From the Ablowitz-Ladik lattice to the Circular β-ensemble

Guido, Mazzuca
mazzuca@kth.se

May 24, 2022

Overview

- Background, and motivations
- Ablowitz-Ladik lattice
- Generalized Gibbs Ensemble
- Circular β ensemble
- Glimpses of proofs

G.M., and T. Grava, *Generalized Gibbs ensemble of the Ablowitz-Ladik lattice, circular β-ensemble and double confluent Heun equation*, arXiv e-print 2107.02303 (2021)

G.M., and R. Memin, *Large Deviations for Ablowitz-Ladik lattice, and the Schur flow*, arXiv e-print 2201.03429 (2022)

Integrable systems

Consider a Poisson manifold $(M, \{\ ,\ \})$, such that $\{\ ,\ \}$ is non-degenerate. Let $\boldsymbol{x} = (x_1, \ldots, x_{2N})$ be coordinates on M. The evolution $\boldsymbol{x}(0) \to \boldsymbol{x}(t)$ according to Hamilton equations with Hamiltonian $H(\boldsymbol{x})$

$$\frac{dx_j}{dt} = \dot{x}_j = \{x_j, H\}, \;\; j = 1, \ldots, 2N$$

is integrable if there are $H_1 = H, H_2, \ldots H_N$ independent conserved quantities ($\dot{H}_k = 0$) that Poisson commute: $\{H_j, H_k\} = 0$. (Liouville)

Lax Pair

The integrable Hamilton equations

$$\dot{x}_j = \{x_j, H\}, \quad j = 1, \ldots, 2N$$

admits a Lax pair formulation if there exist two square matrices $L = L(\boldsymbol{x})$ and $A = A(\boldsymbol{x})$ such that

$$\dot{L} = [A, L] := LA - AL \longleftrightarrow \dot{x}_j = \{x_j, H\}, \quad j = 1, \ldots, 2N$$

Then, $\mathrm{Tr}L^k$, k integer, are constant of motions: $\frac{d}{dt}\mathrm{Tr}L^k = 0$

Gibbs measure

Consider the Gibbs measure

$$\mu = \frac{1}{Z(V)} e^{-\mathrm{Tr}(V(L(\boldsymbol{x})))} \tilde{\mu}\,, \quad \tilde{\mu} = m(x) dx_1, \ldots dx_{2N}\,,$$

here V is a continuous function, and $\tilde{\mu}$ is invariant for the dynamics, thus also μ is invariant.

Gibbs measure

Consider the Gibbs measure

$$\mu = \frac{1}{Z(V)} e^{-\mathrm{Tr}(V(L(\boldsymbol{x})))} \tilde{\mu}\,, \quad \tilde{\mu} = m(x) dx_1, \ldots dx_{2N}\,,$$

here V is a continuous function, and $\tilde{\mu}$ is invariant for the dynamics, thus also μ is invariant.

$$\mu \longrightarrow L$$

thus L becomes a Random Matrix.

Gibbs measure

Consider the Gibbs measure

$$\mu = \frac{1}{Z(V)} e^{-\mathrm{Tr}(V(L(\boldsymbol{x})))} \tilde{\mu}\,, \quad \tilde{\mu} = m(x) dx_1, \dots dx_{2N}\,,$$

here V is a continuous function, and $\tilde{\mu}$ is invariant for the dynamics, thus also μ is invariant.

$$\mu \longrightarrow L$$

thus L becomes a Random Matrix.

- Does L can be reduced to a known family of random matrices? Which is the spectrum of L when $N \to \infty$ (density of states) ?
- How do the correlation functions looks like $S(j,t) = \mathbb{E}(x_j(t)x_\ell(0)) - \mathbb{E}(x_j(t))\mathbb{E}(x_\ell(0))$ behave when $N \to \infty$ and $t \to \infty$?

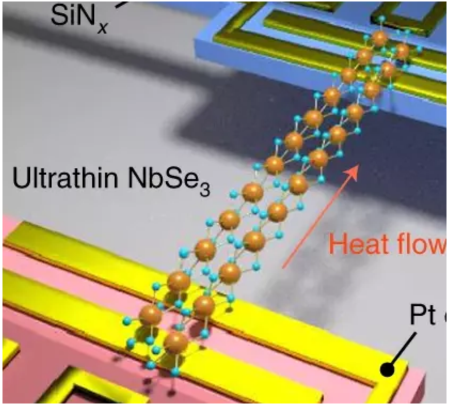
Why:

Correlation functions $\rightarrow$ Transport properties

Why:

Correlation functions $\rightarrow$ Transport properties

Specific 1D phenomenon: conductivity diverges as the length of the chain grows (Anomalous transport).
Surprisingly, this is **measured** experimentally:



(Nature Nanotechnology 2021)

Why:

Correlation functions $\rightarrow$ Transport properties

For a general dynamical system, the computation of a general correlation function $S(j, t)$ as $t, N \rightarrow \infty$ is "utterly out of reach" (Spohn). Rigorous mathematical results in dimension bigger or equal to 3 (Lukkarinen-Spohn 2011).

Why:

Correlation functions $\rightarrow$ Transport properties

For a general dynamical system, the computation of a general correlation function $S(j,t)$ as $t, N \rightarrow \infty$ is "utterly out of reach" (Spohn). Rigorous mathematical results in dimension bigger or equal to 3 (Lukkarinen-Spohn 2011). Numerical simulations show that:

$$S(j,t) \simeq \frac{1}{\lambda t^{\gamma}} f\left(\frac{j - vt}{\lambda t^{\delta}}\right) .$$

- **Non integrable systems**, such as DNLS, FPUT, etc, $\gamma = \delta = \frac{2}{3}$ and $f = F_{TW}$.
- **Non linear integrable systems**, such as Toda, AL, $\gamma = \delta = 1$ and $f = e^{-x^2}$.

Why:

Correlation functions $\rightarrow$ Transport properties

For a general dynamical system, the computation of a general correlation function $S(j,t)$ as $t, N \rightarrow \infty$ is "utterly out of reach" (Spohn). Rigorous mathematical results in dimension bigger or equal to 3 (Lukkarinen-Spohn 2011). Numerical simulations show that:

$$S(j,t) \simeq \frac{1}{\lambda t^{\gamma}} f\left(\frac{j - vt}{\lambda t^{\delta}}\right) .$$

- **Non integrable systems**, such as DNLS, FPUT, etc, $\gamma = \delta = \frac{2}{3}$ and $f = F_{TW}$.
- **Non linear integrable systems**, such as Toda, AL, $\gamma = \delta = 1$ and $f = e^{-x^2}$.
- **Short range harmonic chain**, we can perfectly describe the behaviour of the correlation functions (Mazur;. . . , M - Grava - McLaughlin - Kriecherbauer).The behaviour can be "wild", for different position-time scales the behaviour is described by Airy, Pearcy integral,. . .

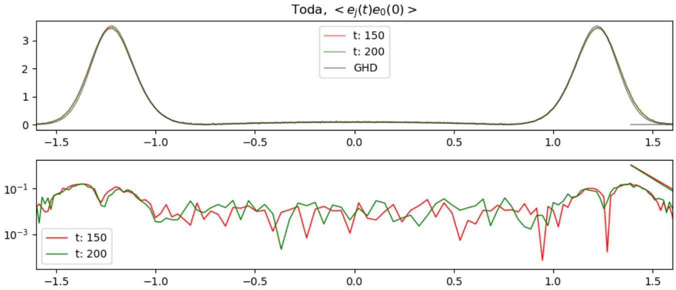
Recent Breakthrough

- H. Spohn was able to characterize the density of states for the GGE of the Toda lattice with polynomial potential in terms of the equilibrium measure of the Gaussian β ensemble at high temperature.

Recent Breakthrough

- H. Spohn was able to characterize the density of states for the GGE of the Toda lattice with polynomial potential in terms of the equilibrium measure of the Gaussian β ensemble at high temperature.

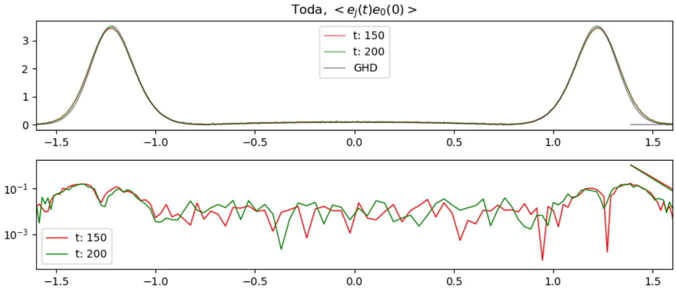
  Applying the theory of Generalized Hydrodynamic, he argued that the decay of correlation functions is ballistic. $(\delta = \gamma = 1)$



Recent Breakthrough

- H. Spohn was able to characterize the density of states for the GGE of the Toda lattice with polynomial potential in terms of the equilibrium measure of the Gaussian β ensemble at high temperature.

  Applying the theory of Generalized Hydrodynamic, he argued that the decay of correlation functions is ballistic. $(\delta = \gamma = 1)$
- A. Guionnet, and R. Memin generalized Spohn results, obtaining a Large deviations principle for the empirical measures with continuous potential.



β-ensemble at high temperature	Integrable System
Gaussian	Toda lattice (Spohn; Guionnet-Memin)
Circular	Defocusing Ablowitz-Ladik lattice (Spohn, Grava-M.; Memin-M.)
Laguerre	Exponential Toda lattice (Gisonni-Grava-Gubbiotti-M.)
Jacobi	Defocusing Schur flow (Spohn; Memin-M.)
Antisymmetric Gaussian	Volterra lattice (Gisonni-Grava-Gubbiotti-M.)

High temperature regime means that we are considering $\beta = \frac{2\alpha}{N}$ for some $\alpha \in \mathbb{R}_+$.

β-ensemble at high temperature	Integrable System
Gaussian	Toda lattice (Spohn; Guionnet-Memin)
Circular	Defocusing Ablowitz-Ladik lattice (Spohn, Grava-M.; Memin-M.)
Laguerre	Exponential Toda lattice (Gisonni-Grava-Gubbiotti-M.)
Jacobi	Defocusing Schur flow (Spohn; Memin-M.)
Antisymmetric Gaussian	Volterra lattice (Gisonni-Grava-Gubbiotti-M.)

High temperature regime means that we are considering $\beta = \frac{2\alpha}{N}$ for some $\alpha \in \mathbb{R}_+$.

The Ablowitz-Ladik lattice

$$i\dot{a}_j = (2a_j - a_{j-1} - a_{j+1}) + |a_j|^2(a_{j-1} + a_{j+1}), \quad j = 1, \dots, N$$

where $a_j \in \mathbb{C}$, and we consider periodic boundary condition, thus $a_{j+N} = a_j$.

The Ablowitz-Ladik lattice

$$i\dot{a}_j = (2a_j - a_{j-1} - a_{j+1}) + |a_j|^2(a_{j-1} + a_{j+1}), \quad j = 1, \dots, N$$

where $a_j \in \mathbb{C}$, and we consider periodic boundary condition, thus $a_{j+N} = a_j$.

- The Ablowitz-Ladik (1973–74) system is the integrable discretization of the defocussing cubic NLS:

$$i\partial_t \psi(x,t) = -\frac{1}{2}\partial_x^2 \psi(x,t) + |\psi(x,t)|^2 \psi(x,t).$$

its integrability was discovered by Ablowitz and Ladik (1974) by making spatial discretization of the Zakharov-Shabat Lax pair for NLS;

The Ablowitz-Ladik lattice

$$i\dot{a}_j = (2a_j - a_{j-1} - a_{j+1}) + |a_j|^2(a_{j-1} + a_{j+1}), \quad j = 1, \ldots, N$$

where $a_j \in \mathbb{C}$, and we consider periodic boundary condition, thus $a_{j+N} = a_j$.

- The Ablowitz-Ladik (1973–74) system is the integrable discretization of the defocussing cubic NLS:

$$i\partial_t \psi(x,t) = -\frac{1}{2}\partial_x^2 \psi(x,t) + |\psi(x,t)|^2 \psi(x,t).$$

  its integrability was discovered by Ablowitz and Ladik (1974) by making spatial discretization of the Zakharov-Shabat Lax pair for NLS;
- For periodic boundary conditions. Finite-gap integration developed by P. Miller, N. Ercolani, I. Krichever and D. Levermore;

The Ablowitz-Ladik lattice

$$i\dot{a}_j = (2a_j - a_{j-1} - a_{j+1}) + |a_j|^2(a_{j-1} + a_{j+1}), \quad j = 1, \ldots, N$$

where $a_j \in \mathbb{C}$, and we consider periodic boundary condition, thus $a_{j+N} = a_j$.

- The Ablowitz-Ladik (1973–74) system is the integrable discretization of the defocussing cubic NLS:

$$i\partial_t \psi(x,t) = -\frac{1}{2}\partial_x^2 \psi(x,t) + |\psi(x,t)|^2 \psi(x,t).$$

  its integrability was discovered by Ablowitz and Ladik (1974) by making spatial discretization of the Zakharov-Shabat Lax pair for NLS;
- For periodic boundary conditions. Finite-gap integration developed by P. Miller, N. Ercolani, I. Krichever and D. Levermore;
- The DNLS is another discretization, but it is not integrable.

Hamiltonian Structure

There are two conserved quantities:

$$K^{(0)} = \prod_{j=1}^{N} \left(1 - |a_j|^2\right) , \quad K^{(1)} := -\sum_{j=1}^{N} a_j \bar{a}_{j+1} .$$

Since $K^{(0)}$ is conserved, this implies that if $|a_j(0)| < 1 \, \forall \, j$, then $|a_j(t)| < 1 \, \forall \, t$.
Thus we can consider $\mathbb{D}^N$ as **phase space**, $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.

Hamiltonian Structure

There are two conserved quantities:

$$K^{(0)} = \prod_{j=1}^{N} \left(1 - |a_j|^2\right) \,, \quad K^{(1)} := -\sum_{j=1}^{N} a_j \bar{a}_{j+1} \,.$$

Since $K^{(0)}$ is conserved, this implies that if $|a_j(0)| < 1 \,\forall\, j$, then $|a_j(t)| < 1 \,\forall\, t$. Thus we can consider $\mathbb{D}^N$ as **phase space**, $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.

$$\{f, g\} = i \sum_{j=1}^{N} (1 - |a_j|^2) \left(\frac{\partial f}{\partial \bar{a}_j} \frac{\partial g}{\partial a_j} - \frac{\partial g}{\partial \bar{a}_j} \frac{\partial f}{\partial a_j} \right)$$

(Ercolani, Lozano)

$$\dot{a}_j = \left\{ a_j, \underbrace{-2 \log\left(K^{(0)}\right) + 2\Re(K^{(1)})}_{:= H_{AL}} \right\}$$

Integrability (N even)

Nenciu, and Simon proved that the AL equations of motion are equivalent to the Lax pair:

$$\dot{\mathcal{E}} = i\,[\mathcal{E}, A(\mathcal{E})]$$

where $\mathcal{E} = \mathcal{L}\mathcal{M}$, such that

$$\mathcal{M} = \begin{pmatrix} -a_1 & & & & \rho_1 \\ & \Xi_3 & & & \\ & & \ddots & & \\ & & & \Xi_{N-1} & \\ \rho_1 & & & & \bar{a}_1 \end{pmatrix}, \qquad \mathcal{L} = \begin{pmatrix} \Xi_2 & & & \\ & \Xi_4 & & \\ & & \ddots & \\ & & & \Xi_N \end{pmatrix},$$

here $\Xi_j = \begin{pmatrix} \bar{a}_j & \rho_j \\ \rho_j & -a_j \end{pmatrix}$ and $\rho_j = \sqrt{1 - |a_j|^2}$.

Structure of periodic CMV matrix

$$
\mathcal{E} = \begin{pmatrix}
* & * & * & & & & & & & * \\
* & * & * & & & & & & & * \\
& * & * & * & * & & & & & \\
& * & * & * & * & & & & & \\
& & & & & \ddots & \ddots & & & \\
& & & & & & * & * & * & * \\
& & & & & & * & * & * & * \\
* & & & & & & & * & * & * \\
* & & & & & & & * & * & *
\end{pmatrix}.
$$

- Periodic CMV (Cantero Moral Velazquez) Matrix:
  - unitary $\lambda_j = e^{i\theta_j}$, $\theta_j \in \mathbb{T}$

Generalized Gibbs Ensemble

In view of the Lax pair:

$$\dot{\mathcal{E}} = i\,[\mathcal{E}, A(\mathcal{E})] \ ,$$

then

$$K^{(\ell)} = \mathtt{Tr}\left(\mathcal{E}^{\ell}\right) \ , \quad \ell = 1, \ldots, N-1$$

are conserved.

Generalized Gibbs Ensemble

In view of the Lax pair:

$$\dot{\mathcal{E}} = i\,[\mathcal{E}, A(\mathcal{E})]\ ,$$

then

$$K^{(\ell)} = \mathtt{Tr}\left(\mathcal{E}^{\ell}\right)\ , \quad \ell = 1, \dots, N-1$$

are conserved.
So we can define the Generalized Gibbs Ensemble as

$$\mu_{AL} = \frac{1}{Z_N^{AL}(V,\beta)} \prod_{j=1}^{N} (1-|a_j|^2)^{\beta-1} \exp(-\mathtt{Tr}(V(\mathcal{E})))\mathrm{d}^2\boldsymbol{a}\,, \quad a_j \in \mathbb{D}$$

here $V(z)$ is a continuous function, $V(z)\ :\ \mathbb{D} \to \mathbb{R}$.
The -1 comes from the Poisson bracket (volume form)

Integrability and Random matrix

$$\mu_{AL} \longrightarrow \mathcal{E}$$

thus $\mathcal{E}$ becomes a Random Matrix.

Integrability and Random matrix

$$\mu_{AL} \longrightarrow \mathcal{E}$$

thus $\mathcal{E}$ becomes a Random Matrix.
Define the empirical measure as

$$\mu_N(\mathcal{E}) = \frac{1}{N}\sum_{j=1}^{N} \delta_{e^{i\theta_j}} \,,$$

where $e^{i\theta_j}$s are the eigenvalues of $\mathcal{E}$.
Study the weak limit of $\mu_N(\mathcal{E})$, or density of states

$$\mu_N(\mathcal{E}) \rightharpoonup \nu_\beta^V$$

The eigenvalues are the fundamental ingredient of the finite-gap integration.

Circular β Ensemble

$$\mathrm{d}\mathbb{P}_C\,(\theta_1,\ldots,\theta_N) = (\mathcal{Z}_N^E(V,\widetilde{\beta}))^{-1}|\Delta(e^{i\theta})|^{\widetilde{\beta}}\exp\left(-\sum_{j=1}^N V(e^{i\theta_j})\right)\mathrm{d}\theta_1\ldots\mathrm{d}\theta_N\,,$$

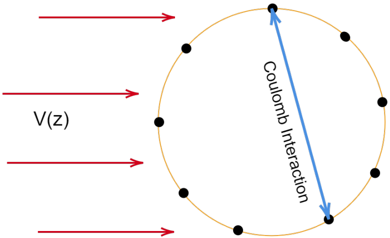
where $\Delta(e^{i\theta}) = \prod_{\ell\neq j}(e^{i\theta_j} - e^{i\theta_\ell})$, $\theta_j \in [-\pi,\pi)$, and $\mathcal{Z}_N^E(V,\widetilde{\beta})$ is the partition function.

Circular β Ensemble

$$\mathrm{d}\mathbb{P}_C(\theta_1,\ldots,\theta_N) = (\mathcal{Z}_N^E(V,\widetilde{\beta}))^{-1}|\Delta(e^{i\theta})|^{\widetilde{\beta}} \exp\left(-\sum_{j=1}^{N} V(e^{i\theta_j})\right) \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_N\,,$$

where $\Delta(e^{i\theta}) = \prod_{\ell\neq j}(e^{i\theta_j} - e^{i\theta_\ell})$, $\theta_j \in [-\pi,\pi)$, and $\mathcal{Z}_N^E(V,\widetilde{\beta})$ is the partition function.

Physical Interpretation: charged particles constrained on the unit circle, subjected to an external potential $V(z)$ at temperature $\widetilde{\beta}^{-1}$



Matrix Representation - Killip, and Nenciu

Matrix Representation - Killip, and Nenciu

Definition

We said that a complex random variable X with values on the unit disk $\mathbb{D}$ is Θ_ν-distributed $(\nu > 1)$ if:

$$\mathbb{E}[f(X)] = \frac{\nu - 1}{2\pi} \int_{\mathbb{D}} f(z)(1 - |z|^2)^{\frac{\nu-3}{2}} \mathrm{d}^2 z\,.$$

if $\nu = 1$ let Θ_1 denote the uniform distribution on the unit circle.

Remark: let $\nu \in \mathbb{N}$, if $\boldsymbol{u}$ is chosen at random according to the surface measure on the unit sphere S^ν in $\mathbb{R}^{\nu+1}$, then $u_1 + iu_2$ is $\Theta_\nu-$distributed.

Theorem (Killip, Nenciu)

Let $a_j \sim \Theta_{\widetilde{\beta}(N-j)+1}$, $\rho_j = \sqrt{1-|a_j|^2}$, and define Ξ_j as

$$\Xi_j = \begin{pmatrix} \overline{a}_j & \rho_j \\ \rho_j & -a_j \end{pmatrix} .$$

for $1 \leq j \leq N-1$ while $\Xi_0 = (1)$ and $\Xi_N = (\overline{a}_N)$ are 1×1 matrices. From these define the $N \times N$ block diagonal matrices as:

$$L = \mathrm{diag}\,(\Xi_1, \Xi_3, \Xi_5, \ldots) \qquad \text{and} \qquad M = \mathrm{diag}\,(\Xi_0, \Xi_2, \Xi_4, \ldots) \; .$$

The eigenvalues of the two CMV matrices $E = LM$ and $\widetilde{E} = ML$ are distributed according to the Circular Beta Ensemble:

$$\mathrm{d}\mathbb{P}_C\,(\theta_1, \ldots, \theta_N) = (\mathcal{Z}_N^E(0, \widetilde{\beta}))^{-1} |\Delta(e^{i\theta})|^{\widetilde{\beta}} \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_N \, , \quad \theta_j \in [-\pi, \pi) \, .$$

Structure of CMV matrix

$$E = \begin{pmatrix} * & * & * & & & & & & \\ * & * & * & & & & & & \\ & * & * & * & * & & & & \\ & * & * & * & * & & & & \\ & & & \ddots & \ddots & \ddots & \ddots & & \\ & & & & & * & * & * & * \\ & & & & & * & * & * & * \\ & & & & & & & * & * \end{pmatrix}.$$

- Pentadiagonal
- Unitary
- finite rank perturbation of $\mathcal{E}$

$$\mathrm{d}\mathbb{P}_C\,(\theta_1,\ldots,\theta_N) = (\mathcal{Z}_N^E(0,\widetilde{\beta}))^{-1}|\Delta(e^{i\theta})|^{\widetilde{\beta}}\mathrm{d}\theta_1\ldots\mathrm{d}\theta_N\,,\quad \theta_j\in[-\pi,\pi)\,,$$

$$\mathrm{d}\mathbb{P}_\alpha(a_1,\ldots,a_N) = (Z_N^E(0,\widetilde{\beta}))^{-1}\prod_{j=1}^{N-1}\left(1-|a_j|^2\right)^{\widetilde{\beta}(N-j)/2-1}\mathrm{d}a_j\mathrm{d}a_N\,.$$

$$\mathrm{d}\mathbb{P}_C(\theta_1,\ldots,\theta_N) = (\mathcal{Z}_N^E(0,\widetilde{\beta}))^{-1}|\Delta(e^{i\theta})|^{\widetilde{\beta}}\mathrm{d}\theta_1\ldots\mathrm{d}\theta_N\,, \quad \theta_j\in[-\pi,\pi)\,,$$

$$\mathrm{d}\mathbb{P}_\alpha(a_1,\ldots,a_N) = (Z_N^E(0,\widetilde{\beta}))^{-1}\prod_{j=1}^{N-1}\left(1-|a_j|^2\right)^{\widetilde{\beta}(N-j)/2-1}\mathrm{d}a_j\mathrm{d}a_N\,.$$

$$\mathrm{d}\mathbb{P}_C(\theta_1,\ldots,\theta_N) = (\mathcal{Z}_N^E(V,\widetilde{\beta}))^{-1}|\Delta(e^{i\theta})|^{\widetilde{\beta}}\exp\left(-\sum_{j=1}^N V(e^{i\theta_j})\right)\mathrm{d}\theta_1\ldots\mathrm{d}\theta_N\,,$$

$$\mathrm{d}\mathbb{P}_\alpha(a_1,\ldots,a_N) = (Z_N^E(V,\widetilde{\beta}))^{-1}\prod_{j=1}^{N-1}\left(1-|a_j|^2\right)^{\widetilde{\beta}(N-j)/2-1}\exp\left(-\mathtt{Tr}(V(E))\right)\mathrm{d}a_j\mathrm{d}a_N.$$

$$\mathrm{d}\mathbb{P}_C\,(\theta_1,\ldots,\theta_N) = (\mathcal{Z}_N^E(0,\widetilde{\beta}))^{-1}|\Delta(e^{i\theta})|^{\widetilde{\beta}}\mathrm{d}\theta_1\ldots\mathrm{d}\theta_N\,,\quad \theta_j\in[-\pi,\pi)\,,$$

$$\mathrm{d}\mathbb{P}_\alpha(a_1,\ldots,a_N) = (Z_N^E(0,\widetilde{\beta}))^{-1}\prod_{j=1}^{N-1}\left(1-|a_j|^2\right)^{\widetilde{\beta}(N-j)/2-1}\mathrm{d}a_j\mathrm{d}a_N\,.$$

$$\mathrm{d}\mathbb{P}_C\,(\theta_1,\ldots,\theta_N) = (\mathcal{Z}_N^E(V,\widetilde{\beta}))^{-1}|\Delta(e^{i\theta})|^{\widetilde{\beta}}\exp\left(-\sum_{j=1}^N V(e^{i\theta_j})\right)\mathrm{d}\theta_1\ldots\mathrm{d}\theta_N\,,$$

$$\mathrm{d}\mathbb{P}_\alpha(a_1,\ldots,a_N) = (Z_N^E(V,\widetilde{\beta}))^{-1}\prod_{j=1}^{N-1}\left(1-|a_j|^2\right)^{\widetilde{\beta}(N-j)/2-1}\exp\left(-\mathtt{Tr}(V(E))\right)\mathrm{d}a_j\mathrm{d}a_N.$$

The last one looks similar to

$$\mu_{AL} = Z_N^{AL}(V,\beta))^{-1}\prod_{j=1}^N(1-|a_j|^2)^{\beta-1}\exp(-\mathtt{Tr}(V(\mathcal{E})))\mathrm{d}^2\boldsymbol{a}\,,\quad a_j\in\mathbb{D}\,,$$

High temperature regime - $\widetilde{\beta} = \frac{2\beta}{N}$

$$\mathrm{d}\mathbb{P}_\alpha(a_1, \ldots, a_N) = \frac{\prod_{j=1}^{N-1} \left(1 - |a_j|^2\right)^{\beta(1-j/N)-1} \exp\left(-\mathtt{Tr}(V(E))\right) \mathrm{d}a_j \mathrm{d}a_N}{Z_N^E\left(V, \frac{2\beta}{N}\right)}.$$

High temperature regime - $\widetilde{\beta} = \frac{2\beta}{N}$

$$\mathrm{d}\mathbb{P}_\alpha(a_1,\dots,a_N) = \frac{\prod_{j=1}^{N-1}\left(1-|a_j|^2\right)^{\beta(1-j/N)-1}\exp\left(-\mathtt{Tr}(V(E))\right)\mathrm{d}a_j\mathrm{d}a_N}{Z_N^E\left(V,\frac{2\beta}{N}\right)}.$$

Theorem (Hardy, and Lambert)

Let $\beta > 0$, and $V : \mathbb{T} \to \mathbb{R}$ continuous. Then

- the sequence $\mu_N(E) = \frac{1}{N}\sum_{j=1}^N \delta_{e^{i\theta_j}}$ satisfies a large deviation principle, and in particular

$$\mu_N(E) \xrightarrow{a.s.} \mu_\beta^V,$$

- $\mu_\beta^V \in \mathcal{P}(\mathbb{T})$, and it is the unique minimizer of the functional

$$I^{(V,\beta)}(\rho) = \int_{\mathbb{T}} V(\theta)\rho(\theta)\mathrm{d}\theta - \beta\int\int_{\mathbb{T}\times\mathbb{T}} \log\sin\left(\left|e^{i\theta}-e^{i\phi}\right|\right)\rho(\theta)\rho(\phi)\mathrm{d}\theta\mathrm{d}\phi$$
$$+\int_{\mathbb{T}}\log\left(\rho(\theta)\right)\rho(\theta)\mathrm{d}\theta + \log(2\pi).$$

Recap

$$\mu_{AL} = (Z_N^{AL}(V,\beta))^{-1} \prod_{j=1}^{N} (1-|a_j|^2)^{\beta-1} \exp(-\mathrm{Tr}(V(\mathcal{E})))\mathrm{d}^2\boldsymbol{a}\,.$$

$$\mathrm{d}\mathbb{P}_\alpha = \left(Z_N^E\left(V,\frac{2\beta}{N}\right)\right)^{-1} \prod_{j=1}^{N-1} \left(1-|a_j|^2\right)^{\beta(1-j/N)-1} \exp\left(-\mathrm{Tr}(V(E))\right) \mathrm{d}a_j \mathrm{d}a_N\,,$$

$$\mu_N(E) \xrightarrow{a.s.} \mu_\beta^V$$

$\mu_\beta^V \in \mathcal{P}(\mathbb{T})$, and it is the unique minimizer of $I^{(V,\beta)}(\rho)$.
The structure of $E, \mathcal{E}$ is similar.

Recap

$$\mu_{AL} = (Z_N^{AL}(V, \beta))^{-1} \prod_{j=1}^{N} (1 - |a_j|^2)^{\beta - 1} \exp(-\mathtt{Tr}(V(\mathcal{E}))) \mathrm{d}^2 \boldsymbol{a} \,.$$

$$\mathrm{d}\mathbb{P}_\alpha = \left(Z_N^E \left(V, \frac{2\beta}{N} \right) \right)^{-1} \prod_{j=1}^{N-1} \left(1 - |a_j|^2\right)^{\beta(1-j/N)-1} \exp\left(-\mathtt{Tr}(V(E))\right) \mathrm{d}a_j \mathrm{d}a_N \,,$$

$$\mu_N(E) \xrightarrow{a.s.} \mu_\beta^V$$

$\mu_\beta^V \in \mathcal{P}(\mathbb{T})$, and it is the unique minimizer of $I^{(V,\beta)}(\rho)$.
The structure of $E, \mathcal{E}$ is similar.

Question

Can we recover, or at least characterize, the density of states ν_β^V, in terms of μ_β^V?

First result

Theorem G.M., and T. Grava

Let $\beta > 0$, $V : \mathbb{T} \to \mathbb{R}$ a Laurent polynomial. Then the mean density of states of the Ablowitz-Ladik lattice ν_β^V can be computed explicitly as

$$\nu_\beta^V = \partial_\beta(\beta \mu_\beta^V)\,,$$

where μ_β^V is the unique minimizer of the functional

$$I^{(V,\beta)}(\rho) = \int_{\mathbb{T}} V(\theta)\rho(\theta)\mathrm{d}\theta - \beta \int\int_{\mathbb{T}\times\mathbb{T}} \log\sin\left(\left|e^{i\theta} - e^{i\phi}\right|\right)\rho(\theta)\rho(\phi)\mathrm{d}\theta\mathrm{d}\phi$$
$$+ \int_{\mathbb{T}} \log\left(\rho(\theta)\right)\rho(\theta)\mathrm{d}\theta + \log(2\pi)\,.$$

- Independently, Spohn obtained the same result.

Generalization

Theorem G.M., and R. Memin

Let $\beta > 0$, $V : \mathbb{T} \to \mathbb{R}$ a continuous and bounded function. Then the mean density of states of the Ablowitz-Ladik lattice ν_β^V can be computed explicitly as

$$\nu_\beta^V = \partial_\beta(\beta\mu_\beta^V)\,,$$

where μ_β^V is the unique minimizer of the functional

$$I^{(V,\beta)}(\rho) = \int_{\mathbb{T}} V(\theta)\rho(\theta)\mathrm{d}\theta - \beta \int\int_{\mathbb{T}\times\mathbb{T}} \log\sin\left(\left|e^{i\theta} - e^{i\phi}\right|\right)\rho(\theta)\rho(\phi)\mathrm{d}\theta\mathrm{d}\phi$$
$$+ \int_{\mathbb{T}} \log\left(\rho(\theta)\right)\rho(\theta)\mathrm{d}\theta + \log(2\pi)\,.$$

M-Grava	M-Memin
Transfer operator technique Moment method	Large deviations principles

M-Grava	M-Memin
Transfer operator technique Moment method	Large deviations principles

LDP

Definition

A sequence of measure μ_ε on a topological vector space $(\mathcal{X}, \mathcal{B})$ satisfy a large deviations principle with rate function $I : \mathcal{X} \to [0, \infty)$ if $\forall \Gamma \in \mathcal{B}$

$$-\inf_{x \in \mathring{\Gamma}} I(x) \leq \liminf_{\varepsilon \to 0} \varepsilon \ln\left(\mu_\varepsilon(\Gamma)\right) \leq \limsup_{\varepsilon \to 0} \varepsilon \ln\left(\mu_\varepsilon(\Gamma)\right) \leq -\inf_{x \in \overline{\Gamma}} I(x)$$

LDP

Definition

A sequence of measure μ_ε on a topological vector space $(\mathcal{X}, \mathcal{B})$ satisfy a large deviations principle with rate function $I : \mathcal{X} \to [0, \infty)$ if $\forall \Gamma \in \mathcal{B}$

$$-\inf_{x \in \mathring{\Gamma}} I(x) \leq \liminf_{\varepsilon \to 0} \varepsilon \ln \left(\mu_\varepsilon(\Gamma)\right) \leq \limsup_{\varepsilon \to 0} \varepsilon \ln \left(\mu_\varepsilon(\Gamma)\right) \leq -\inf_{x \in \overline{\Gamma}} I(x)$$

Remarks:

- Since $\mu_\varepsilon(\mathcal{X}) = 1$, then $\inf_{x \in \mathcal{X}} I(x) = 0$.

LDP

Definition

A sequence of measure μ_ε on a topological vector space $(\mathcal{X}, \mathcal{B})$ satisfy a large deviations principle with rate function $I : \mathcal{X} \to [0, \infty)$ if $\forall \Gamma \in \mathcal{B}$

$$-\inf_{x \in \mathring{\Gamma}} I(x) \leq \liminf_{\varepsilon \to 0} \varepsilon \ln\left(\mu_\varepsilon(\Gamma)\right) \leq \limsup_{\varepsilon \to 0} \varepsilon \ln\left(\mu_\varepsilon(\Gamma)\right) \leq -\inf_{x \in \overline{\Gamma}} I(x)$$

Remarks:

- Since $\mu_\varepsilon(\mathcal{X}) = 1$, then $\inf_{x \in \mathcal{X}} I(x) = 0$.
- Assume that $I(x)$ has a global minimum x_0, let $x_0 \in U$ then there exists an $\alpha > 0$ such that

$$\mu_\varepsilon(U^c) \leq e^{-\frac{\alpha}{\varepsilon}}$$

meaning that all the probability is concentrating around x_0.

Example of LDP

Consider the sequence of atomic measures

$$\mu_N = \frac{1}{N}\sum_{j=1}^{N} \delta_{\lambda_j}\,, \quad \lambda_j \in \Omega$$

If this sequence satisfies an LDP, it means that there exists

$$I \,:\, \mathcal{P}(\Omega) \to [0,\infty)$$

such that $\forall \Gamma \in \mathcal{B}$ (topology of weak convergence)

$$-\inf_{\rho\in\mathring{\Gamma}} I(\rho) \leq \liminf_{N\to\infty} \frac{1}{N} \ln\left(\mathbb{P}\left(\mu_N \in \Gamma\right)\right) \leq \limsup_{N\to\infty} \frac{1}{N} \ln\left(\mathbb{P}\left(\mu_N \in \Gamma\right)\right) \leq -\inf_{\rho\in\overline{\Gamma}} I(\rho)$$

Example of LDP

Consider the sequence of atomic measures

$$\mu_N = \frac{1}{N}\sum_{j=1}^{N} \delta_{\lambda_j}\,, \quad \lambda_j \in \Omega$$

If this sequence satisfies an LDP, it means that there exists

$$I \,:\, \mathcal{P}(\Omega) \to [0,\infty)$$

such that $\forall \Gamma \in \mathcal{B}$ (topology of weak convergence)

$$-\inf_{\rho \in \mathring{\Gamma}} I(\rho) \leq \liminf_{N\to\infty} \frac{1}{N} \ln\left(\mathbb{P}\left(\mu_N \in \Gamma\right)\right) \leq \limsup_{N\to\infty} \frac{1}{N} \ln\left(\mathbb{P}\left(\mu_N \in \Gamma\right)\right) \leq -\inf_{\rho \in \overline{\Gamma}} I(\rho)$$

Assuming that I has a unique minimizer ρ_0, the existence of an LDP implies that

$$\mu_N \xrightarrow{N\to\infty} \rho_0$$

Back to the Theorem

Theorem G.M., and R. Memin

Let $\beta > 0$, $V : \mathbb{T} \to \mathbb{R}$ a continuous and bounded function. Then the mean density of states of the Ablowitz-Ladik lattice ν_β^V can be computed explicitly as

$$\nu_\beta^V = \partial_\beta(\beta\mu_\beta^V),$$

where μ_β^V is the unique minimizer of the functional

$$I^{(V,\beta)}(\rho) = \int_{\mathbb{T}} V(\theta)\rho(\theta)\mathrm{d}\theta - \beta \int\int_{\mathbb{T}\times\mathbb{T}} \log\sin\left(\left|e^{i\theta} - e^{i\phi}\right|\right)\rho(\theta)\rho(\phi)\mathrm{d}\theta\mathrm{d}\phi$$
$$+ \int_{\mathbb{T}} \log\left(\rho(\theta)\right)\rho(\theta)\mathrm{d}\theta + \log(2\pi).$$

Ideas of the proof M.-Memin

We proved a large deviation principle for the family of empirical measures $\mu_N(\mathcal{E}) = \frac{1}{N}\sum_{j=1}^{N} \delta_{e^{i\theta_j}}$, implying that

$$\mu_N(\mathcal{E}) \xrightarrow[N\to\infty]{} \nu_\beta^V \, ,$$

and ν_β^V is the unique minimizer of the functional

$$J^{(V,\beta)} \; : \; \mathcal{P}(\mathbb{T}) \to [0;\infty) \, .$$

Ideas of the proof M.-Memin

We proved a large deviation principle for the family of empirical measures $\mu_N(\mathcal{E}) = \frac{1}{N}\sum_{j=1}^N \delta_{e^{i\theta_j}}$, implying that

$$\mu_N(\mathcal{E}) \xrightarrow[N\to\infty]{} \nu_\beta^V ,$$

and ν_β^V is the unique minimizer of the functional

$$J^{(V,\beta)} \; : \; \mathcal{P}(\mathbb{T}) \to [0;\infty) .$$

How:

- We applied a subadditivity argument to prove an LDP for the Gibbs measure $\mu_{AL} = Z_N^{AL}(0,\beta)^{-1}\prod_{j=1}^N (1-|a_j|^2)^{\beta-1}\mathrm{d}^2\boldsymbol{a}$
- We exploited Varadhan's lemma to generalize the LDP to continuous potential.

Ideas of the proof M.-Memin

We proved a large deviation principle for the family of empirical measures $\mu_N(\mathcal{E}) = \frac{1}{N}\sum_{j=1}^{N} \delta_{e^{i\theta_j}}$, implying that

$$\mu_N(\mathcal{E}) \xrightarrow[N\to\infty]{} \nu_\beta^V ,$$

and ν_β^V is the unique minimizer of the functional

$$J^{(V,\beta)} \; : \; \mathcal{P}(\mathbb{T}) \to [0;\infty) .$$

Moreover, we proved that we can rewrite the functional of Lambert, and Hardy $I^{(V,\beta)}$ (which is minimized by μ_β^V) as

$$I^{(V,\beta)}(\mu) = \lim_{\delta\to 0} \liminf_{q\to\infty} \inf_{\substack{\nu_{\beta/q},\dots,\nu_\beta \\ \frac{1}{q}\sum_i \nu_{i\beta/q} \in B_\mu(\delta)}} \left\{ \frac{1}{q} \sum_{i=1}^{q} J^{(V,i\beta/q)}(\nu_{i\beta/q}) \right\} ,$$

Ideas of the proof M.-Memin

We proved a large deviation principle for the family of empirical measures $\mu_N(\mathcal{E}) = \frac{1}{N}\sum_{j=1}^N \delta_{e^{i\theta_j}}$, implying that

$$\mu_N(\mathcal{E}) \xrightarrow[N\to\infty]{} \nu_\beta^V \,,$$

and ν_β^V is the unique minimizer of the functional

$$J^{(V,\beta)} \ : \ \mathcal{P}(\mathbb{T}) \to [0;\infty)\,.$$

Moreover, we proved that we can rewrite the functional of Lambert, and Hardy $I^{(V,\beta)}$ (which is minimized by μ_β^V) as

$$I^{(V,\beta)}(\mu) = \lim_{\delta\to 0} \liminf_{q\to\infty} \inf_{\substack{\nu_{\beta/q},\dots,\nu_\beta \\ \frac{1}{q}\sum_i \nu_{i\beta/q}\in B_\mu(\delta)}} \left\{ \frac{1}{q}\sum_{i=1}^q J^{(V,i\beta/q)}(\nu_{i\beta/q}) \right\} ,$$

which implies that

$$\int_0^1 \nu_{t\beta}^V dt = \mu_\beta^V \implies \nu_\beta^V = \partial_\beta(\beta\mu_\beta^V)$$

Explicit density

To obtain explicit expression for the density of the AL lattice and the CβE at high temperature, we have to minimize

$$I^{(V,\beta)}(\rho) = \int_{\mathbb{T}} V(\theta)\rho(\theta)\mathrm{d}\theta - \beta \int\int_{\mathbb{T}\times\mathbb{T}} \log \sin\left(|e^{i\theta} - e^{i\phi}|\right)\rho(\theta)\rho(\phi)\mathrm{d}\theta\mathrm{d}\phi + \int_{\mathbb{T}} \log\left(\rho(\theta)\right)\rho(\theta)\mathrm{d}\theta + \log(2\pi)\,.$$

Explicit density

To obtain explicit expression for the density of the AL lattice and the CβE at high temperature, we have to minimize

$$I^{(V,\beta)}(\rho) = \int_{\mathbb{T}} V(\theta)\rho(\theta)\mathrm{d}\theta - \beta \int\int_{\mathbb{T}\times\mathbb{T}} \log\sin\left(|e^{i\theta} - e^{i\phi}|\right)\rho(\theta)\rho(\phi)\mathrm{d}\theta\mathrm{d}\phi$$
$$+ \int_{\mathbb{T}} \log\left(\rho(\theta)\right)\rho(\theta)\mathrm{d}\theta + \log(2\pi)\,.$$

- Hardy, and Lambert proved that if $V(\theta) = 0$, then $\mu_\beta^0 = \frac{1}{2\pi}$, so $\nu_\beta^0 = \frac{1}{2\pi}$

Explicit density

To obtain explicit expression for the density of the AL lattice and the CβE at high temperature, we have to minimize

$$I^{(V,\beta)}(\rho) = \int_{\mathbb{T}} V(\theta)\rho(\theta)\mathrm{d}\theta - \beta \int\int_{\mathbb{T}\times\mathbb{T}} \log\sin\left(|e^{i\theta} - e^{i\phi}|\right)\rho(\theta)\rho(\phi)\mathrm{d}\theta\mathrm{d}\phi + \int_{\mathbb{T}} \log\left(\rho(\theta)\right)\rho(\theta)\mathrm{d}\theta + \log(2\pi)\,.$$

- Hardy, and Lambert proved that if $V(\theta) = 0$, then $\mu_\beta^0 = \frac{1}{2\pi}$, so $\nu_\beta^0 = \frac{1}{2\pi}$
- G.M. and T. Grava focused on the case $V(\theta) = 2\eta\cos(\theta)$, it is important since it is related to the classical Gibbs ensemble, and generalize the result of J Baik, P Deift, K Johansson for the Circular β ensemble.

Theorem - G.M., and T. Grava

Fix $\beta > 0$ and let $V(e^{i\theta}) = \eta \cos\theta$, where $\eta \in \mathbb{R}$. There exists $\varepsilon > 0$ such that for all $\eta \in (-\varepsilon, \varepsilon)$, the mean densities of states of the Circular beta ensemble in the high temperature regime μ_β^V, and the Ablowitz-ladik lattice ν_β^V read

$$\mu_\beta^V(\theta) = \frac{1}{2\pi} + \frac{1}{\pi\beta}\Re\left(\frac{zv'(z)}{v(z)}\right)_{\big|z=e^{i\theta}}, \quad \nu_\beta^V = \frac{1}{2\pi} + \partial_\beta\left(\frac{1}{\pi}\Re\left(\frac{zv'(z)}{v(z)}\right)_{\big|z=e^{i\theta}}\right).$$

where $v(z)$ is the unique analytic solution at 0 of the double confluent Heun equation

$$z^2 v''(z) + \left(-\eta + z(\beta+1) + \eta z^2\right) v'(z) + \eta\beta(z+\lambda)v(z) = 0\,,$$

and λ is determined as the unique solution of the transcendental equation $\xi(\beta, \eta, \lambda) = 0$.

Open problems

Rigorous computation of the correlation functions. Despite having explicit solutions via finite-gap integration, and several insights for the GHD theory (Doyon, Spohn, El), a mathematically rigorous computation of those remains out of reach.

Open problems

Rigorous computation of the correlation functions. Despite having explicit solutions via finite-gap integration, and several insights for the GHD theory (Doyon, Spohn, El), a mathematically rigorous computation of those remains out of reach.

β-ensemble at high temperature	**Integrable System**
Gaussian	Toda lattice (Spohn; Guionnet-Memin)
Circular	Defocusing Ablowitz-Ladik lattice (Grava-M.; Memin-M.)
Laguerre	Exponential Toda lattice (Gisonni-Grava-Gubbiotti-M.)
Jacobi	Defocusing Schur flow (Spohn; Memin-M.)
Antisymmetric Gaussian	Volterra lattice (Gisonni-Grava-Gubbiotti-M.)

Open problems

Rigorous computation of the correlation functions. Despite having explicit solutions via finite-gap integration, and several insights for the GHD theory (Doyon, Spohn, El), a mathematically rigorous computation of those remains out of reach.

β**-ensemble at high temperature**	**Integrable System**
Gaussian	Toda lattice (Spohn; Guionnet-Memin)
Circular	Defocusing Ablowitz-Ladik lattice (Grava-M.; Memin-M.)
Laguerre	Exponential Toda lattice (Gisonni-Grava-Gubbiotti-M.)
Jacobi	Defocusing Schur flow (Spohn; Memin-M.)
Antisymmetric Gaussian	Volterra lattice (Gisonni-Grava-Gubbiotti-M.)
(??)2D β ensemble at high temperature	Focusing Ablowitz-Ladik and focusing mKdV

Thank you for the attention!