)\right)$$

The left tail (uniformly for all $\sigma \in (0, \infty)$) is work in progress.
Thank you very much for your attention!!!