Borel–Lapace multitransform, and integral representations of solutions of qDE's

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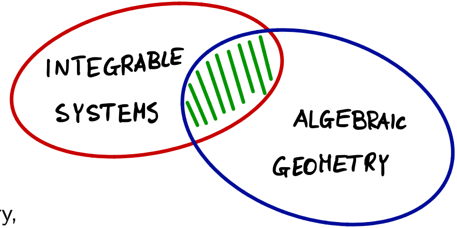


Leitmotiv: To study relations between

- the topology,
- the enumerative/symplectic geometry,
- the complex geometry,

of a smooth projective variety X over $\mathbb{C}$.

Such a study is developed via the analysis of meromorphic connections on $\mathbb{P}^1(\mathbb{C})$, and their isomonodromic deformations.

X smooth projective variety $/\mathbb{C}$ $\rightsquigarrow$ $\dfrac{dY}{dz} = \left(U(t) + \dfrac{1}{z}V(t)\right)Y$

qDE

$t \in QH^\bullet(X)$

Quantum cohomology

- Quantum cohomology
- Frobenius manifolds
- Quantum differential equations

Why to study qDE's?

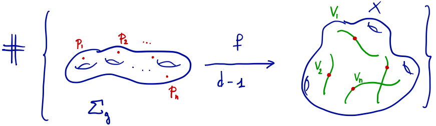
- Monodromy data
- RHB inverse problem
- Dubrovin's Conjecture

Integral representations of solutions of qDE's

- scalar qDE
- Borel–Laplace (α, β)-multitransform
- Integral representations
- Dubrovin's Conjecture for Hirzebruch surfaces

Gromov-Witten theory associates with X a family of Frobenius algebras, its quantum cohomology $QH^\bullet(X)$.

$QH^\bullet(X)$ is a deformation of $H^\bullet(X)$, via counting numbers of rational curves on X



$/\ \mathrm{Aut}(\Sigma_g, p_1, \dots, p_n)$

$$\left\langle [V_1], [V_2], \dots, [V_n] \right\rangle^X_{g,n,d} := \int_{[\overline{\mathcal{M}}_{g,n}(X,d)]^{virt}} ev_1^*[V_1]\, ev_2^*[V_2] \dots ev_n^*[V_n]$$

$$\begin{array}{c} \overline{\mathcal{M}}_{g,n}(X,d) \\ \big\downarrow ev_i \qquad i=1,\dots,n \\ X \end{array}$$

COLLECT $GW_{g=0}$ INVARIANTS INTO A GENERATING FUNCTION

$$\boxed{F_0^X(t) := \sum_{n=0}^{\infty} \sum_{d \in H_2(X,\mathbb{Z})} \sum_{\alpha_1,\dots,\alpha_n=1}^{N} \frac{t^{\alpha_1}\dots t^{\alpha_n}}{n!} \int_{[\overline{\mathcal{M}}_{0,n}(X,d)]^{virt}} \bigcup_{i=1}^{n} ev_i^* T_{\alpha_i}}$$

ASSUMPTION: F_0^X has a non-empty domain of convergence $\Omega \subseteq H^\bullet(X,\mathbb{C})$

Gromov-Witten theory associates with X a family of Frobenius algebras, its quantum cohomology $QH^\bullet(X)$.

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Two remarkable properties

1. QUASI-HOMOGENEITY

$$\boxed{\mathcal{L}_E F_0^X = (3 - \dim_{\mathbb{C}} X) F_0^X + \text{quadratic}}$$

E Euler vector field on Ω

2. WDVV eqs

$$\boxed{\partial^3_{\alpha\beta\gamma} F \cdot \eta^{\gamma\delta} \cdot \partial^3_{\delta\varepsilon\varphi} F = \partial^3_{\varphi\beta\gamma} F \cdot \eta^{\gamma\delta} \cdot \partial^3_{\delta\varepsilon\alpha} F}$$

η Poincaré pairing

$$\eta(\alpha,\beta) := \int_X \alpha \cup \beta$$

$$c_{\alpha\beta\gamma} := \frac{\partial^3 F_X}{\partial t_\alpha \partial t_\beta \partial t_\gamma},$$

$$\frac{\partial}{\partial t_\alpha} * \frac{\partial}{\partial t_\beta} := \sum_\lambda c^\lambda_{\alpha\beta} \frac{\partial}{\partial t_\lambda}$$

COMMUTATIVITY

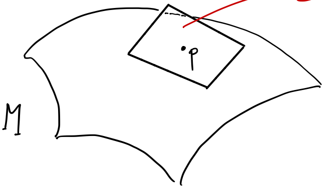
ASSOCIATIVITY

EXISTENCE of UNIT

&

$$\eta\left(\frac{\partial}{\partial t_\alpha} * \frac{\partial}{\partial t_\beta}, \frac{\partial}{\partial t_\gamma}\right) = \eta\left(\frac{\partial}{\partial t_\alpha}, \frac{\partial}{\partial t_\beta} * \frac{\partial}{\partial t_\gamma}\right)$$

Frobenius Manifolds are complex manifolds whose tangent spaces admit a *Frobenius algebra* structure.



FROBENIUS ALGEBRA

$$
\begin{aligned}
x(yz) &= (xy)z,\\
xy &= yx,\\
xe &= x,\\
\eta(x, yz) &= \eta(xy, z)
\end{aligned}
$$

FLAT "METRIC"

Examples coming from:

- Symplectic and Algebraic Geometry
- Singularity Theory

Milestones: Dubrovin, Hitchin, Kontsevich, Manin, Saito, Vafa, Witten, ...

Mirror Symmetry as isomorphism of Frobenius manifolds

$$QH^\bullet(X) \cong (V, f\colon V \to \mathbb{C})$$

To each point of $QH^\bullet(X)$ there is an attached differential equation

$$\frac{dY}{dz} = \left(\underline{U(t)} + \frac{1}{z}\underline{V(t)}\right) Y, \quad z \in \mathbb{C}^*, \quad t \in QH^\bullet(X).$$

$*$-MULTIPLICATION by
The EULER FIELD

$$\mathcal{U}(t) : T_t\Omega \to T_t\Omega$$
$$v \mapsto E * v$$

GRADING OPERATOR

$$V(t) : T_t\Omega \to T_t\Omega$$
$$v \mapsto \frac{2-\dim X}{2}\, v - \nabla_v E$$

To each point of $QH^\bullet(X)$ there is an attached differential equation

$$\frac{dY}{dz} = \left(U(t) + \frac{1}{z} V(t) \right) Y, \quad z \in \mathbb{C}^*, \quad t \in QH^\bullet(X).$$

Its solutions are multivalued, and they manifest a Stokes phenomenon.

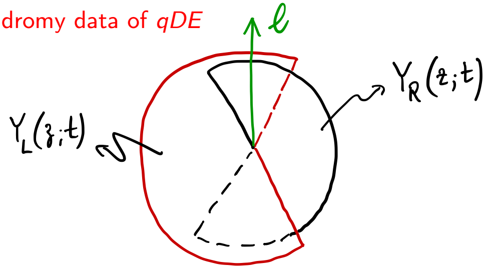
$\to$ Monodromy data of *qDE*

FUCHSIAN SINGULARITY $z=0$

IRREGULAR SINGULARITY $z=\infty$

$Y_0(z;t) \rightsquigarrow$ solution at $z=0$

$Y_F(z;t) \rightsquigarrow$ formal solution with exponential expansion at $z=\infty$



$$Y_R(z;t) = Y_0(z;t) \cdot C(t), \qquad Y_L(z;t) = Y_R(z;t) \cdot S(t)$$

- Isomonodromic property: monodromy data are locally constant wrt t.

RHB inverse problem

FROBENIUS MANIFOLD $QH^\bullet(X)$

MONODROMY DATA (μ, R, S, C)

RHB

The monodromy data define a "system of coordinates" in the space of quasi-homogeneous solutions of WDVV equations

1. Reconstruction of WDVV potential:

Malgrange Theorem ('80s)

on the existence of solutions of families of RHB problems

2. Extension of Malgrange Theorem: C. Sabbah (2021)

based on results of T. Mochizucki

3. Convergence Theorem (C. 2021)

Let $F \in \mathbb{C}[[t^1, \dots, t^n]]$, quasi-homogeneous, WDVV-potential.

If F defines a semisimple Frobenius algebra at $t=0$,

then $F \in \mathbb{C}\{t^1, \dots, t^n\}$.

Refined in [CDG18]

Motivation: tt^*-geometry

Dubrovin Conjecture

$$\begin{array}{c}\text{Symplectic and Enumerative Geometry of } X\text{: } QH^\bullet(X)\\ \Big\downarrow{\scriptstyle qDE}\\ \text{Complex geometry of } X\text{: } \mathcal{D}^b(X)\end{array}$$

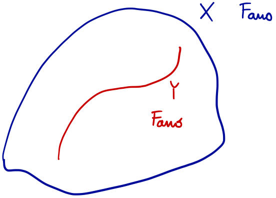
The monodromy data of the *qDE* of X are determined by

- the topology of X (dimension, characteristic classes),
- characteristic classes of exceptional collections in $\mathcal{D}^b(X)$

$$(E_i)_{i=1}^n, \quad \mathrm{Hom}^\bullet(E_i, E_i) \cong \mathbb{C}, \quad \mathrm{Hom}^\bullet(E_j, E_i) = 0, \quad j > i.$$

$$S^{-1} = \left(\chi(E_i, E_j)\right)_{ij}, \quad C = \frac{(\sqrt{-1})^d}{(2\pi)^{\frac{d}{2}}} \hat{\Gamma}_X^- \cup e^{-\pi\sqrt{-1}c_1(X)} \cup \mathrm{Ch}(E_j)$$

Problem: How to find bases of solutions of the qDE?



$$\begin{tikzcd} E \arrow[d] & \\ X & Y \equiv \{\sigma = 0\} \arrow[l, hook'] \end{tikzcd}$$

How to obtain solutions of qDE of Y from solutions of qDE of X?

QUANTUM LEFSCHETZ

Answer in two cases:

- E is a direct sum of fractional powers of the determinant bundle $\det TX$ of X,
- $X = X_1 \times \cdots \times X_h$ is a product of Fano varieties X_i's, and that E is the external tensor product of fractional powers of the determinant bundles $\det TX_i$.

Examples: all Fano complete intersections in products of projective spaces.

$\Rightarrow$ Reduction to a scalar *qDE*

$\hat{\nabla}$ is FLAT

$$\begin{tikzcd} \pi^*TM \arrow[r] \arrow[d] & TM \arrow[d] \\ M \times \mathbb{C}^* \arrow[r, "\pi"'] & M \end{tikzcd}$$

$\hat{\nabla}$ connection on π^*TM

$$\hat{\nabla}_X Y = \nabla_X Y + z \cdot X * Y, \qquad \hat{\nabla}_{\frac{\partial}{\partial z}} Y = \frac{\partial Y}{\partial z} + \mathcal{U}(Y) - \frac{1}{z}\mathcal{V}(Y)$$

e unit vector field.

In the frame $\left(e, \hat{\nabla}_z e, \hat{\nabla}_z^2 e, \dots, \hat{\nabla}_z^{N-1} e\right)$ the qDE is a SCALAR EQUATION

Examples:

- projective space $\mathbb{P}^n$: $\vartheta^n \Phi = (nz)^n \Phi$, $\vartheta := z\frac{d}{dz}$,
- $G(2,4)$: $\vartheta^5\Phi - 1024z^4\vartheta\Phi - 2048z^4\Phi = 0$,
- $\widetilde{\mathbb{P}^2}$:

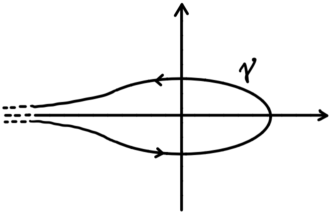
$$(283z - 24)\vartheta^4\Phi + \left(283z^2 - 590z + 24\right)\vartheta^3\Phi + \left(-2264z^2 + 192z + 3\right)\vartheta^2\Phi$$
$$- 4z^2\left(2547z^2 + 350z - 104\right)\vartheta\Phi + z^2\left(-3113z^3 - 9924z^2 + 1476z + 192\right)\Phi = 0.$$

Borel-Laplace (α, β)-multitransform

Let $\alpha, \beta \in (\mathbb{C}^*)^h$, and let $\Phi_1, \dots, \Phi_h \colon \mathbb{C} \setminus \mathbb{R}_- \to \mathbb{C}$ smooth/analytic functions.

$$\mathscr{B}_{\alpha,\beta}[\Phi_1, \dots, \Phi_h](z) := \frac{1}{2\pi i} \int_\gamma \prod_{j=1}^h \Phi_j\left(z^{\frac{1}{\alpha_j \beta_j}} \lambda^{-\beta_j}\right) e^\lambda \frac{d\lambda}{\lambda},$$

$$\mathscr{L}_{\alpha,\beta}\left[\Phi_1, \dots, \Phi_h\right](z) := \int_0^\infty \prod_{i=1}^h \Phi_i(z^{\alpha_i \beta_i} \lambda^{\beta_i}) e^{-\lambda} d\lambda.$$



QUANTUM LEFSCHETZ

LAPLACE (α,β)-MULTITRANSFORM

QUANTUM LERAY-HIRSCH

BOREL (α,β)-MULTITRANSFORM

Theorem (C. 2020)

Assume

$L \to X$ ample line bundle, $\quad E = \bigoplus_{j=1}^{r} L^{\otimes d_j}$, $\quad \det X = L^{\ell}$

then there exists $c \in \mathbb{C}$ such that every solution of the scalar qDE of Y is a $\mathbb{C}$-linear combination of integrals of the form

$$e^{-cz}\mathscr{L}_{\frac{\ell-\sum_{i=1}^{s} d_i}{d_s}, \frac{d_s}{\ell-\sum_{i=1}^{s-1} d_i}} \circ \cdots \circ \mathscr{L}_{\frac{\ell-d_1-d_2}{d_2}, \frac{d_2}{\ell-d_1}} \circ \mathscr{L}_{\frac{\ell-d_1}{d_1}, \frac{d_1}{\ell}}[\Phi]$$

$$= e^{-cz}\int_0^{\infty}\cdots\int_0^{\infty}\Phi\left(z^{\frac{\ell-\sum_{j=1}^{r} d_j}{\ell}}\prod_{i=1}^{r}\zeta_i^{\frac{d_i}{\ell}}\right)e^{-\sum_{i=1}^{r}\zeta_i}\,d\zeta_1\ldots d\zeta_r,$$

where Φ is a solution of the scalar qDE of X.

Remark : $c \neq 0$ only if $\sum d_j = \ell - 1$

Corollary : $Y \subseteq \mathbb{P}^{n-1}$ Fano complete intersection defined by hom. polynomials of degrees $(d_1, \dots, d_\ell)$, then the integral above is

$$\frac{e^{-cz}}{2\pi\sqrt{-1}}\int_\gamma \Gamma(s)^n \prod_{k=1}^{\ell}\Gamma(1-d_k s)\, z^{-(n-\sum_k d_k)s}\, e^{2\pi\sqrt{-1}\, js}\, e^{\pi\sqrt{-1}\, s}\, ds, \quad j = 0, \dots, n-1.$$

Theorem (C. 2020)

Assume

$X = \prod_{j=1}^{h} X_j$, $L_j \to X_j$ ample line bundle, $\det X_j = L_j^{\ell_j}$, $E = \boxtimes_{j=1}^{h} L_j^{\otimes d_j}$

then there exists $c \in \mathbb{C}$ such that every solution of the scalar qDE of Y is a $\mathbb{C}$-linear combination of integrals of the form

$$e^{-cz}\mathscr{L}_{\alpha,\beta}[\Phi_1,\dots,\Phi_h](z) = e^{-cz}\int_0^\infty \prod_{j=1}^{h}\Phi_j\left(z^{\frac{\ell_j-d_j}{\ell_j}}\lambda^{\frac{d_j}{\ell_j}}\right)e^{-\lambda}d\lambda,$$

where $(\alpha,\beta) = \left(\frac{\ell_1-d_1}{d_1},\dots,\frac{\ell_h-d_h}{d_h};\frac{d_1}{\ell_1},\dots,\frac{d_h}{\ell_h}\right)$, Φ_j is a solution of the scalar qDE of X_j.

<u>Remark</u>: $c \neq 0$ only if $d_j = \ell_j - 1$ for some j

<u>Corollary</u>: $Y \subseteq \prod_{i=1}^{h} \mathbb{P}^{n_i-1}$ hypersurface defined by hom. polynomial of multi-degree $(d_1,\dots,d_h)$, then the integral above is

$$\frac{e^{-cz}}{(2\pi\sqrt{-1})^h}\int_{\times\gamma_i}\left[\prod_{i=1}^{h}\Gamma(s_i)^{n_i}\varphi^i_{j_i}(s_i)\right]\Gamma\left(1-\sum_i s_i\right)z^{-\sum_i d_i s_i}\,ds_1\dots ds_h, \qquad \varphi^i_{j_i}(s_i) = \begin{cases}\exp(2\pi\sqrt{-1}\,j_i s_i) & n_i \text{ even}\\ \exp(2\pi\sqrt{-1}\,j_i s_i + \pi\sqrt{-1}\,s_i) & n_i \text{ odd}\end{cases}$$

- The proof is based on Quantum Lefschetz Theorem by A. Givental and T. Coates.

- Mirror Symmetry provides other type of integral representations of solutions of qDE, namely multi-dimensional oscillating integrals.

- The multi-dimensional Mellin-Barnes integrals of the previous theorems have tame asymptotics: study of generalized Faxén integrals

$$I(\lambda; c_1, \dots, c_r) = \int_0^\infty \exp\left[-\lambda\left(x^\mu + \sum_{k=1}^r c_k x^{m_k}\right)\right] dx,$$

$$\mu > m_1 > \cdots > m_r > 0.$$

Application: Dubrovin's conjecture for Hirzebruch surfaces,

$$\mathbb{F}_k := \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-k)), \quad k \in \mathbb{Z}.$$

$\mathbb{F}_k \subseteq \mathbb{P}^2 \times \mathbb{P}^1$
hypersurface
bidegree $(1, k)$.

- Case of $\mathbb{F}_{2k}$: it easily follows from $\mathbb{F}_0 := \mathbb{P}^1 \times \mathbb{P}^1$;
- Case of $\mathbb{F}_{2k+1}$: it is reduced to the case of $\mathbb{F}_1 := \widetilde{\mathbb{P}^2}$.

$$\mathcal{S}(\mathbb{P}^1) := \left\{\Phi\colon\ \vartheta^2\Phi = 4z^2\Phi\right\}, \quad \mathcal{S}(\mathbb{P}^2) := \left\{\Phi\colon\ \vartheta^3\Phi = 27z^3\Phi\right\}$$

$$\mathscr{P}\colon \mathcal{S}(\mathbb{P}^1) \otimes \mathcal{S}(\mathbb{P}^2) \to \mathcal{O}(\widetilde{\mathbb{C}^*})$$

$$\begin{aligned}\mathscr{P}(\Phi_1 \otimes \Phi_2;\ z) &:= e^{-z}\mathscr{L}_{(1,2;\frac{1}{2},\frac{1}{3})}[\Phi_1, \Phi_2;\ z]\\ &= e^{-z}\int_0^{\infty} \Phi_1\left(z^{\frac{1}{2}}\lambda^{\frac{1}{2}}\right)\Phi_2\left(z^{\frac{2}{3}}\lambda^{\frac{1}{3}}\right)e^{-\lambda}d\lambda,\end{aligned}$$

$\Phi_1 \in S(\mathbb{P}^1)$

$$\Phi_1(z) = \sum_{m=0}^{\infty} \left(A_{m,1} + A_{m,0} \log z\right) \frac{z^{2m}}{(m!)^2}$$

$\Phi_2 \in S(\mathbb{P}^2)$

$$\Phi_2(z) = \sum_{n=0}^{\infty} \left(B_{n,2} + B_{n,1} \log z + B_{n,2} \log^2 z\right) \frac{z^{3n}}{(n!)^3}$$

$\rightarrow$ $(A_{0,i}, B_{0,j})$ with $i=0,1$ and $j=0,1,2$ are coordinates on $S(\mathbb{P}^1) \otimes S(\mathbb{P}^2)$.

$$\mathcal{H}: \begin{cases} A_{00} B_{00} = 0 \\ 4 A_{0,1} B_{0,0} = 3 A_{0,0} B_{0,1} \end{cases}$$

$\mathcal{H}$ is iso to solutions of qDE of $\mathbb{F}_1$

$\rightarrow$ Reconstruction of Stokes bases of solutions

The central connection matrix of $\mathbb{F}_{2k+1}$ is

$$C_k = \begin{pmatrix} \frac{1}{2\pi} & -\frac{1}{2\pi} & \frac{1}{2\pi} & -\frac{1}{2\pi} \\ \frac{\gamma}{\pi} & -\frac{\gamma}{\pi} & i + \frac{\gamma}{\pi} & -i - \frac{\gamma}{\pi} \\ \frac{\gamma - 2\gamma k - i\pi}{2\pi} & -\frac{\gamma - 2\gamma k + i\pi}{2\pi} & \frac{-2\gamma k - i(2\pi k + \pi) + \gamma}{2\pi} & \frac{(2k-1)(\gamma + i\pi)}{2\pi} \\ \gamma\left(-i + \frac{2\gamma}{\pi}\right) & \gamma\left(-i - \frac{2\gamma}{\pi}\right) & \frac{2\gamma(\gamma + i\pi)}{\pi} & -\frac{2(\gamma + i\pi)^2}{\pi} \end{pmatrix}.$$

Theorem

Dubrovin conjecture holds true for all Hirzebruch surfaces.

The matrix C_k is the matrix associated with the morphism

$$Д^-_{\mathbb{F}_{2k+1}} : K_0(\mathbb{F}_{2k+1})_{\mathbb{C}} \to H^\bullet(\mathbb{F}_{2k+1}, \mathbb{C}), \quad [\mathscr{F}] \mapsto \frac{1}{2\pi} \widehat{\Gamma}^-_{\mathbb{F}_{2k+1}} \cup e^{-\pi i c_1(\mathbb{F}_{2k+1})} \cup \mathrm{Ch}(\mathscr{F}),$$

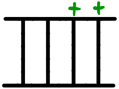
w.r.t. an exceptional basis $\mathfrak{E} := (E_i)_{i=1}^4$ of $K_0(\mathbb{F}_{2k+1})_{\mathbb{C}}$.

$\mathbb{F}_k \equiv \mathcal{O}(-k) \cup \infty$ section

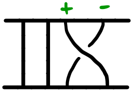
fiber of $\mathcal{O}(-k)$

∞-section

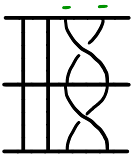
The exceptional collection $\mathfrak{E}$ is obtained from $(\mathcal{O}, \mathcal{O}(\Sigma_2), \mathcal{O}(\Sigma_4), \mathcal{O}(\Sigma_2 + \Sigma_4))$ by applying the following elements of $(\mathbb{Z}/2\mathbb{Z})^4 \rtimes \mathcal{B}_4$:



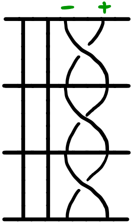
$\mathbb{F}_1$



$\mathbb{F}_3$



$\mathbb{F}_5$



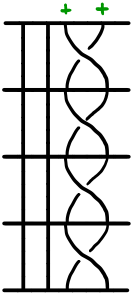
$\mathbb{F}_7$

Different Complex Structures

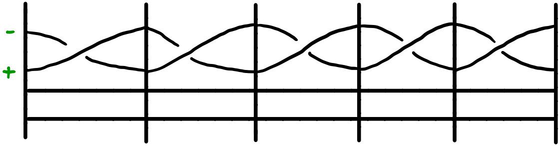
$\Downarrow$

increasing powers of the SAME BRAID

β_{34}^{k}



$\mathbb{F}_9$



$\mathbb{F}_{11}$

Thank you !