Janossy densities of the thinned Airy process and the Schrödinger and (c)KdV equations

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Plan

Outline

The general picture

Probabilistic side

Determinantal point processes defined through *integrable* kernels

Gap probability
· Airy

Integrable Systems' side

Integrable PDEs
· KdV

$\updownarrow$

Painlevé type equations
· PII

Aim

Consider Janossy densities (instead of gap probabilities) of a suitable modification of the Airy DPP on the probabilistic side and see how the connection with integrable systems is realized.

The Airy kernel

We consider the integral operator $\mathcal{K}^{\mathrm{Ai}}$ on $L^2(\mathbb{R})$ acting through the Airy kernel

$$K^{\mathrm{Ai}}(x,y) := \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y} = \int_0^{+\infty} \mathrm{Ai}(x+t)\mathrm{Ai}(y+t)dt,$$

where $\mathrm{Ai}(\cdot)$ stands for the classical Airy function, i.e. a rapidly decaying at $+\infty$ real solution of the Airy equation $f''(x) = xf(x)$.

[Soshnikov, 2000] Hermitian locally trace class operator $\mathcal{K}$ on $L^2(\mathbb{R})$ defines a determinantal point process on $\mathbb{R}$ if and only if $0 \le \mathcal{K} \le 1$. If the corresponding point process exists it is unique.

The Airy point process defined through $\mathcal{K}^{\mathrm{Ai}}$ has largely been studied, with particular focus on the probability distribution function of the largest particle of the process, described by

$$\boxed{F(s) := \det(1 - \mathcal{K}^{\mathrm{Ai}}|_{(s,+\infty)})}$$

Some applications

* **Random Matrix Theory**
  [Forrester 1993, Tracy - Widom, 1994] $F(s) = F_{GUE}(s)$, is the edge scaling limit of the probability distribution of the largest eigenvalue in the Gaussian Unitary Ensemble.

* **Random Permutations**
  [Baik - Deift - Johansson, 1999] $F(s)$ describes certain scaling limit of the probability distribution of the longest increasing subsequence of random permutations with uniform distribution.

* **Fermionic Systems**
  [Eisler, 2013 (and others)] $F(s)$ describes certain scaling limits of the probability distribution of the largest position and momentum of a system of free fermions in harmonic potential at zero temperature.

A well-known characterization

[Tracy - Widom, 1994] The Fredholm determinant $F(s)$ satisfies

$$\frac{d^2}{ds^2} \ln F(s) = -u^2(s)$$

where u is the Hastings-McLeod solution of the Painlevé II equation, i.e.

$$u''(s) = su(s) + 2u^3(s)$$

with $u(s) \sim \mathrm{Ai}(s)$ for $s \to +\infty$.

Remark

(1) Tracy and Widom proof consists in the application to the Tracy-Widom criteria to the specific case of the Airy kernel on a semi-infinite interval.

(2) Since the Airy kernel also enjoy the integrable structure of IIKS type, an alternative proof as been given through a Riemann–Hilbert approach (e.g. [Kapaev - Hubert, 1999]).

And a recent generalization

Consider $s \in \mathbb{R}$ and $\mathcal{K}_s^{\mathrm{Ai}}$ the operator with s-shifted Airy kernel

$$K_s^{\mathrm{Ai}}(x, y) := K^{\mathrm{Ai}}(x + s, y + s.)$$

[Amir - Corwin - Quastel, 2011] Generalization of this formula, for some function $\sigma : \mathbb{R} \to [0, 1]$ and $F_\sigma(s) = \det(1 - \sigma\mathcal{K}_s^{\mathrm{Ai}})$, then

$$\frac{d^2}{ds^2} \ln F_\sigma(s) = -\int_{\mathbb{R}} \varphi^2(r; s)\sigma'(r)dr$$

where φ solves the integro-differential Painlevé II equation

$$\frac{\partial^2}{\partial s^2}\varphi(z; s) = \left(z + s + 2\int_{\mathbb{R}} \varphi^2(r; s)\sigma'(r)dr\right)\varphi(z; s),$$

with $\varphi(z; s) \sim \mathrm{Ai}(z + s)$ for $s \to +\infty$.

Remark

(1) The interest in $F_\sigma(s)$ came from its appearence in relation with the KPZ equation and fermionic systems at finite temperature.

(2) Several authors (re)proved this result, either by generalizing the Tracy-Widom method [Krajenbrink, 2020] or through a Riemann-Hilbert approach in a matrix-valued setting [Cafasso - Claeys - Ruzza, 2021] and in an operator-valued one [Bothner, 2021].

Outline

What are Janossy densities?

Let $\mathcal{K}$ be an integral operator with kernel K defining a DPP on $\mathbb{R}$, I an interval of $\mathbb{R}$ s.t. $\mathcal{K}_I$ is trace-class and take $m \in \mathbb{N}$ and $V = \{v_1, \dots, v_m\} \subset I$.

[Soshnikov, 2000] The m-th Janossy density is defined as the density (w.r.t. the Lebesgue measure) of the probability distribution function of m particles $\{v_i\}_{i=1}^m$ in I. It is given by

$$J_I(v_1, \dots, v_m) = \det(1 - \mathcal{K}_I) \det_{1 \le k,h \le m} (L_I(v_k, v_h)),$$

where L_I is the kernel of the operator defined by

$$\mathcal{L}_I := \mathcal{K}_I(1 - \mathcal{K}_I)^{-1}.$$

Heuristically, it is intended as the infinitesimal probability

$$J_I(v_1, \dots, v_m) = \mathrm{Prob}\,(\text{having exactly } m \text{ particles in } I \text{ each one in } [v_i, v_i + dv_i]).$$

Properly, it is defined as the density of the so called *Janossy measures*.

Remark The degenerate case $V = \emptyset$ goes back to the gap probability, i.e.

$$J_I(\emptyset) = \det(1 - \mathcal{K}_I).$$

Few facts on Janossy densities

- [Fuji - Kanamori - Nishigaki, 2019] The following formula holds

$$\det(1-\mathcal{K}_I)\det_{1\le k,h\le m}(L_I(v_k,v_h)) = \det_{1\le k,h\le m}(K_I(v_k,v_h))\det(1-\mathcal{K}_{I,V})$$

  where $\mathcal{K}_{I,V}$ is the integral operator with kernel $K_{I,V}$ obtained through a finite rank deformation of K_I

$$K_{I,V}(x,y) := \frac{\det\begin{pmatrix} K_I(x,y) & K_I(x,\vec{v}) \\ K_I(\vec{v},y) & K_I(\vec{v},\vec{v})\end{pmatrix}}{\det\left(K_I(\vec{v},\vec{v})\right)}.$$

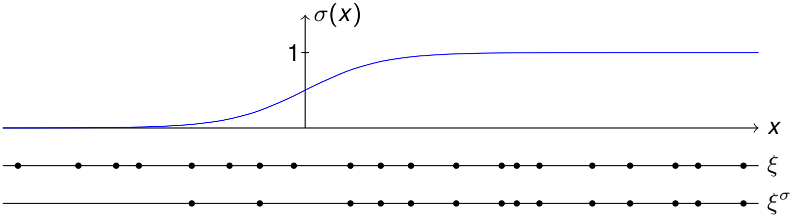
- [Nishigaki, 2021] Application of the *Tracy - Widom method* to Janossy densities for DPP defined by kernels K satysfing the Tracy-Widom criteria $\implies$ expression of this type of Janossy densities (and not only of the gap probability) in terms of solutions of closed systems of differential equations.

Thinning the Airy point process

Let $\sigma : \mathbb{R} \to [0,1]$ a non-decreasing smooth function.

If $\mathcal{K}$ defines a DPP $\mathbb{P}$ and we consider a function σ so that $\sigma\mathcal{K}$ defines the *thinned* process $\mathbb{P}^\sigma$, constructed as follows.

For every random configuration ξ in $\mathbb{P}$, a configuration ξ^σ in $\mathbb{P}^\sigma$ is built by independently eliminating a particle ξ_j in the configuration ξ with probability $1 - \sigma(\xi_j)$ and by keeping it with probability $\sigma(\xi_j)$:



Remark If $\sigma\mathcal{K}_s^{\mathrm{Ai}}$ is trace-class then $\mathbb{P}^{\mathrm{Ai}_{s;\sigma}}$ has a. s. $\#$ particles $< \infty$.

Janossy densities of the thinned Airy DPP

We can then define Janossy densities of the thinned shifted Airy point process

$$J_\sigma(V;s) \equiv J_\sigma(v_1,\dots,v_m;s) = \det(1-\sigma\mathcal{K}_s^{\mathrm{Ai}}) \det_{1\le k,h\le m}(L_{\sigma,s}^{\mathrm{Ai}}(v_k,v_h))$$

where $L_{\sigma,s}^{\mathrm{Ai}}$ is the kernel of the operator $\mathcal{L}_{\sigma,s}^{\mathrm{Ai}}$ defined as

$$\mathcal{L}_{\sigma,s}^{\mathrm{Ai}} := \mathcal{K}_s^{\mathrm{Ai}}\left(1-\sigma\mathcal{K}_s^{\mathrm{Ai}}\right)^{-1}.$$

[Claeys - Glesner, 2021] (Remember Tom Claeys' talk.) The kernel of this operator defines (on $(\mathbb{R},(1-\sigma(x))dx)$) the DPP $\mathbb{P}^\sigma_{|\emptyset}$ obtained by

1. first, σ-marking with 0 and 1 the (shifted) Airy kernel dpp;
2. then, conditioning on empty 1-configuration the marked point process.

Remark Also in this case we have two representations

$$\det(1-\sigma\mathcal{K}_s^{\mathrm{Ai}}) \det_{1\le k,h\le m}(L_{\sigma,s}^{\mathrm{Ai}}(v_k,v_h)) = \det_{1\le k,h\le m}(K_s^{\mathrm{Ai}}(v_k,v_h)) \det(1-\sigma\mathcal{K}_{s,V}^{\mathrm{Ai}}).$$

Statement 1

Theorem

We have

$$J_\sigma(V;s) = \det\left(L_s^\sigma(v_i,v_j)\right)_{i,j=1}^m F_\sigma(s),$$

where

$$L_s^\sigma(\lambda,\mu) = \int_s^{+\infty} \varphi(\lambda;s')\varphi(\mu;s')\,\mathrm{d}s' = \frac{\varphi(\lambda;s)\varphi'(\mu;s) - \varphi'_(\lambda;s)\varphi(\mu;s)}{\lambda-\mu},$$

and

$$F_\sigma(s) = \exp\left(-\int_s^{+\infty}(s'-s)\left(\int_{\mathbb{R}}\varphi(\lambda;s')^2\mathrm{d}\sigma(\lambda)\right)\mathrm{d}s'\right).$$

Here $\varphi(z;s,\emptyset)$ solves the Schrödinger equation

$$\left[\partial_s^2 + 2u(s,\emptyset)\right]\varphi(z;s) = z\varphi(z;s),$$

with potential $u(s,\emptyset) = -\int_{\mathbb{R}}\varphi^2(r,s)\sigma'(r)\mathrm{d}r - \frac{s}{2}$ and with asymptotic behavior for $z\to\infty$ in terms of the Airy function.

Statment 2

Theorem

We have

$$\frac{d^2}{ds^2}\ln J_\sigma(V;s) = u(s,V) + \frac{s}{2} = -\int_{\mathbb{R}} \varphi^2(r;s,V)\,\sigma'(r)dr + 4\pi \sum_{v\in V} \lim_{r\to v} \varphi(r;s,V)\widetilde{\varphi}(r;s,V).$$

Here $\varphi(z;s,V), \widetilde{\varphi}(z;s,V)$ *solve both the Schrödinger equation*

$$\left[\partial_s^2 + 2u(s,V)\right]\begin{pmatrix}\varphi(z;s,V)\\ \widetilde{\varphi}(z;s,V)\end{pmatrix} = z\begin{pmatrix}\varphi(z;s,V)\\ \widetilde{\varphi}(z;s,V)\end{pmatrix},$$

with asymptotic behavior for $z\to\infty$ *in terms of the Airy function.*

Remark

- We have that $\varphi(z;s) = \varphi(z;s,\emptyset)$.
- Moreover, $\varphi(z;s,V)$ and $\widetilde{\varphi}(z;s,V)$ are obtained from a Backlund transformation of $\varphi(z;s,\emptyset)$ and its analogue $\widetilde{\varphi}(z;s,\emptyset)$.
- The blue term can also be computed in terms of $\varphi(z;s,\emptyset), \partial_z\varphi(z;s,V)$ evaluated at $z = v \in V$.

Outline

RH problem for Ψ

(a) $\Psi(\cdot; s, V) : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2\times 2}$ is analytic for all $s \in \mathbb{R}$ and all finite $V \subset \mathbb{R}$.

(b) The continuous boundary values of $\Psi(\cdot; s, V)$ are related by

$$\Psi_+(z; s, V) = \Psi_-(z; s, V) \begin{pmatrix} 1 & 1-\sigma(z) \\ 0 & 1 \end{pmatrix}, \qquad z \in \mathbb{R} \setminus V.$$

(c) For all $v \in V$, as $z \to v$ away from the real axis we have

$$\Psi(z; s, V)(z-v)^{-\sigma_3} = O(1).$$

(d) As $z \to \infty$, we have

$$\Psi(z; s, V) = \left(I + \frac{1}{z} \begin{pmatrix} q(s, V) & ir(s, V) \\ ip(s, V) & -q(s, V) \end{pmatrix} + O(z^{-2}) \right) z^{\frac{1}{4}\sigma_3} A^{-1} \mathrm{e}^{\left(-\frac{2}{3} z^{\frac{3}{2}} - s z^{1/2}\right)\sigma_3} C_\delta$$

for any $\delta \in (0, \frac{\pi}{2})$, where

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \quad C_\delta := \begin{cases} I, & |\arg z| < \pi - \delta, \\ \begin{pmatrix} 1 & 0 \\ \mp 1 & 1 \end{pmatrix}, & \pi - \delta < \pm \arg z < \pi. \end{cases}$$

Few remarks

- When $V = \emptyset$ then $\Psi(z, s; \emptyset)$ is obtained as

$$\Psi(z, s; \emptyset) = \begin{pmatrix} 1 & \frac{is^2}{4} \\ 0 & 1 \end{pmatrix} Y(z; s)\Phi_{\mathrm{Ai}}(z + s),$$

  where $Y(z; s)$ is the solution of the classical RH problem associated to the integrable IIKS kernel $\sigma\mathcal{K}_s^{\mathrm{Ai}}$ and $\Phi_{\mathrm{Ai}}(z + s)$ is the (shifted) solution to the model Airy RH problem.

- The kernel of the operator $\mathcal{L}_{\sigma,s}^{\mathrm{Ai}}$ can be written as

$$L_{\sigma,s}^{\mathrm{Ai}}(v, w; s) = \begin{cases} \dfrac{\left(\Psi^{-1}(w; s, \emptyset)\Psi(v; s, \emptyset)\right)_{2,1}}{2\pi i(v - w)}, & v \neq w, \\[2ex] \dfrac{\left(\Psi^{-1}(v; s, \emptyset)\Psi'(v; s, \emptyset)\right)_{2,1}}{2\pi i}, & v = w. \end{cases}$$

- Finally, the relation between $\Psi(z; s, V)$ and $\Psi(z; s, \emptyset)$ is expressed through an explicit Darboux-Schlesinger transformation that can be seen as a particular case of the ones studied in general in [Bertola - Cafasso, 2014] in relation with DPP.

The Darboux-Schlesinger transformation

For all finite $V = \{v_1, \dots, v_m\} \subset \mathbb{R}$, we denote by $\mathbf{L}(s, V)$ the square matrix of size m with entries

$$\mathbf{L}_{k,h}(s, V) := L^{\mathrm{Ai}}_{\sigma,s}(v_k, v_h; s), \qquad 1 \le k, h \le m.$$

We also denote by $\mathbf{L}^{-1}_{j,i}(s, V)$ be the j, i-entry of the inverse matrix of $\mathbf{L}(s, V)$.

Lemma

We have

$$\Psi(z; s, V) = R(z; s, V)\Psi(z; s, \emptyset)$$

where $R(z; s, V)$ is a rational function of z with poles at $z \in V$ only, explicitly given by

$$R(z; s, V) = I - \frac{1}{2\pi i} \sum_{i,j=1}^{m} \frac{\mathbf{L}^{-1}_{j,i}(s; V)}{z - v_j} \Psi(v_i; s, \emptyset) \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Psi^{-1}(v_j; s, \emptyset).$$

Remark In particular we have that $p(s, V)$ and $p(s, \emptyset)$ are related by

$$p(s, V) - p(s, \emptyset) = \sum_{i,j=1}^{m} \mathbf{L}^{-1}_{j,i}(s, V)\varphi(v_i; s, \emptyset)\varphi(v_j; s, \emptyset).$$

Derivation of the Schrödinger equation

Let

$$\Theta(z;s,V) := \begin{pmatrix} 1 & p(s,V) \\ 0 & 1 \end{pmatrix} e^{\frac{i\pi}{4}\sigma_3}\Psi(z;s,V)e^{-\frac{i\pi}{4}\sigma_3}.$$

Lemma

Let us denote $u(s,V) := -\partial_s p(s,V)$. Then we have

$$u(s,V) = p(s,V)^2 + 2q(s,V)$$

and

$$\frac{\partial\Theta(z;s,V)}{\partial s} = \begin{pmatrix} 0 & z-2u(s,V) \\ 1 & 0 \end{pmatrix}\Theta(z;s,V).$$

It follows that

$$\Theta(z;s,V) = \sqrt{2\pi}\begin{pmatrix} \partial_s\varphi(z;s,V) & \partial_s\widetilde{\varphi}(z;s,V) \\ \varphi(z;s,V) & \widetilde{\varphi}(z;s,V) \end{pmatrix},$$

where

$$\left[\partial_s^2 + 2u(s,V)\right]\begin{pmatrix} \varphi(z;s,V) \\ \widetilde{\varphi}(z;s,V) \end{pmatrix} = z\begin{pmatrix} \varphi(z;s,V) \\ \widetilde{\varphi}(z;s,V) \end{pmatrix},$$

and $\varphi, \widetilde{\varphi}$ have asymptotics in terms of the Airy function.

Relation between the potential and the eigenfunctions

Lemma

We have

$$-\int_{\mathbb{R}} \varphi^2(r;s,V)\,\sigma'(r)dr + 4\pi \sum_{v\in V} \lim_{r\to v} \varphi(r;s,V)\widetilde{\varphi}(r;s,V) = \frac{s}{2} + u(s,V)\,.$$

Remarks

1. The blue term disappears in the case $V=\emptyset$ and we recover the result from [Amir - Corwin - Quastel 2011, Cafasso - Claeys - Ruzza, 2021] that gave rise to the integro-differential Painlevé II equation.
2. Since for all $v\in V$,

$$\varphi(z;s,V) = O(z-v),\ \ \widetilde{\varphi}(z;s,V) = O\left(\frac{1}{z-v}\right), \qquad z\to v,$$

   the summation term is well defined. Moreover, we can actually compute it as

$$-\int_{\mathbb{R}} \varphi^2(r;s,V)\,\sigma'(r)dr + 2\sum_{i,j=1}^{m} \mathbf{L}^{-1}_{j,i}(s,V)\varphi(v_i;s,\emptyset)\partial_z\varphi(z;s,V)|_{\{z=v_j\}} = \frac{s}{2} + u(s,V).$$

3. Similar formula was previously found in [Deift - Trubowitz, 1979] in the study of classical inverse scattering for the Schrödinger equation.

Integro-differential equations

Plugging in the Schrödinger equation the formula obtained for the potential $u(s, V)$, one finds either this system

$$\frac{\partial^2 \varphi(z; s, V)}{\partial s^2} = \left(z + s + 2 \left(\int_{\mathbb{R}} \varphi(r; s, V)^2 \, \sigma'(r) dr - 4\pi \sum_{v \in V} \lim_{r \to v} \varphi(r; s, V) \widetilde{\varphi}(r; s, V) \right) \right) \varphi(z; s, V)$$

$$\frac{\partial^2 \widetilde{\varphi}(z; s, V)}{\partial s^2} = \left(z + s + 2 \left(\int_{\mathbb{R}} \varphi(r; s, V)^2 \, \sigma'(r) dr - 4\pi \sum_{v \in V} \lim_{r \to v} \varphi(r; s, V) \widetilde{\varphi}(r; s, V) \right) \right) \widetilde{\varphi}(z; s, V)$$

Remark The system does not seem to be reducible to a single equation. However, by using the alternative formulation for $u(s, V)$ we can consider only the equation for $\varphi(z; s, V)$ in the form

$$\frac{\partial^2 \varphi(z; s, V)}{\partial s^2} = \left(z + s + 2 \int_{\mathbb{R}} \varphi(r; s, V)^2 \, \sigma'(r) dr - 4 \sum_{i,j=1}^{m} \mathbf{L}_{j,i}^{-1} \varphi(v_i; s, \emptyset) \varphi'(v_j; s, V) \right) \varphi(z; s, V),$$

recalling that $\mathbf{L}_{j,i}^{-1}$ is written in terms of $\varphi(v; s, \emptyset)$, $v \in V$ only and that $\varphi(z; s, \emptyset)$ solves the classical integro-differential PII.

Proof of the first characterization of $J_\sigma(s, V)$

Lemma

We have

$$\frac{d}{ds} L_s^\sigma(v, w) \;=\; -\varphi(v; s, \emptyset)\varphi(w; s, \emptyset).$$

Moreover, from the asymptotic analysis for $s \to +\infty$, we can conclude

$$L_s^\sigma(v, w) \;=\; \int_s^{+\infty} \varphi(v; r, \emptyset)\varphi(w; r, \emptyset)\mathrm{d}r.$$

This is enough to prove the first characterization of $J_\sigma(s, V)$.

1. Recall that $J_\sigma(V; s) = F_\sigma(s)\det_{1\le k,h\le m}(L^{\mathrm{Ai}}_{\sigma,s}(v_k, v_h))$.
2. The characterization of $F_\sigma(s) = \det(1 - \sigma\mathcal{K}_s^{\mathrm{Ai}})$ comes from IIKS theory through

$$-p(s; \emptyset) + \frac{s^2}{4} \;=\; \frac{d}{ds} \ln\det(1 - \sigma\mathcal{K}_s^{\mathrm{Ai}}).$$

thus one more derivative gives

$$u(s, \emptyset) + \frac{s}{2} = \frac{d^2}{ds^2} \ln\det(1 - \sigma\mathcal{K}_s^{\mathrm{Ai}}), \text{ with } u(s, \emptyset) = -\int_{\mathbb{R}} \varphi^2(r, s; \emptyset)\mathrm{d}r.$$

Proof of the second characterization of $J_\sigma(s, V)$

Proposition

We have

$$-p(s; V) + \frac{s^2}{4} = \frac{d}{ds} \ln J_\sigma(V; s).$$

1. Recall that in the case $V = \emptyset$, we already have $-p(s; \emptyset) + \frac{s^2}{4} = \frac{\mathrm{d}}{\mathrm{d}s} \log\det(1 - \sigma \mathcal{K}_s^{\mathrm{Ai}})$.
2. Recall the relation
$$p(s, V) - p(s, \emptyset) = \sum_{i,j=1}^{m} \mathbf{L}_{j,i}^{-1}(s, V)\varphi(v_i; s, \emptyset)\varphi(v_j; s, \emptyset).$$
3. And finally use that
$$\partial_s \log\det \mathbf{L}(s, V) = \sum_{i,j=1}^{m} \mathbf{L}_{j,i}^{-1}(s, V)\partial_s \mathbf{L}_{j,i}(s; V) = -\sum_{i,j=1}^{m} \mathbf{L}_{j,i}^{-1}(s, V)\varphi(v_i; s, \emptyset)\varphi(v_j; s, \emptyset).$$
since we have $\partial_s \mathbf{L}_{j,i}(s; V) = -\varphi(v_j; s, \emptyset)\varphi(v_i; s, \emptyset)$.
4. Thus one more derivative gives
$$\frac{d^2}{ds^2} \ln J_\sigma(V; s) = u(s, V) + \frac{s}{2}.$$

Outline

UCLouvain

The introduction of parameters X, T and (c)Kdv

Now we define $\mathcal{K}^{\mathrm{Ai}}_{X,T}$ the integral operator acting with kernel (depending on $X \in \mathbb{R}$ and $T > 0$)

$$K^{\mathrm{Ai}}_{X,T}(\lambda, \mu) := T^{-1/3} K^{\mathrm{Ai}}\left(T^{-1/3}(\lambda + X), T^{-1/3}(\mu + X)\right),$$

and we consider as before

$$J_\sigma(V; X, T) = \det\left(L^\sigma_{X,T}(v_i, v_j)\right)_{i,j=1}^m \det(1 - \sigma \mathcal{K}^{\mathrm{Ai}}_{X,T}.)$$

Theorem

The function $U = U_\sigma(X, T; V) := \frac{\partial^2}{\partial X^2} J_\sigma(V; X, T)$ *solves the cKdV equation*

$$\frac{\partial U}{\partial T} + \frac{1}{12}\frac{\partial^3 U}{\partial X^3} + U \frac{\partial U}{\partial X} + \frac{U}{2T} = 0.$$

Remark

- KdV and cKdV are algebraically equivalent.
- In case $V = \emptyset$ this recovers the result from [Cafasso - Claeys - Ruzza, 2021].
- In case $\sigma \equiv 0$ these solutions should be compared to the class of *soliton-type* solutions of cKdV found in [Nakamura, 1980].

With X, T parameters

We can characterize them through the solution of a RH problem analogue to the previous one $\widetilde{\Psi}(\zeta, X, T; V)$ now depending on both parameters X, T. In particular, by defining as before

$$\widetilde{\Theta}(\zeta, X, T; V) := \begin{pmatrix} 1 & p(X,T,V) \\ 0 & 1 \end{pmatrix} e^{\frac{i\pi}{4}\sigma_3} \widetilde{\Psi}(\zeta; X, T, V) e^{-\frac{i\pi}{4}\sigma_3} \implies \underbrace{\begin{matrix} \partial_X \widetilde{\Theta} = B\widetilde{\Theta}, \\ \partial_T \widetilde{\Theta} = C\widetilde{\Theta} \end{matrix}}_{\text{cKdV Lax pair}}$$

$\partial_X^2 J_\sigma(V; X, T) = U(X, T; V)$ solves the cKdV equation coming from the Lax pair.

DPP	(c)KdV
gap probability	solution $U(X, T; \emptyset)$
$\downarrow$	$\downarrow$
Janossy density	Darboux-Backlünd transformed one $U(X, T; V)$

Outline

Asymptotics

Recall that

$$J_\sigma(v_1,\dots,v_m;s) = \det(1-\sigma\mathcal{K}_s^{\mathrm{Ai}}) \det_{1\le k,h\le m}(L_{\sigma,s}^{\mathrm{Ai}}(v_k,v_h)) = F_\sigma(s)\rho_{m,s}^\sigma(v_1,\dots,v_m).$$

For the parameter $s \to \pm\infty$ we have decorrelation, in the sense that

$$\rho_{m,s}^\sigma(v_1,\dots,v_m) \sim \rho_{1,s}^\sigma(v_1)\dots\rho_{1,s}^\sigma(v_m).$$

To be further investigated:

- How to combine the (known) asymptotics for the Fredholm determinant and the (unknown) asymptotics for the 1-point correlation function.
- Various X, T asymptotic regimes for the Janossy densities $J_\sigma(V;X,T)$ and for the solution of the (c)KdV equation $U(X,T;V)$.
- Which type of (c)KdV solutions are the ones given by $U(X,T;V)$? We should compare them to the ones found in the case $V=\emptyset$ already in [Cafasso - Claeys - Ruzza, 2021]? Also, compare to the ones previously studied in [Its - Sukhanov, 2020].
- How the *decorrelation* phenomenon reflects on the behavior of the (c)KdV solutions?

Thank you!