Large Genus Asymptotics for Intersection Numbers

Amol Aggarwal

Institute of Advanced Study / Columbia University

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Flat Surfaces

- Consider a finite family of polygons in $\mathbb{R}^2 \simeq \mathbb{C}$
- Form a **flat surface** X from them by gluing (anti-)parallel sides of equal lengths using translations $(z \mapsto c + z)$ and / or reflections $(z \mapsto c - z)$

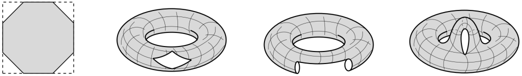

Figure by A. Zorich.

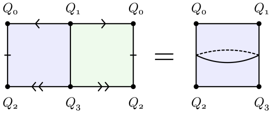

Figure by E. Duryev.

Alternative interpretation: **Meromorphic quadratic differentials**

- Gluing $dz \otimes dz$ on each polygon yields meromorphic quadratic differential on X
  - Quadratic differential taken to ensure gluing is consistent along reflections
  - Locally of the form $f(z) \cdot dz \otimes dz$, and has at most simple poles
- Can be reversed: Every such differential comes from gluing polygons

Question

How does a "random" flat surface of large genus "look?"

- To make sense of this question, we discretize

Square-Tiled Surfaces

Square-tiled surface: Connected flat surface X produced from gluing squares

- Finite collection of $\frac{1}{2} \times \frac{1}{2}$ squares
- Glue pairs of vertical / horizontal sides by translations or reflections

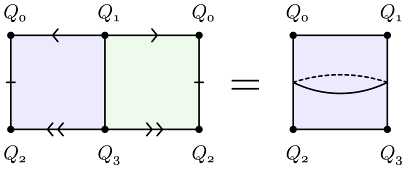

Figure by E. Duryev.

Conical singularities: Points on X that do not have total angle 2π

- Also defined on general flat surface
- Will either have total angle π or $2\pi(m+1)$ for some integer $m \geq 1$
- Conical singularities are poles / zeroes of associated differential q
    - Simple poles at conical singularities with total angle π
    - Zeroes of order m at conical singularities with total angle $2\pi(m+1)$

Discretization Through Square-Tiled Surfaces

$\mathcal{S}_N(g,n)$: Set of square-tiled surfaces with three properties

- At most $4N$ squares
- Genus g
- n Simple poles (conical singularities with total angle π)

For N large, $\mathcal{S}_N(g,n)$ discretizes the set of flat surfaces of genus g with n poles

Question

1. *How many square-tiled surfaces are in $\mathcal{S}_N(g,n)$ for large N and g?*
2. *How does a uniformly random surface in $\mathcal{S}_N(g,n)$ "look" for large N and g?*

First let N tend to ∞

- Yields a discretization of a flat surface

Then let g tend to ∞

- View n as fixed

Enumeration of Square-Tiled Surfaces

- **Fact**: For N large, $\#\mathcal{S}_N(g,n) \sim N^{6g+2n-6} \cdot \operatorname{Vol} \mathcal{Q}_{g,n}$, for some constant $\operatorname{Vol} \mathcal{Q}_{g,n}$

Masur, Veech (1982): The constant $\operatorname{Vol} \mathcal{Q}_{g,n}$ is finite and positive

- $\operatorname{Vol} \mathcal{Q}_{g,n}$: **Volume** of moduli space $\mathcal{Q}_{g,n}$ of meromorphic quadratic differentials
  - $\mathcal{Q}_{g,n}$: Set of pairs (X, q), Riemann surface X and differential q on X
  - Noncompact orbifold of dimension $6g + 2n - 6$
  - Parameterized by **periods**: Sides of polygons (in $\mathbb{R}^2 \simeq \mathbb{C}$) that glue to form (X, q)
    - Square-tiled surfaces have periods in $\frac{1}{2}\mathbb{Z}[i]$, so can be viewed as lattice points in $\mathcal{Q}_{g,n}$
  - Lebesgue measure on these periods pulls back to volume form on $\mathcal{Q}_{g,n}$
- **Analogy**: Volume of a disk approximately determined by number of lattice points in it

Formulas / algorithms to compute $\operatorname{Vol} \mathcal{Q}_{g,n}$

- Eskin–Okounkov (2005): Count branched coverings using representation theory of $\mathfrak{S}_m$
- Mirzakhani (2008): Relate to volumes in hyperbolic geometry
- Delecroix–Goujard–Zograf–Zorich (2019): Relate to ribbon graph counts
- Chen–Möller–Sauvaget (2019), Kazarian (2019), Yang–Zagier–Zhang (2020): Recursions based on intersection theory
- Andersen–Borot–Charbonnier–Delecroix–Giacchetto–Lewański–Wheeler (2019): Topological recursion

Large Genus Asymptotics of Volumes

Almost all formulas / algorithms have **exponential** complexity in genus g

- $\operatorname{Vol} \mathcal{Q}_{2,1} = \frac{29}{840}\pi^8$; $\operatorname{Vol} \mathcal{Q}_{3,0} = \frac{115}{33264}\pi^{12}$; $\operatorname{Vol} \mathcal{Q}_{4,0} = \frac{2106241}{11548293120}\pi^{18}$

Theorem (A., 2020)

Fix $n \geq 0$. As g tends to ∞, we have $\operatorname{Vol} \mathcal{Q}_{g,n} \sim \frac{4}{\pi}\left(\frac{8}{3}\right)^{4g+n-4} 2^n$.

- Based on analysis of formulas from Delecroix–Goujard–Zograf–Zorich (2019) through ribbon graph counts / intersection numbers
  - Will count square-tiled surfaces "associated" with a graph Γ, and then sum over all (exponentially many) graphs
    - The graph Γ contains geometric information about the surface
    - Can be used to analyze statistics of random square-tiled surfaces

Graph Associated With Square-Tiled Surface

Fix a square-tiled surface X

- Horizontal foliation decomposes X into maximal cylinders $\bigcup_{i=1}^{k} C_i$
- Each cylinder C_i admits a waist curve γ_i

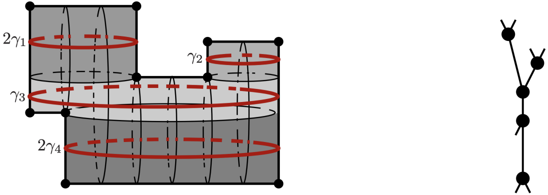

Figure by V. Delecroix, É. Goujard, P. Zograf, and A. Zorich.

- Gives rise to a graph Γ
  - Vertices: Connected components of $S \setminus \bigcup_{i=1}^{k} \gamma_i$
  - Edges: Waist curves γ_i (alternatively, cylinders C_i)
  - Unpaired half-edges: Singularities of total angle π (poles of quadratic differential)
- Can view Γ as the "skeleton" of the square-tiled surface

Topologies of Surfaces

- Let $\mathcal{S}_N^{(\Gamma)}(g,n)$ denote the surfaces in $\mathcal{S}_N(g,n)$ associated with Γ

Delecroix–Goujard–Zograf–Zorich (2019): $\lim_{N\to\infty} N^{6-6g-2n}\mathcal{S}_N^{(\Gamma)}(g,n) = \mathcal{Z}(\Gamma)$ Exists

- Formulas for $\mathcal{Z}(\Gamma)$ through intersection numbers $\langle \tau_{d_1}\cdots\tau_{d_n}\rangle$ of ψ-classes
- Summing over Γ implies an analogous formula for Vol $\mathcal{Q}_{g,n}$

$$\operatorname{Vol}\mathcal{Q}_{2,0} = \frac{32\pi^8}{945}\left(\langle\tau_0^2\tau_3\rangle + 3\langle\tau_0\tau_1\tau_2\rangle + 3\langle\tau_1\rangle\langle\tau_0\tau_2\rangle\right) + \frac{16\pi^8}{3969}\left(\langle\tau_0^4\tau_2\rangle + \langle\tau_0^3\tau_1^2\rangle + \langle\tau_0\tau_2\rangle\langle\tau_0^3\rangle + 2\langle\tau_0^3\tau_1\rangle\langle\tau_1\rangle + 2\langle\tau_0\tau_2\rangle\right) + \frac{2\pi^8}{675}\left(\langle\tau_0^3\tau_1^2\rangle + 2\langle\tau_0^3\tau_1\rangle\langle\tau_1\rangle + \langle\tau_1\rangle^2\langle\tau_0^3\rangle + \langle\tau_1^2\rangle\right) + \frac{2}{2835}\left(9\langle\tau_0^3\tau_1\rangle\langle\tau_0^3\rangle + 4\langle\tau_1\rangle\langle\tau_0^3\rangle^2\right) + \frac{5}{3402}\langle\tau_0^3\rangle^3$$

Figure by V. Delecroix, É. Goujard, P. Zograf, and A. Zorich.

Theorem (A., 2020)

Fix $n \geq 0$. As g tends to ∞, we have $\operatorname{Vol}\mathcal{Q}_{g,n} \sim \frac{4}{\pi}\left(\frac{8}{3}\right)^{4g-4} 2^n.$

- Have formulas for Vol $\mathcal{Q}_{g,n}$ in terms of intersection numbers

Issues

- Must understand **asymptotics of intersection numbers**
- Formula involves sums over graphs, number of which **grows exponentially in g**

Ribbon Graphs

- **Ribbon graph**: Graph (possibly with loops) with cyclic ordering of the edges incident to each vertex
- "Thickening" each edge into a thin rectangle (a "ribbon") gives rise to a surface with boundary
  - Cyclic ordering prescribes orientation to both long sides of any ribbon

Figures by M. Mulase and M. Penkava.

$\mathcal{R}_{g,n}$: Set of ribbon graphs with following properties

- **Trivalent**: Each vertex is of degree 3
- Resulting surface is of genus g, with n boundary components

Metric ribbon graph: Ribbon graph with a positive real number (length) assigned to each edge

- Called **integral** if all edges have integer lengths

Intersection Numbers Through Ribbon Graphs

- Fix $g \in \mathbb{Z}_{\geq 0}$, $n \in \mathbb{Z}_{\geq 1}$, and $R \in \mathcal{R}_{g,n}$

$N_R(b_1, b_2, \ldots, b_n)$: Number of metric ribbon paths with following properties

- Underlying ribbon graph is R
- Lengths of n boundary components are $b_1, b_2, \ldots, b_n$

Proposition (Kontsevich, 1992)

The weighted sum $\sum_{R \in \mathcal{R}_{g,n}} \left| \mathrm{Aut}(R) \right|^{-1} N_R(b_1, b_2, \ldots, b_n)$ *is a polynomial in* $(b_1, b_2, \ldots, b_n)$ *of degree* $6g + 2n - 6$*, with top degree homogeneous part*

$$2^{6-5g-2n} \sum_{|\boldsymbol{d}|=3g+n-3} \langle \tau_{d_1} \cdots \tau_{d_n} \rangle \prod_{i=1}^{n} \frac{b_i^{2d_i}}{d_i!}.$$

- Can be interpreted as combinatorial definition for intersection number $\langle \tau_1 \cdots \tau_n \rangle$

Intersection Numbers of ψ-Classes

Fix $g \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}_{\geq 1}$ with $2g + n \geq 3$

- $\mathcal{M}_{g,n}$: Moduli space of smooth, genus g curves with n marked points
- $\overline{\mathcal{M}}_{g,n}$: Deligne–Mumford compactification of $\mathcal{M}_{g,n}$
  - Moduli space of tuples $(C; x_1, x_2, \ldots, x_n)$, with C a stable genus g curve and $(x_1, x_2, \ldots, x_n)$ a collection of nonsingular points on C
- $\mathcal{L}_i$: Line bundle on $\overline{\mathcal{M}}_{g,n}$ whose fiber over $(C; x_1, x_2, \ldots, x_n) \in \overline{\mathcal{M}}_{g,n}$ is $T^*_{x_i} C$
- $\psi_i = c_1(\mathcal{L}_i)$: First Chern class of $\mathcal{L}_i$
- For any n-tuple $\mathbf{d} = (d_1, d_2, \ldots, d_n) \in \mathbb{Z}^n_{\geq 0}$, define the **intersection number**

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} \prod_{i=1}^{n} \psi_i^{d_i},$$

  which is nonzero only if $|\mathbf{d}| = \sum_{i=1}^{n} d_i = 3g + n - 3$
  - Ubiquitous in mathematics
    - Mathematical physics: Correlations functions for quantum gravity models
    - Algebraic geometry: Invariants in intersection theory
    - Dynamics / geometric topology: Moduli space volumes / multicurve counts

Evaluating Intersection Numbers

- Recursions (on $3g+n-3$)
  - Witten (1991): Initial data $\langle \tau_0^3 \rangle_{0,3} = 1$ and $\langle \tau_1 \rangle_{1,1} = \frac{1}{24}$
  - Kontsevich (1992): Virasoro constraints / Witten's conjecture
    - Witten (1991), Kontsevich (1992): Imply that $\langle \tau_{3g-3} \rangle_{g,1} = \frac{1}{24^g g!}$
    - Alternative proofs: Okounkov–Pandharipande (2001), Mirzakhani (2003), Kazarian–Lando (2006)
- Okounkov (2001), Zhou (2013), Bertola–Dubrovin–Yang (2015): Exact formulas for generating series

Theorem (Virasoro Constraints)

For fixed $k \geq 1$ and $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ with $|\mathbf{d}| = 3g+n-k-3$,

$$(2k+3)!!\langle \tau_{k+1}\tau_{d_1}\cdots\tau_{d_n}\rangle = \sum_{j=1}^{n} \frac{(2k+2d_j+1)!!}{(2d_j-1)!!}\langle \tau_{d_1}\cdots\tau_{d_j+k}\cdots\tau_{d_n}\rangle$$
$$+ \frac{1}{2}\sum_{\substack{r+s=k-1 \\ r,s\geq 0}} (2r+1)!!(2s+1)!!\langle \tau_r\tau_s\tau_{d_1}\cdots\tau_{d_n}\rangle$$
$$+ \frac{1}{2}\sum_{\substack{r+s=k-1 \\ r,s\geq 0}} \sum_{\substack{I\cup J=\{1,2,\ldots,n\} \\ |I\cap J|=0}} (2r+1)!!(2s+1)!!\langle \tau_r\tau_I\rangle\langle \tau_s\tau_J\rangle.$$

Integrability: Generating series of intersection numbers solves KdV equation / is annihilated by differential operators satisfying Virasoro commutation relations

Evaluating Intersection Numbers

$$\langle \tau_{14}^2 \tau_{17} \rangle = \frac{234144996149497743}{4218918642758340433196089344000}$$

$$\langle \tau_{14} \tau_{15} \tau_{19} \rangle = \frac{3672285682784833909}{132376344108779591192325295964160 00}$$

$$\langle \tau_{14} \tau_{16} \tau_{21} \rangle = \frac{473056002226707571430 3}{3923634839384227082940521772377702400 00}$$

$$\langle \tau_{14} \tau_{18} \tau_{19} \rangle = \frac{1352428754551622110128037}{823963316270687687417509572199317504000 00}$$

$$\langle \tau_{14} \tau_{20}^2 \rangle = \frac{321949318817660755336964 9}{452944405858515174454653833403281965056 0000}$$

$$\langle \tau_{15}^2 \tau_{18} \rangle = \frac{218085649505684627}{6248821212588353373053864902656 00}$$

$$\langle \tau_{15} \tau_{16} \tau_{20} \rangle = \frac{1312348909533728275697}{7847269678768454165881043544755404800 0}$$

$$\langle \tau_{15} \tau_{18}^2 \rangle = \frac{5235208082852854847595 7}{2535271742371346730515414068305592320000}$$

$$\langle \tau_{15} \tau_{20} \tau_{22} \rangle = \frac{3308236340083900843297847}{881405333031927277191117588109469270671360 0000}$$

$$\langle \tau_{16}^2 \tau_{19} \rangle = \frac{597847836467457614550 7}{287733221554843319415638263307698176000}$$

$$\langle \tau_{16} \tau_{17} \tau_{21} \rangle = \frac{346584600523292330873017 9}{362543859159102582463704211767699701760 0000}$$

$$\langle \tau_{16} \tau_{19} \tau_{22} \rangle = \frac{904251266254585507550861 9}{193909172670240009820586938408323954769920 000}$$

$$\langle \tau_{17}^3 \rangle = \frac{2257180607411319191 29}{922221863957831151973199561883648000 0}$$

$$\langle \tau_{17} \tau_{18} \tau_{22} \rangle = \frac{135637689940928503997585459}{2610315785945538593738670324727437852672000000}$$

$$\langle \tau_{17} \tau_{21} \tau_{22} \rangle = \frac{1057844571695605417974075443}{3983313971536999428391801945797976821596160 00000}$$

$$\langle \tau_{18} \tau_{19} \tau_{20} \rangle = \frac{123364949487816164600458793}{1770475054815234872274924220249914369638400000}$$

$$\langle \tau_{19}^3 \rangle = \frac{29829012145109284680228630 79}{4072092626075040206232325706574803050168320 0000}$$

$$\langle \tau_{19} \tau_{22}^2 \rangle = \frac{1364569522408068750074196767329}{8602559160929915526502661911508520401324998656000000}$$

$$\langle \tau_{14}^2 \tau_{20} \rangle = \frac{462767707930924427}{2206272401812993186538754932736000}$$

$$\langle \tau_{14} \tau_{15} \tau_{22} \rangle = \frac{1212964108431287644 97}{137190029349098849053864397635584000 00}$$

$$\langle \tau_{14} \tau_{17}^2 \rangle = \frac{93225052365760397479 7}{268981376987700752658849872216064000 0}$$

$$\langle \tau_{14} \tau_{18} \tau_{22} \rangle = \frac{42976490465567970100061137}{731679424848370666426748500112993943552000 00}$$

$$\langle \tau_{14} \tau_{21} \tau_{22} \rangle = \frac{691652202242122316308685 93}{2556075457925891038544100551746088494694400000}$$

$$\langle \tau_{15}^2 \tau_{21} \rangle = \frac{1831184525199503365 03}{142677630523062803016018973541007360 00}$$

$$\langle \tau_{15} \tau_{17} \tau_{19} \rangle = \frac{5380630529333132957260 9}{274654438756895895805836524066439168000 0}$$

$$\langle \tau_{15} \tau_{18} \tau_{21} \rangle = \frac{2495409123960584116915093 9}{292671769939348266570699400045197577420800 00}$$

$$\langle \tau_{15} \tau_{21}^2 \rangle = \frac{1384843119108209483392843}{35256213212770910876470352437877082685440000}$$

$$\langle \tau_{16}^2 \tau_{22} \rangle = \frac{3850940005682440457520 77}{517919798798717974948148873953856716800000}$$

$$\langle \tau_{16} \tau_{18} \tau_{20} \rangle = \frac{12774118133648545681221393 1}{1149781953333153904384890500177561911296000 00}$$

$$\langle \tau_{16} \tau_{20} \tau_{21} \rangle = \frac{431509087822413196943563}{843083359435826129654725819166625890304 0000}$$

$$\langle \tau_{17}^2 \tau_{20} \rangle = \frac{709673229662896735591613}{60423976526517097077284035294616616960 0000}$$

$$\langle \tau_{17} \tau_{19} \tau_{21} \rangle = \frac{176918726009115009061989 23}{2950791758025391453791540367083190616064 00000}$$

$$\langle \tau_{18}^3 \rangle = \frac{2038334958495643903899304 67}{148587206276899889182047387715254154690560000}$$

$$\langle \tau_{18} \tau_{20} \tau_{22} \rangle = \frac{2543559175190251619286112 93}{82413392514558608863278660947544348033024000000}$$

$$\langle \tau_{19}^2 \tau_{22} \rangle = \frac{282617686131066197935026 47}{87103585558449283863598557661122573369344000000}$$

$$\langle \tau_{20}^3 \rangle = \frac{1333817616256765359341572 63}{357124700896420638407540864106025508143104 00000}$$

$$\langle \tau_{14} \tau_{15} \tau_{16} \rangle = \frac{21073049653063010 63}{336308086094164851674773979136000}$$

$$\langle \tau_{14} \tau_{16} \tau_{18} \rangle = \frac{185113176213677518 31}{5646256309945905012080813644185600 0}$$

$$\langle \tau_{14} \tau_{17} \tau_{20} \rangle = \frac{26078728329513180828 13}{17606053766467685628579264363233280000 0}$$

$$\langle \tau_{14} \tau_{19} \tau_{21} \rangle = \frac{1289294713954774197790723 91}{190236650460576373270954610029378425323520000}$$

$$\langle \tau_{15}^3 \rangle = \frac{407865477597219179}{61146924744393609395413450752000}$$

$$\langle \tau_{15} \tau_{16} \tau_{17} \rangle = \frac{784502544481556653 7}{20048301390387633738547816562688000}$$

$$\langle \tau_{15} \tau_{17} \tau_{22} \rangle = \frac{4621128007278705212064 1}{659170653016550149934007657759454003200 00}$$

$$\langle \tau_{15} \tau_{19} \tau_{20} \rangle = \frac{25548236268720775352584411}{2717666435151091046727923000419691790336 0000}$$

$$\langle \tau_{16}^3 \rangle = \frac{1188640218358884251}{28640430557696619626496880803840 00}$$

$$\langle \tau_{16} \tau_{17} \tau_{18} \rangle = \frac{5380630528745543461531 7}{2323999097173734502972462895946792960000}$$

$$\langle \tau_{16} \tau_{19}^2 \rangle = \frac{1745796144893777253808092707}{14947165393331000757003576502308304846848 00000}$$

$$\langle \tau_{16} \tau_{22}^2 \rangle = \frac{3917942858102068055397487 1}{1896816176922380680186572355141893724569600 0000}$$

$$\langle \tau_{17} \tau_{18} \tau_{19} \rangle = \frac{1047477686982113541721818679}{8048473673332077330694233501242933379072 00000}$$

$$\langle \tau_{17} \tau_{20}^2 \rangle = \frac{47418054206895177018804 49}{754091227050933371524504760476815379660800 00}$$

$$\langle \tau_{18}^2 \tau_{21} \rangle = \frac{395915419289200937111873281}{6264757886269292624972808779345850846412800000}$$

$$\langle \tau_{18} \tau_{21}^2 \rangle = \frac{6613327336916101181750 77}{20475377022250586052988487191936483983360 0000}$$

$$\langle \tau_{19} \tau_{20} \tau_{21} \rangle = \frac{7630677525554176931346921 71}{21427482053785238304452451846361530488586240 0000}$$

$$\langle \tau_{20} \tau_{21} \tau_{22} \rangle = \frac{5990793025211453654394615727 1}{34410236643719662106010647646034081605299994624 0000}$$

Figure by M. Bertola, B. Dubrovin, and D. Yang.

If g is are very large, then $\langle \tau_{d_1} \cdots \tau_{d_n} \rangle$ is typically quite intricate

Question

Do these numbers admit a tractable asymptotic behavior as g tends to ∞?

Large Genus Asymptotics for Intersection Numbers

Theorem (A., 2020)

Let $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)$ *satisfy* $|\boldsymbol{d}| = 3g + n - 3$. *Then, as* g *tends to* ∞,

$$\langle \tau_{d_1} \cdots \tau_{d_n} \rangle_{g,n} \sim \frac{(6g+2n-5)!!}{24^g g! \prod_{i=1}^n (2d_i+1)!!}, \qquad \text{if } n = o(\sqrt{g}).$$

In particular, $\langle \boldsymbol{d} \rangle_{g,n} \sim 1$, *uniformly in* $\boldsymbol{d}$ *if* $n = o(\sqrt{g})$, *where*

$$\langle \boldsymbol{d} \rangle_{g,n} = \frac{24^g g! \prod_{i=1}^n (2d_i+1)!!}{(6g+2n-5)!!} \cdot \langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle_{g,n}.$$

Proof is based on a probabilistic interpretation of the Virasoro constraints

- **Universality:** Asymptotically, $\left\langle \tau_{d_1} \cdots \tau_{d_n} \right\rangle_{g,n} \prod_{i=1}^n (2d_i+1)!!$ (typical normalization in literature) is independent of **d**
- Predicted by Delecroix–Goujard–Zograf–Zorich (2019)
- Theorem is **false if** $n \sim c\sqrt{g}$, since then $\langle 3g-2, 1^{n-1} \rangle_{g,n} \sim \exp\left(\frac{n^2}{12g}\right)$

Alternative proof by Guo–Yang (2021) when $n = O(\log g)$

Asymptotic Contribution to the Volume

Theorem (A., 2020)

Fix $n \geq 0$. As g tends to ∞, we have $\operatorname{Vol} \mathcal{Q}_{g,n} \sim \frac{4}{\pi} \left(\frac{8}{3} \right)^{4g-4} 2^n$.

- Now have asymptotics for intersection numbers

Theorem established by analyzing contributions from classes of graphs Γ

1. Graphs with at least two vertices: $o\left(\left(\frac{8}{3} \right)^{4g-4} 2^n \right)$
2. Graphs with one vertex (but possibly several loops): $\frac{4}{\pi} \left(\frac{8}{3} \right)^{4g-4} 2^n \big(1 + o(1) \big)$.

Imply with probability $1 - o(1)$ that Γ has one vertex, as g tends to ∞

- Pinching cylinder waist curves in a random square-tiled surface likely leaves it connected

Single-Vertex Graphs

- Random square-tiled surface $S \in \mathcal{S}_N(g,n)$
  - First let N tend to ∞
  - Then let g tend to ∞
- Underlying graph Γ has one vertex, with probability $1 - o(1)$

There are $g+1$ such graphs

- $\Gamma_g(E)$: Single vertex with E self-edges, for any $E \in [0, g]$
  - Square-tiled surface has E cylinders

Proposition (A., 2020)

Let $Z_k(m) = \sum_{|\boldsymbol{a}|=m} \frac{\zeta(2a_1)\cdots\zeta(2a_k)}{a_1\cdots a_k}$. *Then,*

$$\mathbb{P}\left[\Gamma = \Gamma_g(E)\right] = \frac{\mathcal{Z}(\Gamma_g(E))}{\mathrm{Vol}\,\mathcal{Q}_{g,n}} \sim (6\pi g)^{1/2} \cdot \frac{Z_E(3g)}{2^{E-1}E!}.$$

- Uses asymptotics for intersection numbers of ψ-classes, and further analysis after inserting into formulas for $\mathcal{Z}\left(\Gamma_g(E)\right)$

Statistical Consequences

Proposition (A., 2020)

Let $Z_k(m) = \sum_{|\boldsymbol{a}|=m} \frac{\zeta(2a_1)\cdots\zeta(2a_k)}{a_1\cdots a_k}$. *Then,* $\mathbb{P}[\Gamma = \Gamma_g(E)] = \frac{\mathcal{Z}(\Gamma_g(E))}{\text{Vol}\,\mathcal{Q}_{g,n}} \sim (6\pi g)^{1/2} \cdot \frac{Z_E(3g)}{2^{E-1}E!}$.

Delecroix–Goujard–Zograf–Zorich (2020): Used, with other deep analytic / combinatorial ideas, to study refined geometric statistics at high genus

1. E Converges to a Poisson random variable with parameter $\frac{1}{2}\big(\log(24g)+\gamma\big)$
   - Random square-tiled surface of large genus g has about $\frac{\log g}{2}$ cylinders
   - Slowly divergent number of cylinder / geodesics; surface still remains connected after pinching / cutting along them
2. Law of E is very close to number of cycles in random permutation sampled under a certain multiparameter Ewens measure
   - Cycle of length k weighted by $\frac{1}{2}\zeta(2k)$
3. Distribution of cylinder heights / geodesic multiplcities
   - Square-tiled surface: $\mathbb{P}\big[\text{All cylinder heights} \leq A\big] \approx \sqrt{\frac{A}{A+1}}$
   - Probability all cylinders are one square tall is about $\frac{\sqrt{2}}{2}$
4. ...

Analysis of Virasoro Constraints

Theorem

Let $\mathbf{d} = (d_1, d_2, \ldots, d_n)$ *satisfy* $|\mathbf{d}| = 3g + n - 3$. *Then, as* g *tends to* ∞,

$$\langle \mathbf{d} \rangle_{g,n} \sim 1, \qquad \text{uniformly in } \mathbf{d}, \text{ if } n = o(\sqrt{g}).$$

- Exact if $n = 1$ (Kontsevich, 1992; predicted by Witten, 1991)
- Asymptotic known if $n = 2$ (Delecroix–Goujard–Zograf–Zorich, 2019)

Virasoro constraints:

$$\begin{aligned}
\langle k+1, \mathbf{d} \rangle_{g,n+1} = {} & \frac{1}{6g+2n-3} \sum_{j=1}^{n} (2d_j+1) \langle d_j + k, \mathbf{d} \setminus \{d_j\} \rangle_{g,n} \\
& + \frac{12g}{(6g+2n-3)(6g+2n-5)} \sum_{\substack{r+s=k-1 \\ r,s \geq 0}} \langle r, s, \mathbf{d} \rangle_{g-1,n+2} \\
& + \frac{1}{2} \sum_{\substack{r+s=k-1 \\ r,s \geq 0}} \sum_{\substack{I \cup J = \{1,2,\ldots,n\} \\ |I \cap J| = 0}} \frac{g!}{g'!g''!} \frac{(6g'+2n'-3)!!(6g''+2n''-3)!!}{(6g+2n-3)!!} \\
& \qquad \times \langle r, \mathbf{d}|_I \rangle_{g',n'+1} \langle s, \mathbf{d}|_J \rangle_{g'',n''+1}
\end{aligned}$$

- Blue term: **Decreases n**; Red term: **Increases n**
- Green term: Will be **asymptotically negligible**

Analysis of Virasoro Constraints

Recall

$$\langle k+1, \mathbf{d} \rangle_{g,n+1} = \sum_{\mathbf{d}'} A_{\mathbf{d}'} \langle \mathbf{d}' \rangle_{g,n} + \sum_{\mathbf{d}'} B_{\mathbf{d}'} \langle \mathbf{d}' \rangle_{g-1,n+2} + \text{Asymptotically negligible}$$

- Red term causes issues
  - Understand $\langle \mathbf{d} \rangle_{g,n}$ for small n
  - Repeated use of recursion yields $\langle \mathbf{d}' \rangle_{g',n'}$ with large n', due to red term
  - Red term is not asymptotically negligible
- Partially counteracted by effect of blue term, which reduces n

We will show that the effect of the blue term "dominates" that of the red term

- Comparison to random walk
  - Space variable: n
  - Time variable: Number of applications of recursion

Random Walk Heuristic

- Evaluate $\langle \mathbf{D} \rangle_{g,n+1}$, for $\mathbf{D} = (D_1, D_2, \ldots, D_{n+1}) \in \mathbb{Z}_{\geq 0}^{n+1}$
- Let $k + 1 = \min_{1 \leq i \leq n+1} D_i$, and set $\mathbf{d} = \mathbf{D} \setminus \{k+1\}$, so

$$\langle \mathbf{D} \rangle_{g,n+1} = \langle k+1, \mathbf{d} \rangle_{g,n+1} = \sum_{\mathbf{d}'} A_{\mathbf{d}'} \langle \mathbf{d}' \rangle_{g,n} + \sum_{\mathbf{d}'} B_{\mathbf{d}'} \langle \mathbf{d}' \rangle_{g-1,n+2} + \text{Small term}$$

Define $A = \sum_{\mathbf{d}'} A_{\mathbf{d}'}$ and $B = \sum_{\mathbf{d}'} B_{\mathbf{d}'}$

- Then, it can be shown that $A + B = 1 + O\left(\frac{n}{g}\right) \approx 1$

Random walk interpretation

- Flip coin with heads probability $\frac{A}{A+B} \approx A$ and tails probability $\frac{B}{A+B} \approx B$
  - If heads, then select $\mathbf{d}' \in \mathbb{Z}_{\geq 0}^{n}$ with probability $A^{-1} A_{\mathbf{d}'}$
  - If tails, then select $\mathbf{d}' \in \mathbb{Z}_{\geq 0}^{n+2}$ with probability $B^{-1} B_{\mathbf{d}'}$
- Replace $\mathbf{d}$ with $\mathbf{d}'$ and $(g, n+1)$ with $(g', n'+1)$
  - If heads, then $n' = n - 1$ and, if tails, then $n' = n + 1$
  - Decrease n with probability A, and increase n with probability B
- Repeat many times
- Output $\mathbb{E}\left[\langle \mathbf{d}' \rangle_{g',n'}\right]$ as approximation for $\langle \mathbf{D} \rangle_{g,n+1}$

Random Walk Heuristic

- Under random walk, n decreases with probability A and increases with probability B
- Asymptotic $\langle \mathbf{d} \rangle_{g,n+1} \approx 1$ known for $n \in \{0, 1\}$
  - Kontsevich (1992), predicted by Witten (1991): Exact for $n = 0$
  - Delecroix–Goujard–Zograf–Zorich (2019): Asymptotic for $n = 1$
- Assume initially $n \geq 2$; after many repetitions, wish for n to likely decrease to 1
- Want random walk to have negative drift: $B < A$

Explicit forms of A, B yield, $B \leq \frac{1}{n+1} \leq \frac{1}{3}$

- Since $A + B \approx 1$, this implies $A \geq \frac{2}{3} > B$

Random walk has drift of $B - A \leq -\frac{1}{3}$

- After about $3n$ steps, expect n to decrease to 2
- Suggests that $\mathbb{E}\left[\langle \mathbf{d}' \rangle_{g',n'}\right] \approx \langle \mathbf{d}' \rangle_{g',2} \approx 1$ after $4n$ repetitions

Summary

- Asymptotics on square-tiled surfaces
  - Enumerative: Total number of such objects
    - Closely related to volume Vol $\mathcal{Q}_{g,n}$ of moduli space of quadratic differentials
  - Statistical: Geometry of randomly chosen object
    - Geometry summarized through a graph associated with the surface
    - Statistics given by contribution to volume coming from given graph
    - Expression for this contribution in terms of intersection numbers
- Asymptotics of intersection numbers
  - Based on a probablistic interpretation of terms in Virasoro constraints