Universality for free fermions point processes

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1. Probabilistic model of free fermions on $\mathbb{R}^d$.
2. Relation to random matrix theory.
3. Limits of the correlation functions.
?. Central limit theorem.

1 Model

Quantum state of a particle is described by a wave function $\Psi \in L^2(\mathbb{R}^3)$.

$|\Psi(x)|^2 =$ p.d.f. for the position x

$|\hat{\Psi}(\xi)|^2 =$ p.d.f. for the momentum ξ $(m=1)$

Many-body system $\Psi \in L^2(\mathbb{R}^{3N})$ N particles.

$|\Psi(x_1, \dots, x_N)|^2$ is a probability measure invariant under S_N.

$\Psi \in L^2_A(\mathbb{R}^{3N})$ "Antisymmetric" $\boxed{\text{Fermions}}$

Quantum evolution is given by the Schrödinger eq.

$$i\hbar \frac{d}{dt}\Psi = H_N \Psi$$

Planck cst $\hbar \simeq 10^{-34}$ J·s

Hamiltonian - operator on $L^3_A(\mathbb{R}^{3N})$ $\boxed{H \geq 0}$

<u>Stationary states</u> : $H\Psi = \Lambda\Psi$

energy of the state

$$H_N = \sum_{j=1}^{N}\left(-\hbar^2\Delta_{x_j} + V(x_j)\right) + \frac{\beta}{N}\sum_{i<j} W(x_i, x_j)$$

kinetic energy

trap (external potential $V: \mathbb{R}^d \to \mathbb{R}_+$ continuous)

$\beta = 0$ – non-interacting or free fermions.

H_N has pure point spectrum $\{\Lambda_j\}_{j\in\mathbb{N}_0}$; $\Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq \cdots$

$\hbar \propto N^{-1/d}$; $\Lambda_1 \simeq N\lambda_1$.

<u>Ground state</u> : $\langle\Psi | H_N \Psi\rangle = \lambda_1$ <u>unique</u>

<u>Slater det</u>: $\Psi(x_1, \dots, x_N) = \frac{1}{\sqrt{N!}} \det_{N\times N}\left(\phi_i(x_j)\right)$

$\{\phi_i\}$ orthonormal family in $L^2(\mathbb{R}^3)$

$$\underbrace{(-\hbar^2\Delta + V)}_{H}\phi_j = \varepsilon_j\phi_j \quad ; \quad \varepsilon_1(\hbar) \leq \varepsilon_2(\hbar) \leq \cdots$$

Fix $\mu > 0$ and let $N = \#\{i : \varepsilon_i(\hbar) \leq \mu\}$.

Fermi energy

<u>Limit as $\hbar \to 0$.</u>

$\hbar$ = "microscopic scale".

$$\boxed{N \simeq \left(\frac{\omega_d}{2\pi\hbar}\right)^d Z_\mu}$$

Probability on $\mathbb{R}^{3N}$: $\mathbb{P}_N(x_1, \dots, x_N) = \frac{1}{N!} \left| \det_{N\times N} (\phi_i(x_j)) \right|^2$

$$N = \mathrm{Tr}(\Pi_N) \sim \left(\frac{\omega_d}{2\pi\hbar}\right)^d Z_\mu \text{ as } \hbar \to 0 \qquad = \frac{1}{N!} \det_{N\times N} [\Pi_N(x_i, x_j)]$$

$$\boxed{\Pi_N(x,y) = \sum_{i=1}^{N} \phi_i(x) \phi_j(y)}$$

- Like an orthogonal polynomial ensemble.

$\to$ $\mathbb{P}_N$ is a det. point process on $\mathbb{R}^d$ with kernel Π_N

1) The correlation fcts of the measure $\mathbb{P}_N$ are

$$\rho_k(x_1, \dots, x_k) = \det_{k\times k} [\Pi_N(x_i, x_j)] \quad \text{for } k \in \mathbb{N}.$$

2) For $g : \mathbb{R}^d \to \mathbb{R}_+$

$$\mathbb{E}_N \left[\prod_{j=1}^{N} g(x_j) \right] = \det_{L^2(\mathbb{R}^d)} [1 + (g-1) \widehat{\Pi}_N].$$

3)

$$\boxed{\widehat{\Pi}_N = \mathbb{1}_{\{H \leq \mu\}}} \quad \text{orthogonal proj.}$$

- No recurrence relations for $(\phi_i)_{i \in \mathbb{N}}$

(2) Random matrices ($d=1$)

Proposition: The GUE is the ground state of a free Fermi gas confined by $V(x)=x^2$ on $\mathbb{R}$ with $\begin{cases} \hbar = 1/2N \\ \mu = 1 \end{cases}$.

Idea: Hermite functions satisfy the ODE

$$\left(-\frac{1}{4N^2}\Delta + x^2\right)\phi_j = \frac{j-\frac{1}{2}}{N}\phi_j \quad ; \quad j \in \mathbb{N}$$

Thm: For GUE, as $\hbar \to 0$ (or $N\to\infty$)

- $\mu_N = \frac{1}{N}\sum_{j=1}^{N}\delta_{x_j} \longrightarrow \frac{2}{\pi}\sqrt{1-x^2}\,dx$.
- For $x\in(-1,1)$; $\frac{1}{\rho(x)N}K_N\left(x+\frac{u}{\rho(x)N}, x+\frac{v}{\rho(x)N}\right) \longrightarrow K_{\text{sine}}(u,v)$.
- For $x\in\{\pm1\}$; $\frac{1}{\rho(x)N}K_N\left(x\pm\frac{u}{(2N)^{2/3}}, x\pm\frac{v}{(2N)^{2/3}}\right) \longrightarrow K_{\text{Airy}}(u,v)$.

$$\boxed{\widehat{K_{\text{sine}}} = \mathbb{1}_{\{-\Delta\leq\pi\}}} \qquad \boxed{\widehat{K_{\text{Airy}}} = \mathbb{1}_{\{-\Delta+x\leq 0\}}}$$

Analogous results for free fermions in a general trap? Universality?

[Eisler], [Dean, Majumdar, Le Doussal, Schehr, etc]

[Berman], [Bornemann], [Hanin, Zelditch], [Helffer - Robert]

Weyl's law

$$\begin{cases} \text{Number of particles} \quad N = \mathrm{Tr}(\Pi_N) \sim \left(\frac{\omega_d}{2\pi\hbar}\right)^d Z_\mu \quad \text{as } \hbar \to 0 \\ N^{-1}\Pi_N(x,x) \longrightarrow \rho(x) = \frac{1}{Z_\mu}(\mu - V(x))_+^{d/2} \quad \text{as } \hbar \to 0. \end{cases}$$

Thm 1 (Law of Large number - Density of states)

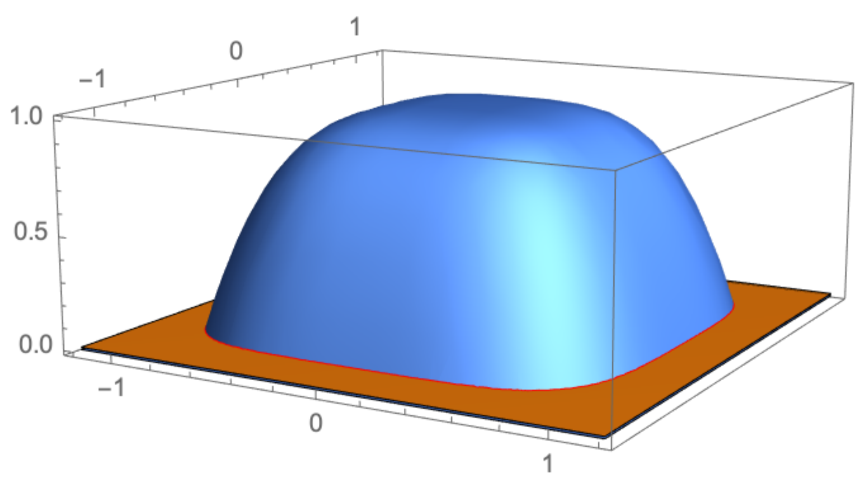
For any $\varepsilon > 0$,

$$\mathbb{P}_N\left[\mathrm{dist}\left(\frac{1}{N}\sum_{j=1}^N \delta_{x_j}, \rho\right) \geq \varepsilon\right] \leq C_\varepsilon e^{-cN\varepsilon^2}.$$

Ex: 1) $V(x) = x^2$ on $\mathbb{R}$ and $\mu = 1$ (GUE)

$$\rho(x) = \frac{2}{\pi}\sqrt{1-x^2}.$$

2) $V(x) = |x|^4$ on $\mathbb{R}^2$ and $\mu = 1$



Semiclassical analysis

$$H = -\hbar^2\Delta + V = \mathrm{OP}_\hbar\left((x,\xi)\to|\xi|^2+V(x)\right)$$

$$\mathrm{OP}_\hbar(a)(x,y)\mapsto \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d} e^{i\frac{(x-y)\cdot\xi}{\hbar}}\, a\left(\tfrac{x+y}{2},\xi\right)d\xi$$

- If $a\in C_c^\infty(\mathbb{R}^{2d})$, $\mathrm{OP}_\hbar(a)$ is trace-class.
- In general a is a symbol:

$$a(x,\xi;\hbar) = \underbrace{a_0(x,\xi)+\hbar\, a_1(x,\xi)+\hbar^2 a_2(x,\xi)+\dots}_{a^2(x,\xi;\hbar)}$$

For any $k\in\mathbb{N}_0$,

$$\mathrm{OP}_\hbar(a)(x,y)\mapsto \frac{1}{(2\pi\hbar)^d}\left\{\int_{\mathbb{R}^d} e^{i\frac{(x-y)\cdot\xi}{\hbar}}\, a^k\left(\tfrac{x+y}{2},\xi,\hbar\right) + \underbrace{\mathcal{O}(\hbar^{k+1})}_{\text{in trace norm}}\right\}$$

<u>Functional calculus:</u> For any $\varphi\in\mathcal{C}_c^\infty(\mathbb{R}\to\mathbb{R}_+)$

$$\varphi(-\hbar^2\Delta+V) = \mathrm{OP}_\hbar(a)$$

where $a_0(x,\xi)=\varphi(|\xi|^2+V(x))$ & a_k are supported on a small neighborhood of $\{(x,\xi):\varphi(|\xi|^2+V(x))>0\}$.

Approximation of free fermions kernel:

$$\widehat{\Pi}_N = \mathbb{1}_{\{H\le\mu\}}\ (x,y)\mapsto \frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d} e^{i\frac{(x-y)\cdot\xi}{\hbar}}\, \mathbb{1}_{\{|\xi|^2+V(\frac{x+y}{2})\le\mu\}}\, d\xi$$

Weyl's law

$$\Pi_N(x,x) \simeq \frac{1}{(2\pi\hbar)^d}\int \mathbb{1}_{\{|\xi|^2+V(x)\leq\mu\}}\,d\xi = \left(\frac{\omega_d}{2\pi\hbar}\right)^d (\mu-V(x))_+^{d/2}$$

$$\omega_d = \left(\int_{\mathbb{R}^d}\mathbb{1}_{\{|\xi|\leq 1\}}\,d\xi\right)^{1/d} \quad ; \quad \mathrm{Tr}(\hat{\Pi}_N) \simeq \left(\frac{\omega_d}{2\pi\hbar}\right)^d \underbrace{\int (\mu-V(x))_+^{d/2}\,dx}_{Z_\mu}$$

Thm 2 (Limits of correlation functions)

- For $z\in\{V<\mu\}$, as $\hbar\to 0$,

$$\left\{(x_j-z)(N\rho(z))^{\frac{1}{d}}\right\}_{j=1}^N \xrightarrow{d} \text{determinantal point process on } \mathbb{R}^d \text{ with op. } \hat{K}_d = \mathbb{1}_{\{-\Delta\leq\frac{2\pi}{\omega_d}\}}.$$

- For $z\in\{V=\mu,\ \nabla V\neq 0\}$, as $\hbar\to 0$,

$$\left\{(x_j-z)\,\hbar^{2/3}\right\}_{j=1}^N \xrightarrow{d} \text{determinantal point process on } \mathbb{R}^d \text{ with op. } \hat{K}_{Ai,d} = \mathbb{1}_{\{-\Delta+\nabla V(z)\cdot x\leq 0\}}.$$

Bulk kernel:

$$K_d(u,v) = \int_{\mathbb{R}^d}\mathbb{1}_{\{|\xi|\leq\frac{1}{\omega_d}\}}\,e^{2\pi i\,\xi\cdot(u-v)}\,d\xi = \frac{J_{d/2}\left(\frac{2\pi|u-v|}{\omega_d}\right)}{(\omega_d|u-v|)^{d/2}}$$

In 1d, $J_{1/2}(r) = \sqrt{\frac{2}{\pi r}}\sin(r)$.

$$|K_d(u,v)|^2 \sim \frac{1}{\omega_d|u-v|^{d+1}}\cdot\frac{\sin^2(\cdot)}{\pi^2} \quad \text{as } |u-v|\to\infty.$$

<u>Sketch</u>: The kernel of the rescaled point process $\left\{ \frac{x_j - z}{\varepsilon} \right\}_{j=1}^{N}$ is

$$\varepsilon^d \Pi_N(z+\varepsilon u, z+\varepsilon v) \simeq \frac{\varepsilon^d}{(2\pi\hbar)^d} \int_{\mathbb{R}^d} e^{i\varepsilon \frac{(u-v)\cdot\xi}{\hbar}} \mathbb{1}_{\{|\xi|^2 + V(z+\frac{u+v}{2}\varepsilon) \leq \mu\}} d\xi$$

$\boxed{\varepsilon = \hbar/a}$

$$\simeq \frac{1}{(2\pi a)^d} \int_{\mathbb{R}^d} e^{i\frac{(u-v)\cdot\xi}{a}} \mathbb{1}_{\{|\xi|^2 \leq \underbrace{\mu - V(z)}_{>0 \text{ if } z \in \text{bulk}} + O(\hbar)\}} d\xi$$

$\boxed{2\pi a = \omega_d \sqrt{\mu - V(z)}}$

$$\simeq \underbrace{\int_{\mathbb{R}^d} e^{i2\pi(u-v)\cdot\xi} \mathbb{1}_{\{|\xi| \leq \frac{1}{\omega_d}\}} d\xi}_{K_d(u,v) \text{ -- density 1 on } \mathbb{R}^d}$$

At the edge, $\boxed{V(z) = \mu}$, the operator associated with the rescaled point process $\left\{ \frac{x_j - z}{\varepsilon} \right\}_{j=1}^{N}$ is

$$\mathbb{1}_{\{-\hbar^2\varepsilon^{-2}\Delta + V(z+\varepsilon\,\cdot) \leq \mu\}} \simeq \underbrace{\mathbb{1}_{\{-\Delta + \nabla V(z)\cdot x \leq 0\}}}_{\equiv \mathbb{1}_{\{-\Delta + x_1 \leq 0\}} \text{ if } \boxed{\nabla V(z) \neq 0}}$$

$\boxed{\varepsilon = \hbar^{2/3}}$

<u>Edge kernel</u>:

$$K_{Ai,d}(x,y) = \int_0^\infty Ai(x_1+s)\,Ai(x_2+s)\, \underbrace{\frac{J_{\frac{d-1}{2}}\left(\sqrt{s}\,|x^\perp - y^\perp|\right)}{\left(2\pi|x^\perp - y^\perp|\right)^{\frac{d-1}{2}}}}_{=1 \text{ if } d=1} s^{\frac{d-1}{2}} ds$$

$x = (x_1, x^\perp)$ & $y = (y_1, y^\perp)$

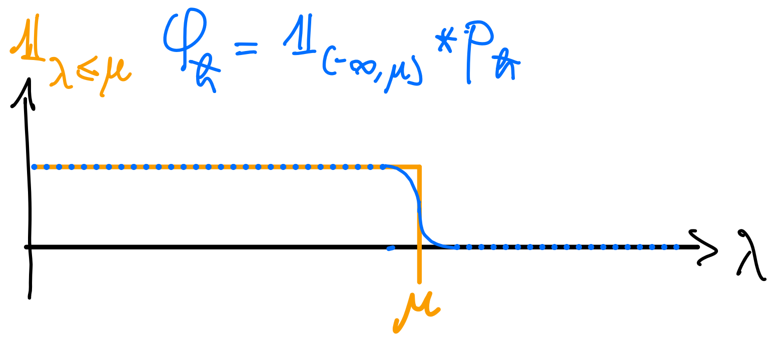
Assumptions: $\mu > 0$ is fixed, $V : \mathbb{R}^d \to \mathbb{R}_+$

- $\{V \leq \mu + \delta\}$ is compact & $V \in C^\infty$ on this set.
- $V \in L^1_{loc}$.

Idea: For $\chi \in \mathcal{C}_c^\infty(\mathbb{R}_+ \to \mathbb{R}_+)$, $t \in [-\tau, \tau]$, $k \in \mathbb{N}$,

$$\chi(H)\, e^{itH/\hbar} : (x,y) \mapsto \frac{1}{(2\pi\hbar)^d} \left(\int_{\mathbb{R}^d} e^{i\frac{\phi_t(x,\xi) - y\cdot\xi}{\hbar}} a_t^k(x,y,\xi;\hbar)\, d\xi + O(\hbar^{k+1}) \right)$$

$$\boxed{\begin{aligned} \phi_0(x,\xi) &= x\cdot\xi, \\ \partial_t \phi_t &= V + |\partial_x \phi|^2 \end{aligned}} \qquad a_0^0(x,x,\xi) = \chi\left(|\xi|^2 + V(x)\right)$$



$$\begin{cases} \rho_\hbar(\cdot) = \hbar^{-1} \rho(\cdot\, \hbar^{-1}) \\ \rho \text{ is a probability measure} \\ \text{on } \mathbb{R} \text{ with } \operatorname{supp}(\rho) \subset [-\tau, \tau] \end{cases}$$

Replace $\hat{\Pi}_N \equiv \mathbb{1}_{\{H \leq \mu\}}$ by $\varphi_\hbar(H)$.

$$\varphi_\hbar(H) : (x,y) \mapsto \frac{1}{(2\pi\hbar)^{d+1}} \left(\int_{\mathbb{R}^{d+2}} \mathbb{1}_{(\lambda \leq \mu)}\, e^{i\frac{\phi_t(x,\xi) - y\cdot\xi - \lambda t}{\hbar}} a_t^k(x,y,\xi;\hbar)\, \hat{\rho}(t)\, d\xi\, dt\, d\lambda + O(\hbar^{k+1}) \right)$$

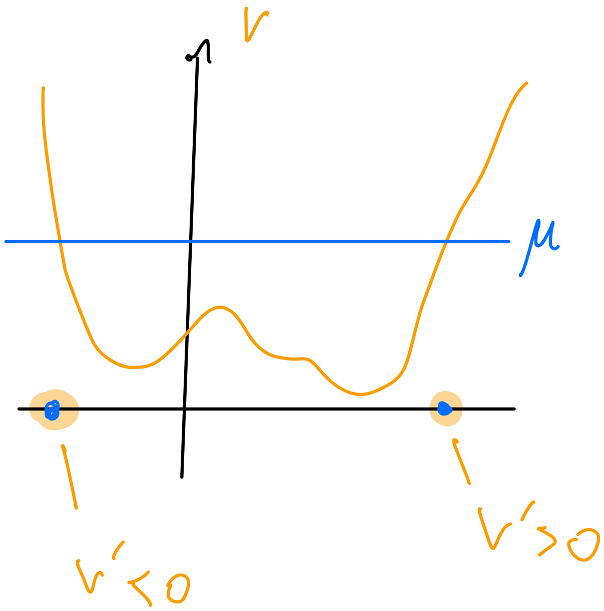
For small t: $\quad \phi_t(x,\xi) = x\cdot\xi + t\left(V(x) + |\xi|^2\right) + O(t^2)$.

CLT in dimension 1

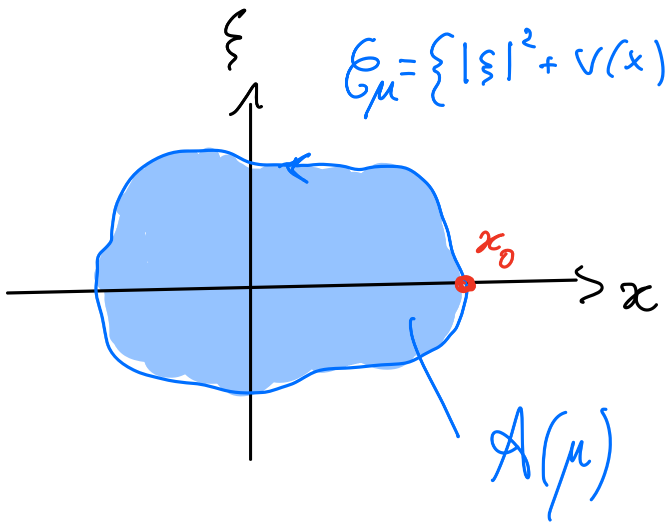
$$\mathbb{E}\left[e^{X(f)}\right] = \det\left[1 + (e^{f}-1)\Pi_N\right] \qquad X := \sum_{j=1}^{N} \delta_{x_j}$$

Thm 3: Central limit theorem

$$\log\det\left(1+(e^{f}-1)\Pi_N\right) = \operatorname{Tr}(f\Pi_N) + \frac{1}{2}\underbrace{\Sigma(f)}_{\text{quadratic fn}} + \cdots$$



$$\mathcal{E}_\mu = \{ |\xi|^2 + V(x) = \mu \}$$



Hamilton's equation of motion

$$\begin{cases} \dfrac{\partial x}{\partial t} = \dfrac{\partial H}{\partial \xi} = 2\xi \\[2ex] \dfrac{\partial \xi}{\partial t} = -\dfrac{\partial H}{\partial x} = -V'(x) \end{cases} \qquad \begin{aligned} &\mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ \Phi_t :\ &(x,\xi) \longrightarrow (x_t, \xi_t) \end{aligned}$$

The flow of Φ is periodic on $\mathcal{E}_\mu$ with period $T_\mu = T$.

For a smooth symbol $a : \mathbb{R}^2 \to \mathbb{R}$, define

$$\hat{a}_k(\mu) = \int_0^T e^{-i\frac{2\pi k t}{T}} a \circ \Phi_t(x_0, 0) \frac{dt}{T} \quad \text{for } k \in \mathbb{Z}.$$

$$\boxed{\Sigma(f) = \sum_{k \in \mathbb{N}_0} k |\hat{a}_k(f)|^2.}$$

Ideas:

- Make a unitary conjugation:
  $$H = -\hbar^2 \Delta + V \longleftrightarrow \mathcal{H} = -\hbar^2 \Delta + x^2.$$
- Find a representation of $\hat{\Pi}_N = \mathbb{1}_{\{H \leq \mu\}}$ as an approximately Toeplitz matrix.
- Apply strong Szegő theorem.

Thank you!

Fluctuations of linear statistics

Thm 2 (Concentration bounds)

Let $f \in C_c^\infty(\{V<\mu\})$. For $t \geq 0$,

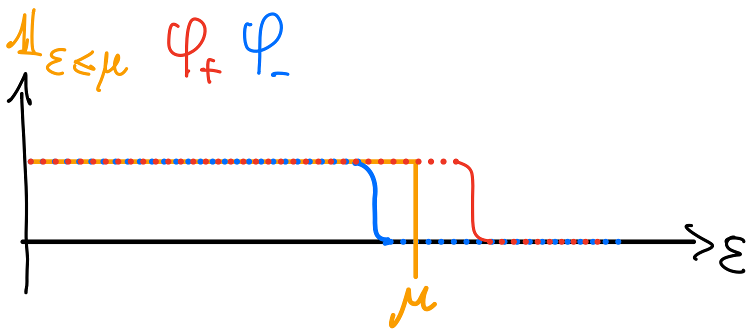
$$\mathbb{P}_N\left[\,\left|\sum_{j=1}^N f(x_j) - \mathbb{E}\sum_{j=1}^N f(x_j)\right| \geq \sqrt{N\hbar}\,t\,\right] \leq 2\,e^{-ct^2}.$$

Thm 3 (Central limit theorem)

For $d \geq 2$, as $\hbar \to 0$,

$$\frac{\sum_{j=1}^N f(x_j) - \mathbb{E}\sum_{j=1}^N f(x_j)}{\sqrt{\operatorname{Var}\sum_{j=1}^N f(x_j)}} \xrightarrow{d} \mathcal{N}_{0,1}.$$

Weyl's law



$$H = -\hbar^2\Delta + V$$

If $f : \mathbb{R}_+ \to \mathbb{R}_+$,

$$\mathrm{Tr}\left[f\,\varphi_-(H)\right] \leq \int f(x)\,\Pi_N(x,x)\,dx = \mathrm{Tr}\left[f\,\Pi_N\right] \leq \mathrm{Tr}\left[f\,\varphi_+(H)\right]$$

$$\frac{1}{N}\mathrm{Tr}\left[f\,\varphi_\pm(H)\right] = \frac{N^{-1}}{(2\pi\hbar)^d}\left(\int_{\mathbb{R}^d\times\mathbb{R}^d} a_0(x,\xi)\,f(x)\,d\xi\,dx + \mathcal{O}(\hbar)\right)$$

$$\simeq \frac{N^{-1}}{(2\pi\hbar)^d}\int_{\mathbb{R}^d} f(x)\left(\int_{\mathbb{R}^d}\varphi_\pm\left(V(x)+|\xi|^2\right)d\xi\right)dx$$

$$\simeq \frac{N^{-1}\omega_d^d}{(2\pi\hbar)^d}\int_{\mathbb{R}^d} f(x)\left(\mu - V(x)\right)_+^{d/2}\,dx$$

$$\Rightarrow \frac{N^{-1}\omega_d^d}{(2\pi\hbar)^d} \sim \frac{1}{Z_\mu} \text{ and } \frac{\mathrm{Tr}\left[f\,\Pi_N\right]}{N} = \frac{\mathbb{E}_N\left[\sum f(x_j)\right]}{N} \longrightarrow \int_{\mathbb{R}^d} f(x)\frac{\left(\mu - V(x)\right)^{d/2}}{Z_\mu}\,dx$$

$$\frac{\phi_t(x_0+x\hbar,\xi)-(x_0+y\hbar)\cdot\xi-\lambda t}{\hbar}=(x-y)\cdot\xi+t\,\frac{V(x_0)+|\xi|^2-\lambda+O(t)}{\hbar}$$

Stationary point : $\begin{cases} V(x_0)+|\xi|^2-\lambda=0 \\ t=0 \end{cases}$

$$\varphi_\hbar(H)(x_0+\hbar x,\ x_0+\hbar y)$$

$$\simeq\frac{1}{(2\pi\hbar)^d}\int_{\mathbb{R}^d}\mathbb{1}(|\xi|^2\leq\mu-V(x_0))\,e^{i(x-y)\cdot\xi}\,a_0^0(x_0,x_0,\xi)\,d\xi$$

Let $\mathcal{D} = \partial_t(\frac{\cdot}{t^2+x})$ By making repeated integration by parts, it also holds for $x \geq -\epsilon^{-1}$ and for any $k \in \mathbb{N}$,

$$\int_{\mathbb{R}} e^{\mathbf{i}(t^3/3+t)} \partial_t \Big(\frac{e^{\mathbf{i}(x-1)t}}{t^2+1} (1-\chi(\epsilon t)) \Big) \mathrm{d}t = \int_{\mathbb{R}} e^{\mathbf{i}(t^3/3+t)} \mathcal{D}^k \Big(e^{\mathbf{i}(1-x)t} \partial_t \Big(\frac{e^{\mathbf{i}(x-1)t}}{t^2+1} (1-\chi(\epsilon t)) \Big) \Big) \mathrm{d}t$$

where we verify by induction that

$$\mathcal{D}^k \Big(e^{\mathbf{i}(1-x)t} \partial_t \Big(\frac{e^{\mathbf{i}(x-1)t}}{t^2+1} (1-\chi(\epsilon t)) \Big) \Big) = \begin{cases} 0 & \text{if } |t| \leq \epsilon^{-1} \\ \mathcal{O}_k\big((t^2+1)^{-1}(\epsilon^{-2}+x)^{-k}\big) & \text{if } |t| \geq \epsilon^{-1} \end{cases}.$$

This proves the claim. □

Note that taking $\epsilon = 1$, Lemma A.20 and Proposition A.16 (applied with $\Phi(t) = t$ and $\hbar = 1/x$) imply that as $x \to +\infty$,

$$\mathrm{Ai}(x) = \mathcal{O}(x^{-\infty}). \tag{A.21}$$

In fact, applying the steepest descent method to the integral (A.20), one obtains the well-known asymptotics $\mathrm{Ai}(x) \sim \frac{e^{-\frac{2}{3}x^{3/2}}}{2\sqrt{\pi}x^{1/4}}$ as $x \to +\infty$.

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