Perturbative connection formulas for Heun equations

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- We are going to consider linear ODEs of the form

$$\psi''(z) = V(z)\,\psi(z) \qquad \text{w/ rational } V(z)\,.$$

- Simplest model cases correspond to ODEs for classical special functions (hypergeometric, Bessel, Airy, ...).
- Next-to-simplest cases correspond to Heun's equation and its degenerations (Mathieu equation, cubic and quartic quantum oscillator, ...)
- Heun accessory parameters were conjecturally related to quasiclassical limit of Virasoro conformal blocks [Zamolodchikov, '86]
- Recently, Heun connection problem was also conjecturally solved in terms of quasiclassical conformal blocks [Bonelli, Iossa, Lichtig, Tanzini, '21]

Goal: understand how the perturbative expansions following from the **Trieste formula** can be computed without CFT

Outline

1. Heun equations
2. CFT heuristics and Trieste formula
3. Darboux-Polya method and Schäfke-Schmidt formula
4. Perturbative solution of the (reduced confluent) Heun connection problem

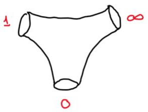
Hypergeometric equation

Consider $\psi''(z) = V(z)\,\psi(z)$, with

$$V(z) = \frac{\theta_0^2 - \frac{1}{4}}{z^2} + \frac{\theta_1^2 - \frac{1}{4}}{(z-1)^2} + \frac{\theta_\infty^2 - \theta_0^2 - \theta_1^2 + \frac{1}{4}}{z(z-1)}$$

Three regular singular points:

$$V(z) \sim \begin{cases} \frac{\theta_0^2 - \frac{1}{4}}{z^2} & \text{as } z \to 0, \\ \frac{\theta_1^2 - \frac{1}{4}}{z^2} & \text{as } z \to 1, \\ \frac{\theta_\infty^2 - \frac{1}{4}}{z^2} & \text{as } z \to \infty. \end{cases}$$



- local coordinate near ∞ is $\xi = \frac{1}{z}$
- three 2nd order poles of the quadratic differential $V(z)\,dz^2$ on $\mathbb{CP}^1$
- two 2nd order poles at $0, \infty$ correspond to $V(z) = \frac{\theta^2 - \frac{1}{4}}{z^2}$ (Euler's equation)

Frobenius solutions provide eigenbases of the operator of analytic continuation around singular points $z = 0, 1, \infty$. Their asymptotics is determined by the exponents $\theta_{0,1,\infty}$ of local monodromy, e.g.

$$\psi_{0,\pm}(z) = z^{\frac{1}{2}\mp\theta_0}(1-z)^{\frac{1}{2}-\theta_1}\,{}_2F_1\left[\begin{array}{c}\frac{1}{2}\mp\theta_0-\theta_1-\sigma_\infty, \frac{1}{2}\mp\theta_0-\theta_1+\sigma_\infty \\ 1\mp 2\theta_0\end{array};z\right]$$

$$= z^{\frac{1}{2}\mp\theta_0}\left[1+O(z)\right] \qquad \text{as } z\to 0,$$

$$\psi_{1,\pm}(z) = (1-z)^{\frac{1}{2}\mp\theta_1} z^{\frac{1}{2}-\theta_0}\,{}_2F_1\left[\begin{array}{c}\frac{1}{2}-\theta_0\mp\theta_1-\sigma_\infty, \frac{1}{2}-\theta_0\mp\theta_1+\sigma_\infty \\ 1\mp 2\theta_1\end{array};1-z\right]$$

$$= (1-z)^{\frac{1}{2}\mp\theta_1}\left[1+O(1-z)\right] \text{ as } z\to 1.$$

The exponents are encoded into the Riemann scheme

0	1	∞
$\frac{1}{2}-\theta_0$	$\frac{1}{2}-\theta_1$	$\frac{1}{2}-\theta_\infty$
$\frac{1}{2}+\theta_0$	$\frac{1}{2}+\theta_1$	$\frac{1}{2}+\theta_\infty$

(normal form)

0	1	∞
0	0	a
$1-c$	$c-a-b$	b

(canonical form)

Eigenbases of Frobenius solutions are related by

$$\begin{pmatrix} \psi_{0,+}(z) \\ \psi_{0,-}(z) \end{pmatrix} = \mathsf{Q} \begin{pmatrix} \psi_{1,+}(z) \\ \psi_{1,-}(z) \end{pmatrix},$$

where the connection matrix is given by

$$\mathsf{Q} = \begin{pmatrix} \frac{\Gamma(1-2\theta_0)\Gamma(2\theta_1)}{\Gamma\left(\frac{1}{2}-\theta_0+\theta_1+\theta_\infty\right)\Gamma\left(\frac{1}{2}-\theta_0+\theta_1-\theta_\infty\right)} & \frac{\Gamma(1-2\theta_0)\Gamma(-2\theta_1)}{\Gamma\left(\frac{1}{2}-\theta_0-\theta_1+\theta_\infty\right)\Gamma\left(\frac{1}{2}-\theta_0-\theta_1-\theta_\infty\right)} \\ \frac{\Gamma(1+2\theta_0)\Gamma(2\theta_1)}{\Gamma\left(\frac{1}{2}+\theta_0+\theta_1+\theta_\infty\right)\Gamma\left(\frac{1}{2}+\theta_0+\theta_1-\theta_\infty\right)} & \frac{\Gamma(1+2\theta_0)\Gamma(-2\theta_1)}{\Gamma\left(\frac{1}{2}+\theta_0-\theta_1+\theta_\infty\right)\Gamma\left(\frac{1}{2}+\theta_0-\theta_1-\theta_\infty\right)} \end{pmatrix}$$

Remark: it suffices to compute one of the 4 coefficients of Q, the others can be obtained it by symmetries — sign flips of θ_0, θ_1. (This holds in a much greater generality!)

Three problems:

1. Reconstruction of the equation with prescribed monodromy

   trivial in the hypergeometric case, parameters $\theta_{0,1,\infty}$ are directly related to the local monodromy exponents

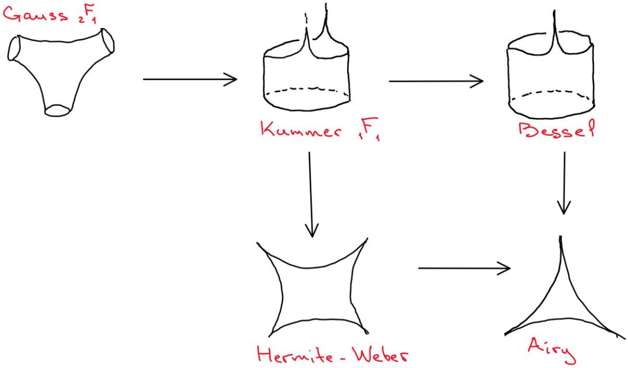
2. Constructing explicit bases of solutions

   series/integral representations for ${}_2F_1$'s

3. Connection problem

   solved in terms of gamma functions

Hypergeometric degeneration scheme



- $\sharp$(cusps) $+ 2 =$ order of the pole of $V(z)\,dz^2$
- colliding two holes increases $\sharp$(cusps) by 2

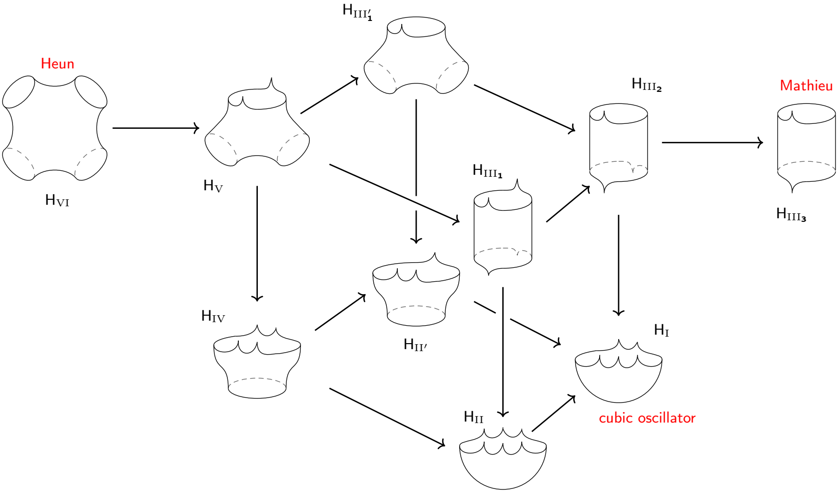
Heun equation

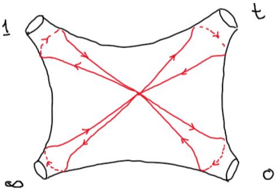
Potential:

$$V(z) = \frac{\theta_0^2 - \frac{1}{4}}{z^2} + \frac{\theta_1^2 - \frac{1}{4}}{(z-1)^2} + \frac{\theta_t^2 - \frac{1}{4}}{(z-t)^2} + \frac{\theta_\infty^2 - \theta_0^2 - \theta_1^2 - \theta_t^2 + \frac{1}{2}}{z(z-1)} + \frac{(1-t)\,\mathcal{E}}{z(z-1)(z-t)}$$

- 4 regular singular points 0, 1, ∞, $t \Rightarrow$ 4 exponents θ_k
- 1 accessory parameter $\mathcal{E}$: not fixed by local monodromy

Degeneration scheme for Heun equations



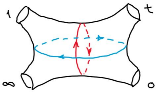


Space of monodromy data:

$$\mathcal{M} = \left\{ M_{0,1,\infty,t} \in \mathrm{SL}\,(2,\mathbb{C}) : M_\infty M_1 M_t M_0 = \mathbf{1}, \mathrm{Tr}\, M_k = -2\cos 2\pi\theta_k \right\} / \sim$$

- $\dim \mathcal{M} = 2$; $(\mathcal{E}, t)$ can be seen as a pair of local coordinates on $\mathcal{M}$
- another possibility is to use trace functions such as

$$\mathrm{Tr}\, M_0 M_t = 2\cos 2\pi\sigma, \quad \mathrm{Tr}\, M_1 M_t = 2\cos 2\pi\sigma'$$



direct problem

equation data: $(\mathcal{E}, t)$ $\longrightarrow$ monodromy data: σ's

equation data: $(\mathcal{E}, t)$ $\longleftarrow$ monodromy data: σ's

inverse problem

- any choice of a coordinate σ on $\mathcal{M}$ makes $\mathcal{E}$ a function of t depending on σ

"Intermediate" problem:

find $\mathcal{E}(t \,|\, \sigma) =$ reconstruct Heun equation from prescribed monodromy (σ) and singularity position (t)

Solved in terms of quasiclassical conformal blocks by Zamolodchikov conjecture

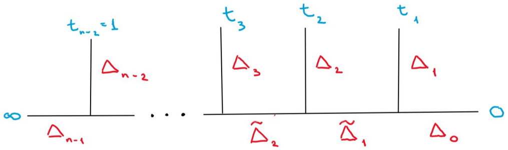
2. Conformal block heuristics

Virasoro conformal blocks

Fix $n \geq 4$ distinct points $t_0, \ldots, t_{n-1}$ on $\mathbb{CP}^1$, using projective invariance to choose

$$t_0 = 0, \qquad t_{n-2} = 1, \qquad t_{n-1} = \infty$$

and assuming that $0 < |t_1| < |t_2| < \ldots < |t_{n-3}| < 1$. Conformal block is a multivariate series assigned to a trivalent graph with n external edges, such as



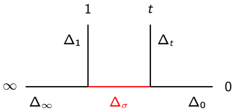
$$\mathcal{Z}\left(\mathbf{t}, \boldsymbol{\Delta}, \tilde{\boldsymbol{\Delta}}\right) = \prod_{\ell=1}^{n-3} t_\ell^{\tilde{\Delta}_\ell - \tilde{\Delta}_{\ell-1} - \Delta_\ell} \sum_{\mathbf{k} \in \mathbb{N}^{n-3}} \mathcal{Z}_{\mathbf{k}}\left(\boldsymbol{\Delta}, \tilde{\boldsymbol{\Delta}}\right) \left(\frac{t_1}{t_2}\right)^{k_1} \left(\frac{t_2}{t_3}\right)^{k_2} \cdots \left(\frac{t_{n-3}}{t_{n-2}}\right)^{k_{n-3}}$$

Remarks:

- to every edge is assigned a weight $\Delta \in \mathbb{C}$
- coefs $\mathcal{Z}_{\mathbf{k}}\left(\boldsymbol{\Delta}, \tilde{\boldsymbol{\Delta}}\right)$ are fixed by the Virasoro commutation relations $\Longrightarrow$ rational functions of weights and central charge c.
- thanks to the AGT relation, there is an explicit combinatorial representation of $\mathcal{Z}$ in terms of a sum over tuples of partitions.
- the series is convergent and analytic properties of $\mathcal{Z}$ in each variable can be described using elementary braiding and fusion transformations.

Simplest nontrivial case: 4-point conformal block

$\mathcal{Z}(t) =$



$= t^{\Delta_\sigma - \Delta_t - \Delta_0} \sum_{k=0}^{\infty} \mathcal{Z}_k t^k$

- depends on 5 conformal weights Δ_k and the central charge c
- a generalization of the Gauss ${}_2F_1$ with 3 more parameters

Quasiclassical limit

Liouville parameterization:

$$c = 1 + 6\left(b + b^{-1}\right)^2, \qquad \Delta = \alpha\left(b + b^{-1} - \alpha\right).$$

We trade the central charge c and conformal weights Δ's for b and α's and consider the scaling limit

$$\alpha \to \infty, \qquad b \to 0, \qquad \alpha b \to \tfrac{1}{2} + \theta.$$

Zamolodchikov conjecture

1. Conformal blocks have WKB type asymptotics

$$\mathcal{Z}\left(\mathbf{t}; \{\alpha_k\}\right) \sim \exp b^{-2} \mathcal{F}\left(\mathbf{t}; \{\theta_k\}\right)$$

The series $\mathcal{F}\left(\mathbf{t}; \{\theta_k\}\right)$ is called quasiclassical conformal block.

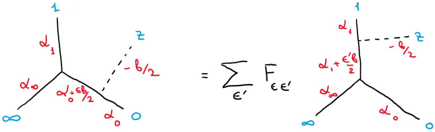
2. The 4-point spherical quasiclassical conformal block is related to Heun accessory parameter function $\mathcal{E}\left(t \,|\, \sigma\right)$ by

$$\mathcal{E} = t \frac{\partial \mathcal{F}}{\partial t},$$

where external θ_k's are Heun monodromy exponents and σ is similarly related to rescaled intermediate momentum.

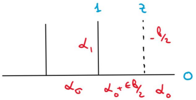
Degenerate fields

- Special fusion relations:



$$, \qquad \epsilon, \epsilon' = \pm.$$

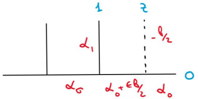
- BPZ (Belavin-Polyakov-Zamolodhikov) constraints:

$\mathcal{D}_{\mathrm{BPZ}}$  $= 0$

  - a linear PDE in position of fields
  - 2nd order in z, 1st order in positions of other fields
  - 3+1 points: hypergeometric equation in z

“Explanations” of Zamolodchikov conjecture and Trieste formula

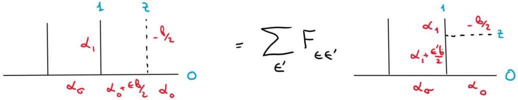
Plugging the WKB ansatz for the asymptotics of $4+1$ conformal blocks



$$\sim \psi(z) \exp b^{-2}\mathcal{F}(t)$$

into the BPZ constraint, the corresponding PDE becomes Heun’s ODE for the amplitude $\psi(z)$, with the accessory parameter given by $t\partial_t\mathcal{F}$.

NB: The connection formulas for ψ follow from the exact fusion transformations



They involve the limit of the hypergeometric connection coefficients $F_{\epsilon\epsilon'}$ ($\Rightarrow$ gamma functions) and derivatives of quasiclassical conformal blocks $\partial_{\theta_0}\mathcal{F}$, $\partial_{\theta_1}\mathcal{F}$ with respect to external momenta, just as in [Bonelli, Iossa, Lichtig, Tanzini, ’21].

3. Darboux-Polya method and Schäfke-Schmidt formula

Example. Consider the Taylor expansion of $u(z) = (1-z)^{-\theta}$ around $z = 0$:

$$u(z) = \sum_{k=0}^{\infty} u_k z^k, \qquad u_k = \frac{(\theta)_k}{k!} = \frac{\Gamma(k+\theta)}{\Gamma(\theta)\Gamma(k+1)}$$

The coefficients have the large k behavior

$$a_k = \frac{k^{\theta-1}}{\Gamma(\theta)}\left[1 + O\left(k^{-1}\right)\right] \qquad \text{as } k \to \infty.$$

It depends on the exponent θ which hints that such asymptotics can capture the critical behavior of $u(z)$ at the branch point $z = 1$.

Darboux theorem (1878)

Let $u(z)$ be analytic in a neighborhood of $z = 0$. Suppose it has exactly one singularity $z = 1$ inside a disk $|z| = R > 1$. If $u(z)$ can be written in the form

$$u(z) = v(z) + (1 - z)^{-\theta} w(z), \qquad \theta \notin \mathbb{Z}$$

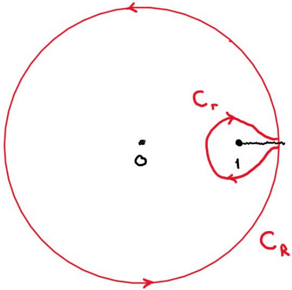
with $v(z)$, $w(z)$ analytic in a neighborhood of $z = 1$, then the coefficients of the Taylor expansion $u(z) = \sum_{k=0}^{\infty} u_k z^k$ at $z = 0$ have the asymptotics

$$u_k = \frac{w(1)}{\Gamma(\theta)} k^{\theta-1} \left[1 + O\left(k^{-1}\right)\right] \qquad \text{as } k \to \infty.$$

Proof idea:

$$u_k = \frac{1}{2\pi i} \oint_{C_R \cup C_r} z^{-k-1} u(z)\, dz$$

- the contribution of C_R is at most $O\left(R^{-k}\right)$
- plug the expression of $u(z)$ into $\oint_{C_r}$
- $\oint_{C_r} z^{-k-1} v(z)\, dz = 0$
- it suffices to estimate the asymptotics of $\oint_{C_r} z^{-k-1} (1 - z)^{-\theta} w(z)\, dz$



Application to connection problem

Consider a linear ODE $\psi''(z) = V(z)\,\psi(z)$ with potential

$$V(z) = \frac{\theta_0^2 - \frac{1}{4}}{z^2} + \frac{\theta_1^2 - \frac{1}{4}}{(z-1)^2} + \frac{\tilde{V}(z)}{z(z-1)}$$

where $\tilde{V}(z)$ is holomorphic inside $|z| = R > 1$. Introduce normalized Frobenius solutions

$$\hat{\psi}_{0,\pm}(z) = \frac{z^{\frac{1}{2}\mp\theta_0}}{\Gamma(1\mp 2\theta_0)} \sum_{k=0}^{\infty} \tau_k^{0,\pm} z^k,$$

$$\hat{\psi}_{1,\pm}(z) = \frac{(1-z)^{\frac{1}{2}\mp\theta_1}}{\Gamma(1\mp 2\theta_1)} \sum_{k=0}^{\infty} \tau_k^{1,\pm} (1-z)^k,$$

with $\tau_0^{0,\pm} = \tau_0^{1,\pm} = 1$, and the Wronkians

$$\mathrm{Q}_{\epsilon\epsilon'} = \mathrm{Q}\left(\epsilon\theta_0, \epsilon'\theta_1\right) = -W\left(\hat{\psi}_{0,\epsilon}, \hat{\psi}_{1,\epsilon'}\right), \qquad \epsilon, \epsilon' = \pm.$$

The connection matrix relating the 2 bases is given by

$$\begin{pmatrix} \hat{\psi}_{0,+}(z) \\ \hat{\psi}_{0,-}(z) \end{pmatrix} = \frac{\pi}{\sin 2\pi\theta_1} \begin{pmatrix} \mathrm{Q}_{+-} & -\mathrm{Q}_{++} \\ \mathrm{Q}_{--} & -\mathrm{Q}_{-+} \end{pmatrix} \begin{pmatrix} \hat{\psi}_{1,+}(z) \\ \hat{\psi}_{1,-}(z) \end{pmatrix}$$

Theorem [Schäfke, Schmidt, '80]

Write the solution $\hat{\psi}_{+,0}$ as

$$\hat{\psi}_{+,0}(z) = \frac{z^{\frac{1}{2}-\theta_0}(1-z)^{\frac{1}{2}-\theta_1}}{\Gamma(1-2\theta_0)}u(z), \quad \text{with } u(z) = 1 + \sum_{k=1}^{\infty} u_k z^k.$$

Then

$$\boxed{Q(\theta_0,\theta_1) = \lim_{k\to\infty}\frac{\Gamma(k+1)}{\Gamma(k-2\theta_1)}u_k}$$

Proof. Direct corollary of the Darboux theorem, with $v(z)$ and $w(z)$ coming from the Frobenius solutions at $z=1$.

4. Application to reduced confluent Heun equation

Schäfke-Schmidt theorem applies to all Heun equations with two Fuchsian singularities: the usual, confluent, and reduced confluent Heun. In the last case,

$$V(z) = \frac{\theta_0^2 - \frac{1}{4}}{z^2} + \frac{\theta_1^2 - \frac{1}{4}}{(z-1)^2} + \frac{\sigma_\infty^2 - \theta_0^2 - \theta_1^2 + \frac{1}{4} - tz}{z(z-1)}$$

- σ_∞ is the accessory parameter
- we look for perturbative expansion of the Q-function (connection matrix) in t
- for $t = 0$:
  - the potential reduces to hypergeometric one with exponents

0	1	∞
$\frac{1}{2} \pm \theta_0$	$\frac{1}{2} \pm \theta_1$	$\frac{1}{2} \pm \sigma_\infty$

  - the coefficients u_k are given by

$$u_k^{(t=0)} = \frac{\left(\frac{1}{2} - \theta_0 - \theta_1 + \sigma_\infty\right)_k \left(\frac{1}{2} - \theta_0 - \theta_1 - \sigma_\infty\right)_k}{k!\,(1 - 2\theta_0)_k}$$

  - the Q-function reads

$$\mathrm{Q}^{(t=0)}(\theta_0, \theta_1) = \left[\Gamma\left(\tfrac{1}{2} - \theta_0 - \theta_1 + \sigma_\infty\right)\Gamma\left(\tfrac{1}{2} - \theta_0 - \theta_1 - \sigma_\infty\right)\right]^{-1}$$

Introducing rescaled coefficients $a_k = u_k / u_k^{(t=0)}$, the Schäfke-Schmidt theorem can be reformulated as follows.

Proposition. The Q-function of the reduced confluent Heun equation is given by

$$\mathrm{Q}\left(\theta_0, \theta_1, \sigma_\infty, t\right) = \frac{a_\infty\left(\theta_0, \theta_1, \sigma_\infty, t\right)}{\Gamma\left(\frac{1}{2} - \theta_0 - \theta_1 + \sigma_\infty\right)\Gamma\left(\frac{1}{2} - \theta_0 - \theta_1 - \sigma_\infty\right)},$$

where $\{a_k\}$ satisfy the 3-term recurrence relation

$$a_{k+1} - a_k = -t\beta_k a_{k-1}$$

subject to initial conditions $a_{-1} = 0$, $a_0 = 1$, with

$$\beta_k = -\frac{k\left(k - 2\theta_0\right)}{\left(\left(k - \frac{1}{2} - \theta_0 - \theta_1\right)^2 - \sigma_\infty^2\right)\left(\left(k + \frac{1}{2} - \theta_0 - \theta_1\right)^2 - \sigma_\infty^2\right)}$$

Formal solution:

$$a_\infty = \det\begin{pmatrix} 1 & -1 & & & \\ -t\beta_1 & 1 & -1 & & \\ & -t\beta_2 & 1 & -1 & \\ & & -t\beta_3 & 1 & \ddots \\ & & & \ddots & \ddots \end{pmatrix} = 1 - t\sum_{k=1}^{\infty}\beta_k + t^2\sum_{k' \geq k+2}^{\infty}\beta_k\beta_{k'} + \ldots$$

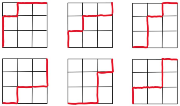
Exponentiation gives a perturbative series involving only 1-fold sums:

$$-\ln a_\infty = \sum_{n=1}^{\infty} \frac{\mathrm{Tr}\, A^{2n}}{2n} t^n, \qquad A = \begin{pmatrix} 0 & 1 & & & \\ \beta_1 & 0 & 1 & & \\ & \beta_2 & 0 & 1 & \\ & & \beta_3 & 0 & \cdot \\ & & & \cdot & \cdot \end{pmatrix}$$

We have, for example,

$$\mathrm{Tr}\, A^2 = \sum_{k=1}^{\infty} 2\beta_k, \qquad \mathrm{Tr}\, A^4 = \sum_{k=1}^{\infty} \left(4\beta_k\beta_{k+1} + 2\beta_k^2\right),$$

$$\mathrm{Tr}\, A^6 = \sum_{k=1}^{\infty} \left(6\beta_k\beta_{k+1}\beta_{k+2} + 6\beta_k^2\beta_{k+1} + 6\beta_k\beta_{k+1}^2 + 2\beta_k^3\right), \quad \ldots$$



Walks of type $(1, 2)$

In general, $\mathrm{Tr}\, A^{2n} = \sum_{k=1}^{\infty} \sum_{\lambda \vdash n}^{2^{n-1}} \mathcal{N}_\lambda \cdot \beta_k^{\lambda_1} \beta_{k+1}^{\lambda_2} \ldots \beta_{k+\ell}^{\lambda_\ell}$, where λ runs over all compositions (ordered partitions) of n and $\mathcal{N}_\lambda$ are positive integers counting staircase walks of a certain type λ.

NB: Since β_k is rational in k, all sums can be computed in terms of expressions rational in $\theta_{0,1}$, σ_∞ and polygamma functions $\Longrightarrow$ we recover the predictions of Trieste formula!

Proposition. We have

$$\ln a_\infty = \sum_{k=1}^{\infty} \ln \left(1 - \cfrac{t\beta_k}{1 - \cfrac{t\beta_{k+1}}{1 - \cfrac{t\beta_{k+2}}{1 - \dots}}} \right)$$

Proof. The determinant

$$D_k = \det \begin{pmatrix} 1 & -1 & & & \\ -t\beta_k & 1 & -1 & & \\ & -t\beta_{k+1} & 1 & -1 & \\ & & -t\beta_{k+2} & 1 & \cdot \\ & & & \cdot & \cdot \end{pmatrix}$$

satisfies a linear 3-term recurrence relation $D_k - D_{k+1} = -t\beta_k D_{k+2}$. It can be transformed into a nonlinear 2-term Riccati equation for D_k/D_{k+1}, which is solved by the above infinite fraction. It remains to write

$$\ln a_\infty = \sum_{k=1}^{\infty} \ln \frac{D_k}{D_{k+1}}.$$

Remark. This also implies

$$\mathcal{N}_\lambda = \frac{2n}{\lambda_1} \prod_\ell \binom{\lambda_\ell + \lambda_{\ell+1} - 1}{\lambda_{\ell+1}}$$

Conclusions

- We have developed a systematic approach to perturbative solution of the connection problem for Heun equations between two Fuchsian singularities.
- It confirms Trieste formula expressing the connection coefficients in terms of quasiclassical Virasoro conformal blocks.
- It would be interesting to extend the method to irregular singularities and compare with CFT predictions of [Bonelli, Iossa, Lichtig, Tanzini, '21].
- A rigorous proof using extended symplectic structure of [Bertola, Korotkin, '19] ?

THANK YOU!