Modular properties of isomonodromic tau-functions

Harini Desiraju (University of Sydney)

joint work with F. Del Monte, P. Gavrylenko (arXiv: 220X.XXXX)

Excursions in Integrability

SISSA, Trieste

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Plan

1. Introduction
2. tau-function on a torus
3. Connection constant

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Pants decomposition

Any punctured Riemann surface can be decomposed into pairs of pants.

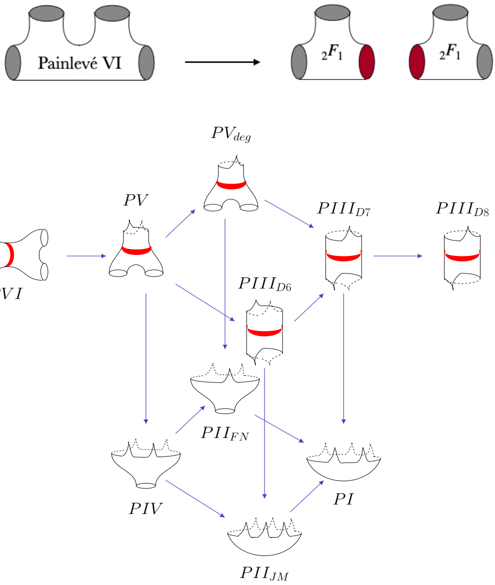
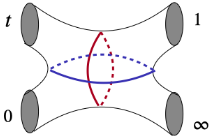


Figure: (CMR) Confluence diagram for Painlevé equations: GL'16

Choice of pants

Construction with a different choice of pairs of pants enables asymptotic analysis at different asymptotes.

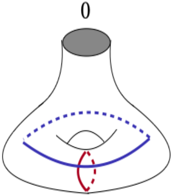


$$= \frac{\mathcal{T}(t\to 0)}{\mathcal{T}(t\to\infty)} = \text{connection constant for Painlevé VI}$$

A brief history

1. 1977, 1991; B. McCoy, C. Tracy, T. Wu: connection problem for a special solution of Painlevé III.
2. 2013; A. Its, O. Lisovyy, Y. Tykhyy: Painlevé III connection problem through the Kyiv formula.
3. 2015; A. Its, A. Prokhorov: Painlevé III connection problem through Riemann-Hilbert methods.
4. 2016; A. Its, O. Lisovyy, A. Prokhorov: Connection problem for Painlevé VI, Painlevé II using Hamiltonian methods.
5. 2019; M. Bertola, D. Korotkin: Symplectic interpretation of the connection constant for a sphere with simple poles.

Pictorial summary of the results



$$= \frac{\mathcal{T}(\tau \to i\infty)}{\mathcal{T}(\tau \to 0)} = \text{connection constant for 1 pt torus}$$

Disclaimer

Everything works for a torus with any number of punctures*[1]!

[1] * regular

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Minimal setup

Consider the equation of motion of nonautonomous Calogero-Moser system

$$(2\pi i)^2 \frac{d^2 Q(\tau)}{d\tau^2} = m^2 \wp'(2Q|\tau),$$

which arises as the consistency condition of the following system of equations

$$\partial_z Y(z,\tau) = A(z,\tau)Y(z,\tau); \quad 2\pi i \partial_\tau Y(z,\tau) = B(z,\tau)Y(z,\tau).$$

For the purposes of this talk we will only need the following information

$$A(z,\tau) = 2\pi i \frac{dQ(\tau)}{d\tau}\sigma_3 + m\frac{\theta_1'(0|\tau)}{\theta_1(z|\tau)} \begin{pmatrix} 0 & \frac{\theta_1(z+2Q(\tau)|\tau)}{\theta_1(-2Q(\tau)|\tau)} \\ \frac{\theta_1(z-2Q(\tau)|\tau)}{\theta_1(2Q(\tau)|\tau)} & 0 \end{pmatrix}.$$

tau-function and the Hamiltonian

Isomonodromic tau-function $\mathcal{T}_{CM}(\tau)$ = generator of the Hamiltonian H:

$$2\pi i \partial_\tau \log \mathcal{T}_{CM}(\tau) := H(\tau),$$

where the Hamiltonian of the Calogero-Moser system is the a-cycle contour integral

$$H(\tau) = \oint_a dz \frac{1}{2} \operatorname{Tr} A^2(z,\tau) = \left(2\pi i \frac{dQ(\tau)}{d\tau}\right)^2 - m^2 \wp(2Q(\tau)|\tau) + 4\pi i m^2 \partial_\tau \log \eta(\tau),$$

where $\eta(\tau)$ is Dedekind's eta function

$$\eta(\tau) := \left(\frac{\theta_1'(0|\tau)}{2\pi}\right)^{1/3}.$$

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$\mathcal{T}_{CM}$ be written as a Fredholm determinant, in terms of the monodromy data of the system.

Fredholm determinant advantages

- ☑ Allows us to write the transcendent $Q(\tau)$ explicitly
- ☑ Fredholm determinant $=$ Fourier transform of $c = 1$ conformal blocks $=$ Nekrasov-Okounkov functions
- ☐ Connection constant $=$ modular transformation of $c = 1$ conformal block on a punctured torus.
- ☐ The distribution of the poles of the transcendent $=$ zero locus of the Fredholm determinant.

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Monodromy data

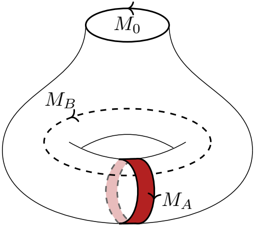
The Lax matrix A behaves as

$$A(z+1,\tau) = A(z,\tau), \qquad A(z+\tau,\tau) = e^{-2\pi i Q(\tau)\sigma_3} A(z,\tau) e^{2\pi i Q(\tau)\sigma_3}.$$

In turn, the monodromies around the a,b-cycles, and the puncture at $z=0$ read

$$Y(z+1,\tau) = Y(z,\tau) M_A, \qquad Y(z+\tau,\tau) = e^{2\pi i Q(\tau)\sigma_3} Y(z,\tau) M_B,$$

$$Y(e^{2\pi i} z,\tau) = Y(z,\tau) M_0,$$



1. The monodromy matrices satisfy the constraint

$$M_0 = M_A^{-1} M_B^{-1} M_A M_B.$$

2. Explicitly, the monodromies are

$$M_A = e^{2\pi i a \sigma_3}, \qquad M_0 \sim e^{2\pi i m \sigma_3},$$

$$M_B = e^{2\pi i \rho} \begin{pmatrix} \frac{\sin \pi(2a-m)}{\sin 2\pi a} e^{-i\nu/2} & \frac{\sin \pi m}{\sin 2\pi a} \\ -\frac{\sin \pi m}{\sin 2\pi a} & \frac{\sin \pi(2a+m)}{\sin 2\pi a} e^{i\nu/2} \end{pmatrix}.$$

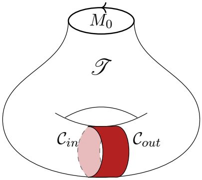
Note: The parameter

1. m is the parameter in the equation of motion of the Calogero-Moser system,
2. a, ν give the monodromy data,
3. ρ is a symmetry parameter.

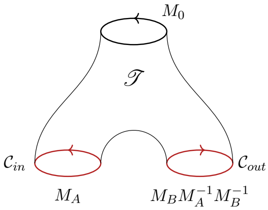
Pants decomposition

Cutting the torus along its a-cycle gives us a pair of pants $\mathscr{T}$ and its monodromies follow from the relation

$$M_A M_0 M_B^{-1} M_A^{-1} M_B = 1 = (M_A) M_0 (M_B^{-1} M_A M_B)^{-1} := M_{in} M_0 M_{out},$$



(a) 1 point Torus



(b) Pair of pants

Linear system

$$\partial_z \Phi(z) = -2\pi i \left(A_- + \frac{A_0}{1 - e^{2\pi i z}} \right) \Phi(z),$$

with

$$A_- = a\sigma_3, \qquad A_0 = G_0 m \sigma_3 G_0^{-1}, \qquad A_+ = -A_- - A_0 = -G_+ a \sigma_3 G_+^{-1}.$$

The diagonalisation matrix

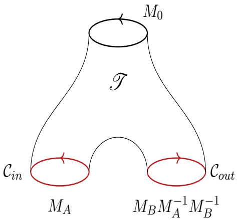
$$G_+ = \frac{1}{2a} \begin{pmatrix} r(m+2a) & m \\ m & r^{-1}(m-2a) \end{pmatrix} \begin{pmatrix} e^{\frac{-i\nu}{2}} & 0 \\ 0 & e^{\frac{i\nu}{2}} \end{pmatrix},$$

$$r = e^{i\nu} \frac{\Gamma(1+2a)\Gamma(1+m-2a)}{\Gamma(1-2a)\Gamma(1+m+2a)}.$$

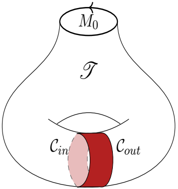
The matrix G_0 has a similar expression in terms of the monodromy data.

Cauchy operators

(in cylindrical coordinates)



$$\left(\mathcal{P}_{\oplus} f\right)(z) := \oint_{\mathcal{C}_{in} \cup \mathcal{C}_{out}} \frac{\Phi(z)\Phi(w)^{-1}}{1-e^{-2\pi i(z-w)}} f(w) dw$$



$$\left(\mathcal{P}_{\Sigma} f\right)(z) := \oint_{\mathcal{C}_{in} \cup \mathcal{C}_{out}} Y(z)\Xi(z,w)Y(w)^{-1} f(w) dw,$$

the Cauchy kernel on the torus $\Xi(z,w)$ is

$$\Xi(z,w) = \frac{\theta_1'(0)}{\theta_1(z-w)} \operatorname{diag}\left(\frac{\theta_1(z-w+Q(\tau)-\rho)}{\theta_1(Q(\tau)-\rho)}, \frac{\theta_1(z-w-Q(\tau)-\rho)}{\theta_1(Q(\tau)+\rho)}\right)$$

Fredholm determinant and the Hamiltonian

Theorem (F. Del Monte, H.D, P. Gavrylenko; 2020)

The logarithmic derivative of the Fredholm determinant gives back the Hamiltonian

$$2\pi i \partial_\tau \log \det_{\mathcal{H}_+} \left[\mathcal{P}_\Sigma^{-1} \mathcal{P}_\oplus \right] = H_{CM} - (2\pi i)^2 a^2 - \frac{(2\pi i)^2}{6} + 2\pi i \frac{d}{d\tau} \log \left(\frac{\theta_1(Q-\rho)\theta_1(Q+\rho)}{\eta(\tau)^2} \right).$$

Remember: $H_{CM} = 2\pi i \partial_\tau \log \mathcal{T}_{CM}$.

Plan

Derivative w.r.t monodromy data

Starting with the tau-function

$$\mathcal{T}_{CM}(\tau) = \det_{\mathcal{H}_+}\left[\mathcal{P}_{\Sigma}^{-1}\mathcal{P}_{\oplus}\right] e^{2i\pi\tau\left(a^2+\frac{1}{6}\right)} \frac{\eta(\tau)^2}{\theta_1\left(Q-\rho\right)\theta_1\left(Q+\rho\right)}.$$

The nontrivial term in the derivative of $\mathcal{T}_{CM}$ w.r.t the monodromy data comes from the Fredholm determinant.

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The nontrivial term in the derivative of $\mathcal{T}_{CM}$ w.r.t the monodromy data comes from the Fredholm determinant.

With $d_\mathcal{M} = d_m + d_a + d_\nu$,

$$d_\mathcal{M} \log \det_{\mathcal{H}_+}\left[\mathcal{P}_\Sigma^{-1}\mathcal{P}_\oplus\right] = \mathrm{Tr}\left(P d_\mathcal{M} Q\right) + d_\mathcal{M}\log\left(\theta_1(Q-\rho)\theta_1(Q+\rho)\right) - \left(\mathrm{Tr}\, m d_\mathcal{M} G_0 G_0^{-1} + \mathrm{Tr}\, a d_\mathcal{M} G_+ G_+^{-1}\right),$$

where m, a are the monodromy data, P, Q are the dynamical variables, ρ is an arbitrary parameter, G_0, G_+ are diagonalizing matrices.

Derivative of the tau-function

The derivative w.r.t the time τ, the parameter m, and the monodromy data a, ν is

$$d \log \mathcal{T}_{CM} = H_\tau d\tau - 2\pi i P d_{\mathcal{M}} Q - \left(\operatorname{Tr} m d_{\mathcal{M}} G_0 G_0^{-1} - \operatorname{Tr} a d_{\mathcal{M}} G_+ G_+^{-1} \right).$$

A similar expression ('dual tau-function') can be obtained by cutting the torus along the B-cycle.

Dual pants

The dual three-point problem has the form

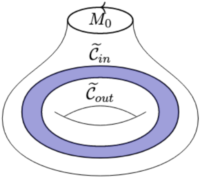
$$\partial_z \widetilde{\Phi}(z) = \left(-2\pi i \widetilde{A}_- - 2\pi i \frac{\widetilde{A}_0}{1 - e^{2\pi i z}} \right) \widetilde{\Phi}(z),$$

with

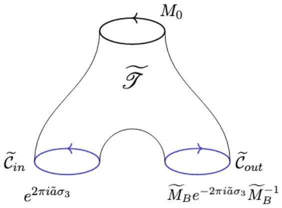
$$\widetilde{A}_- = \widetilde{a}\sigma_3, \qquad \widetilde{A}_0 = \widetilde{G}_0^{-1} m \sigma_3 \widetilde{G}_0, \qquad \widetilde{A}_+ = -\widetilde{A}_- - \widetilde{A}_0 = -\widetilde{G}_+^{-1} \widetilde{a}\sigma_3 \widetilde{G}_+,$$

and

$$2\cos 2\pi\widetilde{a} = \frac{1}{\sin 2\pi a} \left[e^{-\frac{i\nu}{2}} \sin\left(\pi(2a - m)\right) + e^{\frac{i\nu}{2}} \sin\left(\pi(2a + m)\right) \right].$$



(a) Dual pants decomposition



(b) Dual trinion

All in all

- By cutting along the B-cycle we get

$$d\log\widetilde{\mathcal{T}}_{CM} = H_\tau d\tau - 2\pi i P d_{\widetilde{\mathcal{M}}} Q - \left(\mathrm{Tr}\, m d_{\widetilde{\mathcal{M}}} \widetilde{G}_0 \widetilde{G}_0^{-1} - \mathrm{Tr}\, \widetilde{a} d_{\widetilde{\mathcal{M}}} \widetilde{G}_+ \widetilde{G}_+^{-1}\right).$$

  and the above expression is suitable for $\tau \to 0$ asymptotics.
- By cutting along the A-cycle we get

$$d\log\mathcal{T}_{CM} = H_\tau d\tau - 2\pi i P d_{\mathcal{M}} Q - \left(\mathrm{Tr}\, m d_{\mathcal{M}} G_0 G_0^{-1} - \mathrm{Tr}\, a d_{\mathcal{M}} G_+ G_+^{-1}\right),$$

  and it is suitable for $\tau \to i\infty$ asymptotics.

Connection constant

$$
\begin{aligned}
d\log\Upsilon := d\log\frac{\mathcal{T}_{CM}}{\widetilde{\mathcal{T}}_{CM}} &= \left(\operatorname{Tr} m d_{\mathcal{M}} G_0 G_0^{-1} - \operatorname{Tr} a d_{\mathcal{M}} G_+ G_+^{-1}\right) \\
&\quad - \left(\operatorname{Tr} m d_{\widetilde{\mathcal{M}}} \widetilde{G}_0 \widetilde{G}_0^{-1} - \operatorname{Tr} \widetilde{a} d_{\widetilde{\mathcal{M}}} \widetilde{G}_+ \widetilde{G}_+^{-1}\right).
\end{aligned}
$$

Theorem (F. Del Monte, H.D, P. Gavrylenko; In preparation)

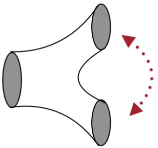
The logarithmic derivative of the connection constant

$$
\begin{aligned}
-d\log\Upsilon(M) = d\log&\left(\frac{G(1+m-2a)G(1-m-2a)G(1+2\widetilde{a})G(1-2\widetilde{a})}{G(1+2a)G(1-2a)G(1+m-2\widetilde{a})G(1-m-2\widetilde{a})}\right) \\
&- 2a\, d\log\sin\pi(2a+m) + 2\widetilde{a}\, d\log\sin\pi(2\widetilde{a}+m) \\
&+ a\, d\log\nu - \widetilde{a}\, d\log\widetilde{\nu}.
\end{aligned}
$$

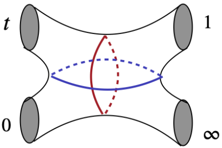
where m is the monodromy around the puncture, a,ν are the A,B-cycle monodromies, $\widetilde{a},\widetilde{\nu}$ are the dual monodromies around the A,B-cycles.

Moore Seiberg formalism

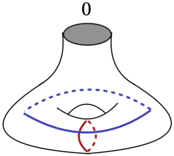
Transition functions of vector bundles on *any* punctured Riemann surface are determined by the following data



1. Braiding



2. Fusion



3. Modularity

References

1. F. Del Monte, H.D, and P. Gavrylenko (), 2020. Isomonodromic tau functions on a torus as Fredholm determinants, and charged partitions. arXiv:2011.06292.
2. F. Del Monte, H.D, and P. Gavrylenko (), (*In preparation*). On the modularity of tau functions and conformal blocks arXiv:220X.XXXX.
3. A. Its, O. Lisovyy, and A. Prokhorov, 2016. Monodromy dependence and connection formulae for isomonodromic tau functions. Duke Mathematical Journal, 167(7), pp.1347-1432.