# QUANTUM KDV & QUASIMODULAR FORMS

EXCURSIONS IN INTEGRABILITY

TRIESTE, 26 MAY 2022

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# Plan

1) Dispersionless quantum KdV (Eliashberg, Dubrovin)

2) Functions of partitions and quasimodular forms (Bloch–Okounkov, Zagier, van Ittersum)

3) Quantum KdV and quasimodular forms (van Ittersum – R)

# Hamiltonian PDEs

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Phase space: $\mathcal{M} = \{ u : \mathbb{R} \to \mathbb{C},\ \text{smooth},\ u(x+2\pi) = u(x) \}$

Poisson bracket: $\{F, G\} = \int_{-\pi}^{\pi} \frac{\delta F}{\delta u(x)} \left( \partial_x \frac{\delta G}{\delta u(x)} \right) \frac{dx}{2\pi}$, $\quad F, G : \mathcal{M} \to \mathbb{C}$

Hamiltonian PDE: $\partial_t u(x) = \{ u(x), G \} = \partial_x \frac{\delta G}{\delta u(x)}$

Fourier coordinates: $\phi = (\phi_k)_{k \in \mathbb{Z}} \longmapsto u(x) = \sum_{k \in \mathbb{Z}} \phi_k e^{ikx}$

$$\phi_k = \int_{-\pi}^{\pi} e^{-ikx} u(x) \frac{dx}{2\pi} \Rightarrow \frac{\delta \phi_k}{\delta u(x)} = e^{-ikx} \Rightarrow \{\phi_k, \phi_\ell\} = \int_{-\pi}^{\pi} e^{-i(k+\ell)x} (-i\ell) \frac{dx}{2\pi} = ik\, \delta_{k,-\ell}$$

($\phi_0$ Casimir, $\phi_k, \phi_{-k}$ canonically conjugate)

# KdV hierarchy



Commuting Hamiltonian flows (on $\mathcal{M}$)

$$\partial_{t_k} u(x) = \left\{ u(x), G_k^{KdV} \right\}, \qquad \left\{ G_k^{KdV}, G_l^{KdV} \right\} = 0$$

$$G_k^{KdV} = \int_{-\pi}^{\pi} g_k^{KdV}(u(x)) \frac{dx}{2\pi} \qquad (k = 1 \Rightarrow \text{KdV eq.})$$

$$g_{-2}^{KdV} = 1, \quad g_{-1}^{KdV} = u_0, \quad g_0^{KdV} = \frac{u_0^2}{2} + \varepsilon u_2, \quad g_1^{KdV} = \frac{u_0^3}{6} + \varepsilon u_0 u_2 + \frac{u_4}{2}$$

$$\ldots, \quad g_k^{KdV} = \frac{1}{k!} u_0^k + O(\varepsilon), \quad \ldots$$

($\varepsilon = 0$ : "dispersionless")

$$(u_0, u_1, u_2, \ldots) = (u, u_x, u_{xx}, \ldots)$$

# Quantization (I)

Promote function(al)s $F : \mathcal{M} \to \mathbb{C}$ to operators $\hat{F}$ on $\Lambda$:

$$\widehat{\{F, G\}} = \frac{1}{i\hbar}[\hat{F}, \hat{G}]$$

Let's fix: $\Lambda = \mathbb{C}[p]$, $p = (p_1, p_2, p_3, \ldots)$

$$(\hat{\phi}_k f)(p) = \begin{cases} p_k f(p) & k > 0 \\ c f(p) & k = 0 \\ \hbar |k| \dfrac{\partial f(p)}{\partial p_{|k|}} & k < 0 \end{cases} \qquad \left( [\hat{\phi}_k, \hat{\phi}_l] = -\hbar k \delta_{k,-l} \right)$$

Normal order: $:\hat{\phi}_{a_1} \cdots \hat{\phi}_{a_l}: = \prod_{a_i \geq 0} \hat{\phi}_{a_i} \prod_{a_i < 0} \hat{\phi}_{a_i} \Rightarrow$ well-defined...

# Quantization (II)

To a differential polynomial

$$g(u) \in \mathbb{C}[u], \qquad u = (u_0, u_1, u_2, \dots) = (u, u_x, u_{xx}, \dots)$$

we associate an operator on $\Lambda$

$$\overline{g} = \int_{-\pi}^{\pi} :g(\hat{u}(x)): \frac{dx}{2\pi}, \quad \text{where } \hat{u}(x) = \sum_{k \in \mathbb{Z}} \hat{\phi}_k e^{ikx}$$

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E.g. $\frac{1}{2}\overline{u^2} = \frac{1}{2}\sum_{k\in\mathbb{Z}} :\hat{\phi}_k \hat{\phi}_{-k}: = \frac{c^2}{2} + \sum_{k>0} k p_k \frac{\partial}{\partial p_k}$ (degree operator)

$\left[L, \overline{\frac{u^2}{2}}\right] = 0 \iff L : \Lambda_k \to \Lambda_k, \quad \Lambda = \bigoplus_{k\geq 0} \Lambda_k$ (eigenspaces of $\sum_{k>0} k p_k \frac{\partial}{\partial p_k}$)

# Quantization problem for integrable hierarchies



**Fact:** $\left[\overline{g_k^{KdV}}, \overline{g_\ell^{KdV}}\right] = O(\hbar) \neq 0$ in general.

<u>Problem</u>: find $g_k^{qKdV} \in \mathbb{C}[u, \varepsilon, \hbar]$ s.t.

$$g_k^{qKdV}\Big|_{\hbar=0} = g_k^{KdV} \quad \& \quad \left[\overline{g_k^{qKdV}}, \overline{g_\ell^{qKdV}}\right] = 0$$

<u>Theorem</u> (Eliashberg, 2000 – byproduct of Symplectic Field Theory)

For $\varepsilon = 0$, a solution is given explicitly by

$$\sum_{k \geq -2} z^{k+2} g_k^{qKdV}(\varepsilon=0) = \frac{1}{S(\hbar^{1/2} z)} \exp\left(z S(i\hbar^{1/2} z \partial_x) u_0\right), \quad S(\xi) = \frac{\sinh(\xi/2)}{\xi/2}$$

# Dispersionless spectrum

Theorem (Dubrovin, 2014): We have

$$\overline{g_k^{qKdV}}(\varepsilon=0)\, S_\lambda\left(P/\hbar^{1/2}\right) = \left[\sum_{j=0} \frac{\hbar^{j/2}\, c^{k+2-j}}{(k+2-j)!}\, Q_j(\lambda)\right] \cdot S_\lambda\left(P/\hbar^{1/2}\right)$$

where:

$S_\lambda$ = Schur functions

$$Q_0(\lambda) = 1, \quad Q_j(\lambda) = \beta_j + \frac{1}{(j-1)!} \sum_{i=1}^{+\infty} \left[\left(\lambda_i - i + \frac{1}{2}\right)^j - \left(-i+\frac{1}{2}\right)^j\right], \quad \sum_{j=1}^{+\infty} \beta_j \xi^j = \frac{\xi/2}{\sinh(\xi/2)}$$

$\left(\lambda = (\lambda_1, \lambda_2, \dots)\right.$ a "partition": $\lambda_i \in \mathbb{Z}_{\geq 0}$, $\lambda_i \geq \lambda_{i+1}$, $\left.|\lambda| = \sum_{i=1}^{+\infty} \lambda_i < +\infty\right)$

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$Q_\lambda \longrightarrow$ "shifted symmetric functions" $\left\{\begin{array}{l}\text{ASYMPTOTIC REP. THEORY} \\ \text{ENUMÉRATIVE GEOMETRY} \\ \text{(H/GW THEORY,} \\ \text{SIEGEL-VEECH CONSTANTS)}\end{array}\right.$

# Quantum KdV

As a special case of a much more general construction by Buryak and Rossi, exploiting the geometry of "double ramification cycles" in the moduli spaces of curves, we have

**Theorem** (Buryak-Rossi, 2015): the quantization problem for the full dispersive KdV hierarchy is solved by

$$g^{qKdV}_{-2} = 1, \quad \frac{\partial g^{qKdV}_k}{\partial u_0} = g^{qKdV}_{k-1}, \quad \partial_x (D-1)\, g^{qKdV}_k = (R_1 + R_2)\, g^{qKdV}_{k-1}$$

$$R_1 g := \sum_{l=0}^{+\infty} \frac{\partial g}{\partial u_l} \partial_x^{l+1}\left(\frac{u_0^2}{2} + \varepsilon u_2\right) - \frac{\hbar}{2} \sum_{l,m=0}^{+\infty} \frac{\partial^2 g}{\partial u_l \partial u_m} \frac{(l+1)!\,(m+1)!}{(l+m+1)!} u_{l+m+3}, \quad D = \varepsilon\partial_\varepsilon + 2\hbar\partial_\hbar + \sum_{i\geq 0} u_i \partial_{u_i}$$

$$R_2 g := -\frac{\hbar}{2} \sum_{l,m,i=0}^{+\infty} \frac{\partial^2 g}{\partial u_l \partial u_m} \frac{B_{2i+2}}{2i+2} \left[ (-1)^{l+i} \binom{m+1}{2i-l} + (-1)^{m+i} \binom{l+1}{2i-m} \right] u_{l+m-2i+1}$$

# Modular forms

Holomorphic functions $\phi : \mathbb{H} = \{\operatorname{Im} z > 0\} \to \mathbb{C}$ such that

$$\phi\left(\frac{az+b}{cz+d}\right) = (cz+d)^k \phi(z), \quad \forall z \in \mathbb{H} \quad (k = \text{"weight"})$$

$\forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$ (+ growth conditions $\begin{cases} \phi(z) = \mathcal{O}(1) \text{ at } \operatorname{Im} z \to +\infty \\ \phi(z) = \mathcal{O}\left(\frac{1}{(\operatorname{Im} z)^k}\right) \text{ at } \operatorname{Im} z \to 0 \end{cases}$)

$$\begin{cases} \phi(z+1) = \phi(z) \\ \phi(-1/z) = z^k \phi(k) \end{cases} \implies \phi(z) = \sum_{n \geq 0} a_n q^n, \quad q = \exp(2\pi i z)$$

$\hookrightarrow$ interesting in various domains of Mathematics

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Fact: space of modular forms $= M = \bigoplus_{k \geq 0} M_k = \mathbb{C}[\underset{\text{wt}=4}{G_4}, \underset{\text{wt}=6}{G_6}]$

where $G_k := -\frac{B_k}{2k} + \sum_{n=1}^{+\infty} q^n \sum_{d|n} d^{k-1}$ ("Eisenstein series")

# Quasimodular forms

It is natural to add $G_2$ :

$$\widetilde{M} = \bigoplus_{k\geq 0} \widetilde{M}_k = \mathbb{C}[\underset{\substack{\uparrow \\ wt=2}}{G_2}, G_4, G_6]$$

$G_2$ is not modular ("quasimodular")

$$G_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 G_2(z) + \frac{ic(cz+d)}{4\pi}, \quad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

$\widetilde{M}$ carries an $sl_2$-action ; $W$ = weight, $D = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{dz}$

$\partial$ derivation on $\widetilde{M}$ defined by $\partial G_2 = -\frac{1}{2}$, $\partial G_4 = \partial G_6 = 0$ :

$$[W, D] = 2D, \quad [W, \partial] = -2\partial, \quad [\partial, D] = W$$

# Functions of partitions & quasimodularity (I)



Partitions of integers are related to modular forms.

<u>Example</u>: $\sum_{n\geq 0} q^n \,\#\{\text{partitions } \lambda \text{ s.t. } |\lambda| = n\} = \sum_{n\geq 0} q^n \dim \Lambda_n$

$$= \prod_{k\geq 1} (1-q^k)^{-1} =: q^{1/24}/\eta(q) \quad (\text{"Dedekind eta"})$$

and $\eta(q)^{24} \in M_{12}$.

# Functions of partitions & quasimodularity (II)



More recently (DIJKGRAAF, BLOCH-OKOUNKOV, KANEKO-ZAGIER):

$$f : \mathcal{P} \to \mathbb{C} \implies \langle f \rangle_q = \frac{\sum_{n\geq 0} q^n \sum_{|\lambda|=n} f(\lambda)}{\sum_{n\geq 0} q^n \#\{\lambda \in \mathcal{P} : |\lambda| = n\}}$$

($\mathcal{P}$: set of all partitions)

<u>Theorem</u> (Bloch-Okounkov, 2000).

Assign $\deg Q_k = k$.

If $f$ is a polynomial in $Q_0, Q_1, Q_2, \dots$ of homogeneous degree $k \Rightarrow \langle f \rangle_q \in \tilde{M}_k$.

# Functions of partitions & quasimodularity (III)



Even more recently (ZAGIER, VAN ITTERSUM):

$$S_k(\lambda) := -\frac{B_k}{2k} + \sum_{i=1}^{+\infty} \lambda_i^{k-1} \qquad k \geq 2, \text{ even}$$

("SYMMETRIC FUNCTIONS OF PARTITIONS")

Theorem (van Ittersum, 2020):

Assign $\deg S_k = k$.

If $f$ is a polynomial in $S_2, S_4, S_6, \dots$ of homogeneous degree $k$ $\Rightarrow$ $\langle f \rangle_q \in \tilde{M}_k$.

# Back to qKdV

**Natural problem**: describe eigenvalues of qKdV Hamiltonians

**A coarser question**: $\langle \text{eigenvalues}(\varepsilon=0)\rangle_q$ is quasimodular of homogeneous weight by the Theorems of Dubrovin and Bloch-Okounkov.

$$\boxed{\text{Does quasimodularity survive the } \varepsilon\text{-deformation?}}$$

A first positive indication: in the $\varepsilon \to \infty$ limit the eigenvalues behave as $\dfrac{\varepsilon^K \, S_{2K+2}(\lambda)}{(-4)^K \, (2K+1)!!}$

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$$\varepsilon = 0 : Q_K(\lambda) \xleftrightarrow[\text{qKdV}]{} \varepsilon = \infty : S_K(\lambda)$$

# Quasimodularity of differential polynomials

For $g \in \mathbb{C}[u] \Rightarrow \{\bar{g}\}_q := \dfrac{\sum_{n\geq 0} q^n \operatorname{tr}_{\Lambda_n} \bar{g}}{\sum_{n\geq 0} q^n \dim \Lambda_n}$

<u>Theorem</u> (van Ittersum - R, 2022): Let $\mathcal{B} : \mathbb{C}[u] \to \mathbb{C}[u]$

$$\mathcal{B} := \exp\left( \frac{\hbar}{2} \sum_{i,j=0}^{+\infty} (-1)^{\frac{i-j}{2}} \frac{B_{i+j+2}}{i+j+2} \frac{\partial}{\partial u_i} \frac{\partial}{\partial u_j} \right)$$

If $\mathcal{B} g$ is homogeneous w.r.t. $\deg u_k = k+1$, $\deg \hbar = 0$,

then: $\{\bar{g}\}_q \in \left(\widetilde{M}[c,\hbar]\right)_k$ $(\deg c = +1, \deg \hbar = 0)$.

(Proof largely based on previous work by van Ittersum)

# Application to qKdV (I)



| $k$ | $g_k^{qKdV}$ vs $\mathcal{B}\, g_k^{qKdV}$ |
| --- | --- |
| $-2$ | $1$ |
| $-1$ | $u_0$ |
| $0$ | $\frac{u_0^2}{2} - \frac{\hbar}{2} + \varepsilon u_2$ |
| $1$ | $\frac{u_0^3}{6} - \frac{\hbar}{24} u_0 - \frac{\hbar}{24} u_2 + \varepsilon\left(u_0 u_2 - \frac{\hbar}{120}\right) + \varepsilon^2 \frac{u_4}{2}$ |
| $2$ | $\frac{u_0^4}{24} - \frac{\hbar}{24} u_0 u_2 - \frac{\hbar}{48} u_0^2 + \frac{7\hbar}{5760} + \varepsilon\left(\frac{1}{2} u_0^2 u_2 - \frac{\hbar}{30} u_4 - \frac{\hbar}{24} u_2 - \frac{\hbar}{120} u_0\right)$ $+ \varepsilon^2\left(\frac{1}{2} u_0 u_4 + \frac{7}{10} u_2^2 - \frac{\hbar}{210}\right) + \varepsilon^3 \frac{u_6}{6}$ |

# Application to qKdV (II)

We can answer our question!

**Theorem**: $\left\{ \overline{g_k^{qKdV}(\varepsilon)} \right\} \in \left(\widetilde{M}[c, \varepsilon, \hbar]\right)_{k+2}$

$(\deg c = +1, \deg \varepsilon = -1, \deg \hbar = 0)$

**Proof**: use criterion in the last slide.

# Outlook

→ Quasimodularity of homogeneous weight constrains differential polynomials. Quantum KdV (& ILW) satisfy the constraint. Simplification of hamiltonian densities.

→ Quantum KdV interpolates between shifted symmetric and symmetric functions of partitions, preserving quasimodularity.

→ The result generalizes to the ILW hierarchy.

# Future directions

→ Conjecture: eigenvalues of qKdV are Taylor series in $\varepsilon$ with shifted symmetric coefficients (i.e. coefficients are polynomial in $Q_0, Q_1, Q_2, \dots$)

→ Compare with BAE found by Bonelli-Sciarappa-Tanzini-Vasko

→ Mirror symmetry? B-model?

→ What for other Cohomological Field Theories of rank 1? Higher rank?

Thank you!