Integrable differential equations for the KPZ fixed point with narrow-wedge initial condition

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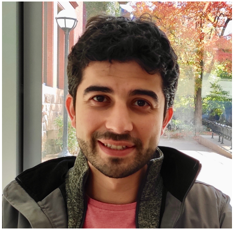
May 2022
PIICQ Workshop @ SISSA



Andrei Prokhorov   Guilherme Silva

1. Baik, Liu, and Silva, **On limiting one-point distributions of the periodic TASEP,** Annales de l'Institut Henri Poincaré 2021
2. Baik, Prokhorov, and Silva, **Integrable systems governing KPZ fixed points**, in preparation

Plan

1. KPZ fixed point and known integrable differential equations
2. Multi-time distributions
3. Results: Integrable DEs for cubic admissible functions
4. Discussions

Part 1. KPZ fixed point and known integrable differential equations

KPZ fixed point

The KPZ fixed point is a 2d random field

$$\mathcal{H}(\gamma, \tau) \qquad \text{for } (\gamma, \tau) \in \mathbb{R} \times \mathbb{R}_+$$

that is the conjectured universal limit of the height fluctuations for the KPZ universality class (random growth, directed polymers, interacting particle systems, ...). It was constructed by Matetski-Quastel-Remenik 2021

Consider three things:

- One-point distributions
- Equal-time, multi-position distributions
- Multi-time, multi-position distributions

Narrow-wedge initial condition

- Assume narrow-wedge initial condition
- $\epsilon^{-1}\mathcal{H}(\epsilon^2\gamma, \epsilon^3\tau) \stackrel{d}{=} \mathcal{H}(\gamma, \tau)$ for all $\epsilon > 0$
- One-point marginal

$$\mathcal{H}(0,1) \stackrel{d}{=} \mathsf{TW}$$

  $\beta = 2$ Tracy-Widom distribution.
- Equal-time process

$$\mathcal{H}(\gamma,1) + \gamma^2 \stackrel{d}{=} \mathcal{A}_2(\gamma)$$

  Airy$_2$ process by Prähofer & Spohn in 2002.
- Multi-time distributions were computed by Johansson & Rahman 2021, and Liu 2022 (formula later)
- Fredholm determinant formulas

Known integrable DEs

The one-point distribution $F(\tau, \gamma, h) = \mathbb{P}(\mathcal{H}(\gamma, \tau) \leq h)$ has 3 variables, but due to the invariance,

$$\mathbb{P}(\mathcal{H}(\gamma, \tau) \leq h) = \mathbb{P}(\mathcal{H}(0,1) \leq \xi) = F_{\mathrm{TW}}(\xi) \quad \text{with} \quad \xi = \frac{h}{\tau^{1/3}} + \frac{\gamma^2}{\gamma^{4/3}}$$

From the formula of the Tracy-Widom distribution,

$$\frac{\partial^2}{\partial \xi^2} \log \mathbb{P}(\mathcal{H}(0,1) \leq \xi) = -u(\xi)^2$$

where u solves the Painlevé II equation $u'' = \xi u + 2u^3$

The equal-time, multi-position distribution function

$$\mathbb{P}\left(\bigcap_{i=1}^{m}\{\mathcal{H}(\gamma_i, 1) + \gamma_i^2 \le h_i\}\right) = \mathbb{P}\left(\bigcap_{i=1}^{m}\{\mathcal{A}_2(\gamma_i) \le h_i\}\right)$$

depends on $2m$ variables, $\gamma_1, \cdots, \gamma_m$ (positions), $h_1, \cdots, h_m$ (heights).

- Tracy & Widom 2005 obtained a matrix ODE system with respect to $\partial = \partial_{h_1} + \cdots + \partial_{h_m}$ (formula later)
- Adler & van Moerbeke 2005 considered $m = 2$ case and obtained a PDE in 3 variables h_1, h_2, and $\gamma = \gamma_2$ (with $\gamma_1 = 0$.) (formula later)
- Wang 2009 extended the result of Adler-van Moerbeke to general m. Bertola & Cafasso 2012 RHP for Airy process

Consider the equal-time, multi-location distribution and include the time variable (i.e. time-scaled Airy process)

$$F(\tau, \gamma_1, \cdots, \gamma_m, h_1, \cdots, h_m) := \mathbb{P}\left(\bigcap_{i=1}^{m}\{\mathcal{H}(\gamma_i, \tau) \leq h_i\}\right)$$

is a function of $2m+1$ variables. Quastel & Remenik 2022 obtained the matrix Kadomtsev-Petviashvili (KP) equation. (formula later)

- When $m = 1$, it becomes a scalar equation in 3 variables τ, γ_1, h_1. A self-similar solution in the variable $\xi = \frac{h}{\tau^{1/3}} + \frac{\gamma^2}{\gamma^{4/3}}$ turns the scalar KP to the Painlevé II equation.
- For $m > 1$, it is not clear if the KP reduces to Tracy-Widom ODE system or Adler-van Moerbeke PDE if we scale out τ.
- Quastel-Remenik obtained KP for general initial conditions

This talk is about

extending the differential equations for the equal-time distributions,

1) Tracy-Widom ODE system
2) Adler-van Moerbeke PDE
3) KP equation of Qastel-Remenik

to multi-time cases.

Part 2. Multi-time distributions of the KPZ fixed point

Liu 2022 obtained

$$\mathbb{P}\left(\bigcap_{i=1}^{m}\{\mathcal{H}(\gamma_i,\tau_i)\le \mathsf{h}_i\}\right)=\oint\cdots\oint \det(1-\mathrm{K})\prod_{i=1}^{m-1}\frac{\mathrm{d}\zeta_i}{2\pi\mathrm{i}(1-\zeta_i)\zeta_i}$$

with an explicit operator K acting on a union of contours. The kernel of K is a bit complicated and the first result is that we can change it to a somewhat algebraically simpler formula.

Theorem

For every ζ, the $3m$-variable function $D(\tau,\gamma,h|\zeta)=\det(1-\mathrm{K})$ is (strongly) cubic admissible whose definition is given in the next slide with the parameters

$$t_i=-\frac{\tau_i}{3},\quad y_i=\gamma_i,\quad x_i=h_i$$

Cubic admissible functions

- Consider $3m$ real parameters "times", "positions", "heights"

$$t = (t_1, \cdots, t_m), \qquad y = (y_1, \cdots, y_m), \qquad x = (x_1, \cdots, x_m)$$

Define $(m+1) \times (m+1)$ matrix

$$\Delta_{t,y,x}(z) = \Delta(z) = \mathrm{diag}(e^{t_1 z^3 + y_1 z^2 + x_1 z}, \cdots, e^{t_m z^3 + y_m z^2 + x_m z}, 1)$$

Let $\mathcal{H} = L^2(\Omega)$ where Ω is a union of "nice" contours.

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- Definition We call $D : \mathcal{O} \subset \mathbb{R}^{3m} \to \mathbb{C}$ cubic admissible on $\mathcal{O}$ if

$$D(t, y, x) = \det(1 - K)_{\mathcal{H}}$$

where

$$K(u,v) = \frac{f(u)^T g(v)}{u - v} \quad \text{with} \quad \begin{cases} f(u) = \Delta(u)_{t,y,x} U(u) \\ g(v) = \Delta(v)^{-1}_{t,y,x} V(v) \end{cases}$$

for $u \neq v$ and $K(u,u) = 0$ for $u, v \in \Omega$. Here, $U(u)$ and $V(v)$ are $(m+1)$-dim column vectors that do not depend on t, y, x, and satisfy $U_i(u) V_i(u) = 0$ for all $i = 1, \cdots, m+1$.

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- For the KPZ fixed point, D is strongly cubic admissible in the sense that U and V are constants on each component of Ω.

$K(u,v) = \frac{f(u)^T g(v)}{u-v}$ is an IIKS integrable operator introduced by Its, Izergin, Korepin and Slavnov 1990. If $1-K$ is invertible,

$$Y(z) = I - \int_\Omega \frac{((1-K)^{-1}f)(u)g(u)^T}{u-z}\mathrm{d}u$$

solves the normalized RHP $Y_+ = Y_- J$ on Ω with

$$J(z) = I - 2\pi\mathrm{i} f(z)g(z)^T = \Delta(z)J_0(z)\Delta(z)^{-1}$$

where $J_0(z) = I - 2\pi\mathrm{i}U(z)V(z)^T$ does not depend on t, y, x

- For example, when $m=1$, $J(z) = \begin{bmatrix} a & be^{t_1 z^3 + y_1 z^2 + x_1 z} \\ ce^{-t_1 z^3 - y_1 z^2 - x_1 z} & d \end{bmatrix}$

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$W := Y\Delta$ satisfies $W_+ = W_- J_0$. Then, $\partial_t W_+ = (\partial_t W_-)J_0$. Thus, $(\partial_t W)W^{-1}$ is entire. By Liouville's theorem, we get a Lax equation

$$\partial_t W(z) = P(z)W(z)$$

for a polynomial $P(z)$ whose coefficients are given in terms of $Y_1, Y_2, \cdots$. Similarly, $\partial_x W(z) = Q(z)W(z)$. From $\partial_t\partial_x W = \partial_x\partial_t W$, we obtain a zero curvature equation

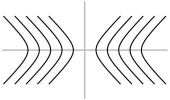
$$\partial_x P + PQ = \partial_t Q + QP$$

Features of the RHP

(i) $J_0(z)$ is a general complex matrix with no symmetry

(ii) $(m+1) \times (m+1)$

(iii) The contours are unions of multiple contours



If D is strongly cubic admissible, $J_0(z)$ is, furthermore, a constant on each component of $\Omega \Rightarrow$ additional Lax equation for ∂_z

Part 3. Results: 5 differential equations for cubic admissible functions

The variables $(t, y, x) \in \mathbb{R}^{3m}$. Write the following $(m+1) \times (m+1)$ complex matrix as a block form

$$Y_1(t, y, x) = \int_\Omega ((1-K)^{-1} f)(u) g(u)^T \mathrm{d}u = \begin{bmatrix} \mathsf{q} & \mathsf{p} \\ \mathsf{r} & \mathsf{s} \end{bmatrix}$$

where q is $m \times m$. Let

$$\partial_\mathsf{t} = \sum_{i=1}^m \partial_{\mathsf{t}_i}, \qquad \partial_\mathsf{y} = \sum_{i=1}^m \partial_{\mathsf{y}_i}, \qquad \partial_\mathsf{x} = \sum_{i=1}^m \partial_{\mathsf{x}_i}$$

Then,

$$\partial_\mathsf{x} \log\det(1-K) = -\mathrm{Tr}(\mathsf{q}) = \mathsf{s}$$

and $\partial_\mathsf{x}\mathsf{q} = -\mathsf{pr}$ and $\partial_\mathsf{x}\mathsf{s} = \mathsf{rp}$.

Note: p and r are $m \times 1$ and $1 \times m$ complex matrices

(a) (x and y) Coupled matrix nonlinear Schrödinger (NLS) with complex time $y \mapsto \mathrm{i}y$

$$\partial_\mathrm{y}\mathrm{p} = \partial_\mathrm{x}^2\mathrm{p} + 2\mathrm{prp}$$
$$\partial_\mathrm{y}\mathrm{r} = -\partial_\mathrm{x}^2\mathrm{r} - 2\mathrm{rpr}$$

(scalar NLS is $\mathrm{i}\phi_t = -\frac{1}{2}\phi_{xx} \pm |\phi|^2\phi$) Also appeared in Krajenbrink & le Doussal 2021 in their study of weak noise theory of the KPZ equation

(b) (x and t) Coupled matrix modified KdV (mKdV) equations

$$\partial_\mathrm{t}\mathrm{p} = \partial_\mathrm{x}^3\mathrm{p} + 3(\partial_\mathrm{x}\mathrm{p})\mathrm{rp} + 3\mathrm{pr}(\partial_\mathrm{x}\mathrm{p})$$
$$\partial_\mathrm{t}\mathrm{r} = \partial_\mathrm{x}^3\mathrm{r} + 3(\partial_\mathrm{x}\mathrm{r})\mathrm{pr} + 3\mathrm{rp}(\partial_\mathrm{x}\mathrm{r})$$

(scalar mKdV is $u_t + u_{xxx} - 6u^2u_x = 0$)

(c) $(x, y, \text{ and } t)$ The $m \times m$ matrix

$$\mathrm{u} := \mathrm{pr}$$

satisfies the matrix Kadomtsev-Petviashvili (KP) equation

$$-4\partial_{\mathrm{t}}\mathrm{u} + \partial_{\mathrm{x}}^3\mathrm{u} + 6\partial_{\mathrm{x}}(\mathrm{u}^2) - 3\partial_{\mathrm{y}}^2\mathrm{q} + 6[\mathrm{u}, \partial_{\mathrm{y}}\mathrm{q}] = 0$$

with $\partial_{\mathrm{x}}\mathrm{q} = -\mathrm{u}$.

Scalar KP also holds for $\mathrm{v} := \mathrm{rp}$ with $\partial_{\mathrm{x}}\mathrm{s} = \mathrm{v}$.

When $t_1 = \cdots = t_m$, this is the same matrix KP equation obtained by Quastel & Remenik for the equal-time KPZ fixed point.

(d) (matrix ODE system) Suppose that D is strongly cubic admissible. Define the differential

$$\mathrm{f}' := \sum_{i=1}^{m} \mathrm{t}_i \partial_{\mathrm{x}_i} \mathrm{f} = \frac{\mathrm{d}}{\mathrm{d}\xi} \mathrm{f}(\mathrm{x}_1 + \mathrm{t}_1 \xi, \cdots, \mathrm{x}_m + \mathrm{t}_m \xi)$$

Then,

$$3Y_1'' = 2[Y_1', \mathrm{y}] + [[Y_1, \mathrm{t}], 3Y_1' - 2[Y_1, \mathrm{y}] - \mathrm{x}]$$

where $\mathrm{t} = \mathrm{diag}(t_1, \cdots, t_m, 0)$ and so on.

For the equal-time case $\mathrm{t}_1 = \cdots = \mathrm{t}_m = -1/3$, it becomes

$$\begin{aligned} &\mathrm{q}' + \mathrm{pr} = 0, \\ &\mathrm{p}'' - 2\mathrm{yp}' + 2\mathrm{prp} + 2[\mathrm{y}, \mathrm{q}]\mathrm{p} - \mathrm{xp} = 0, \\ &\mathrm{r}'' + 2\mathrm{r}'\mathrm{y} + 2\mathrm{rpr} + 2\mathrm{r}[\mathrm{y}, \mathrm{q}] - \mathrm{rx} = 0. \end{aligned}$$

where $f' = \sum_{i=1}^{m} \partial_{x_i} f$. This is the same ODE system obtained by Tracy & Widom for the Airy_2 process

(e) (Adler-van Moerbeke PDE) When $m = 2$, a strongly cubic admissible function $D(\mathrm{t}, \mathrm{y}, \mathrm{x}) = \det(1 - \mathrm{K})$ depends on 6 parameters, $\mathrm{t}_1, \mathrm{t}_2, \mathrm{y}_1, \mathrm{y}_2, \mathrm{x}_1, \mathrm{x}_2$. Consider

$$t_1 = t_2 = -1/3, \quad y_1 = 0$$

Then, $\det(1 - \mathrm{K})$ depends on 3 parameters E, W, y where

$$\mathrm{x}_1 = \frac{E + W}{2}, \quad \mathrm{x}_2 = \frac{E - W}{2} - y_2^2, \quad \mathrm{y}_2 = y.$$

Adler-van Moerbeke 2005 showed that for the Airy_2 process, $M := \log\det(1 - K)$ satisfies

$$\left(y^2(\partial_E^2\partial_W - \partial_W^3) + W(\partial_E\partial_W^2 - \partial_E^3) + 2y\partial_E\partial_W\partial_y\right) M \\ + 8\partial_E\partial_W M\partial_E^3 M - 8\partial_E^2 M\partial_E^2\partial_W M = 0.$$

We could derive this PDE from the Lax equations of a 3×3 RHP with a help of symbolic computations using Maple.

Part 4. Discussions

- For the KPZ fixed point, Liu 2022

$$\mathbb{P}\left(\bigcap_{i=1}^{m}\{\mathcal{H}(\gamma_i,\tau_i)\le \mathrm{h}_i\}\right)=\oint\cdots\oint \det(1-\mathrm{K})\prod_{i=1}^{m-1}\frac{\mathrm{d}\zeta_i}{2\pi\mathrm{i}(1-\zeta_i)\zeta_i}$$

- When $\tau_1=\cdots=\tau_m$, the LHS is the multi-point distribution Airy$_2$ process for which Tracy-Widom, Adler-van Moerbeke, Quastel-Remenik obtained DEs. On the other hand, our result obtained DEs for $\det(1-\mathrm{K})$. They solve the same DEs.
- Equating them, we find, for example, that when $\tau_1=\cdots=\tau_m=:\tau$, there are real/complex $m\times m$ matrix KP solutions $q(\tau,\gamma,h)$ and $q(\tau,\gamma,h|\zeta)$ such that

$$D(\tau,\gamma,h)=\oint\cdots\oint D(\tau,\gamma,h|\zeta)\prod_{i=1}^{m-1}\frac{\mathrm{d}\zeta_i}{2\pi\mathrm{i}(1-\zeta_i)\zeta_i}$$

where

$$D(\tau,\gamma,h|\zeta)=\exp\left[-\sum_{i=1}^{m}\int_0^\infty \mathrm{Tr}\, q_i(\tau,\gamma,h+\xi a|\zeta)\mathrm{d}\xi\right],\quad a=(1,2,\cdots,m)$$

with q_i being the $i\times i$ upper left blocks of q.

Periodic KPZ fixed point

- Periodic KPZ fixed point is the conjectured limit for KPZ universality class models on a ring as the ring size and time both tend to infinity in a critical way
- Interpolates the KPZ fixed point and the Brownian motion - proven for one point function by Baik, Liu, & Silva 2021. In particular, the one-point marginal is not TW, and it depends on time.
- The field is not yet constructed, but multi-time distributions were obtained Baik & Liu 2019.
- Cubic admissible function and a discrete RHP with infinitely many poles
- Result: (a) coupled NLS with complex time (b) coupled mKdV (c) KP
- But, not (d) Tracy-Widom ODE (e) Adler-van Moerbeke PDE
- For one-point distribution, the equations were already obtained in Baik-Liu-Silva 2021.

Summary

- (periodic) KPZ fixed point and cubic admissible functions $D(t, y, x) = \det(1 - K)$, a function of $3m$ variables
- 5 integrable DEs
  - coupled matrix NLS with complex time
  - coupled matrix mKdV
  - Tracy-Widom type system of matrix DEs
  - matrix KP extending Quastel-Remenik
  - Adler-van Moerbeke PDE
- Proof uses IIKS integrable operator and $(m+1) \times (m+1)$ RHP with complex jump without symmetry

Thank you for your attention