# Kyiv formula, its applications and generalizations

Max Planck Institute for Mathematics (Bonn) $\rightarrow$ University of Geneva
~~Skoltech, HSE, Moscow, Russia~~

PIICQ workshop "Excursions in Integrability"

May 25, 2022
Trieste

# Plan

# Isomonodromic deformations and Painlevé equations: the timeline

- 1910+$\epsilon$: Painlevé equations coming from classification problem, isomonodromic deformations (Painlevé, Schlesinger, Fuchs).
- 1977+: Holonomic quantum fields (Jimbo, Miwa, Sato).
- Painlevé equations in the Ising model (McCoy, Tracy, Wu).
- 1982+: Asymptotic and connection problems for Painlevé equations (Jimbo; Its, Kapaev, Novokshenov).
- 1984+: Conformal field theory (Belavin, Polyakov, Zamolodchikov).

# Isomonodromic deformations and Painlevé equations: the timeline

- 1994+: Gap probabilities and Painlevé equations (Tracy, Widom).
- 1996+: Algebraic solutions of Painlevé VI (Dubrovin, Mazzocco), relation to Frobenius manifolds and topological field theory (Dubrovin; Manin).
- 2000: Nekrasov partition functions for $\mathcal{N} = 2$ SUSY gauge theory.
- 2009: AGT conjecture relating Nekrasov functions to conformal blocks.
- 2012: Gamayun, Iorgov, Lisovyy formula for generic Painlevé VI solution (Kyiv formula).
- 2012+: Modern development.

# Simplest example of conjecture (GIL'13)

Painlevé $\mathrm{III}_3$ equation:

$$\boxed{w''(t)-\frac{w'(t)^2}{w(t)}+\frac{w'(t)}{t}+\frac{2w(t)^2}{t^2}-\frac{2}{t}=0}$$

Its generic solution $w(t)=-t^{1/2}\frac{\tau(t)^2}{\tau_1(t)^2}$:

$$\tau(t)=\sum_{n\in\mathbb{Z}}e^{4\pi in\eta}t^{(\sigma+n)^2}\mathcal{B}(\sigma+n,t),$$

$$\tau(t)=\sum_{n\in\frac{1}{2}+\mathbb{Z}}e^{4\pi in\eta}t^{(\sigma+n)^2}\mathcal{B}(\sigma+n,t),$$

where $\mathcal{B}(\sigma,t)$ are irregular $c=1$ Virasoro conformal blocks.

Useful parameterization of the central charge:
$c=1+6\frac{(\epsilon_1+\epsilon_2)^2}{\epsilon_1\epsilon_2}=1+6\left(b+b^{-1}\right)^2$

# Initial conjecture (GIL'12)

Generic tau function of the Painlevé VI equation:

$$\tau(t) = \sum_{n\in\mathbb{Z}} e^{4\pi i n\eta} t^{(\sigma+n)^2-\theta_0^2-\theta_t^2} \mathcal{B}(\sigma+n, \vec{\theta}, t).$$

It is related to isomonodromic deformations of the $2\times 2$ linear problem

$$\frac{dY(z)}{dz} = A(z)Y(z) = \left(\frac{A_0}{z} + \frac{A_t}{z-t} + \frac{A_1}{z-1}\right) Y(z),$$

$\mathrm{tr}\, A_k^2 = 2\theta_k^2$ are 4 parameters of equation,

$$\partial_t \log\tau(t)\Big|_{\text{Monodromies} = \text{const}} = \frac{1}{2} \underset{z=t}{Res}\, A(z)^2 dz.$$

# Historical remark

Vadim Knizhnik could do some of this back in 1987, and this would still be Kyiv formula

**Vadim Genrikhovich Knizhnik** (Russian: Вади́м Ге́нрихович Кни́жник; 20 February 1962, Kiev – 25 December 1987, Moscow) was a Soviet physicist of Jewish and Russian descent.

$$\frac{\partial Y}{\partial z} = \sum_{i=1}^{l} \frac{A_i}{z - a_i} Y \qquad (\text{IV}.2)$$

with given monodromy matrices $M_i$

$$\hat{\pi}_{a_i} Y(z) = Y(z) M_i, \qquad (\text{IV}.3)$$

where $Y(z)$ represents the fundamental matrix of the solutions of (2).

This connection arises as follows. Consider the Green's function for analytic fields $f$ and $\varphi$ with spins $j$ and $1-j$ on a surface $X$ specified in the form of a covering of the $z$-plane with branch points $a_i$, $i = 1,\ldots,l$:

$$Y^{km}(z, z_0) = (z_0 - z)\left\langle \varphi^{(k)}(z_0) f^{(m)}(z) \prod_i V_{\mathbf{q}_i}(a)\right\rangle \times \left\langle \prod_i V_{\mathbf{q}_i}(a_i)\right\rangle^{-1} \qquad (\text{IV}.4)$$

$(k, m = 0, \ldots, N-1)$

where the upper index on the fields $\varphi$ and $f$ represents the number of the sheet and the operators $V_{\mathbf{q}_i}(a_i)$ correspond to branch points as in Sec. 9 and 10. We assume, for the sake of simplicity, that the charges $\mathbf{q}_i$ are chosen so that

$$\tau(a_1 \ldots a_l) \overset{\text{def}}{=} \left\langle \prod_{i=1}^{l} V_{\mathbf{q}}(a_i)\right\rangle \neq 0 \qquad (\text{IV}.5)$$

Sov. Phys. Usp. **32** (11), November 1989

# Plan

# Higher rank, many points

- $N \times N$ connection matrix is

$$A(z) = \frac{A_0}{z} + \sum_{k=1}^{n-2} \frac{A_k}{z - z_k}$$

- All $A_k$ should have the form "identity + rank 1": $A_k = b_k \mathbb{I} + u_k \otimes v_k$.
- In this case isomonodromic tau function is described by multi-point $W_N$ conformal blocks (subsets of PG, Iorgov, Lisovyy).
- Isomonodromic tau function:

$$\tau = \sum_{\{\vec{w}_k \in Q_{A_N}\}} e^{4\pi i \sum_{k=1}^{n-3} (\vec{\eta}_k, \vec{w}_k)} \mathrm{F}(\vec{\sigma}_{n-3} + \vec{w}_{n-3}, \dots, \vec{\sigma}_1 + \vec{w}_1; \{z_k\}),$$

  where $Q_{A_N}$ is $\mathfrak{sl}_N$ root lattice.
- Conformal block of $W_N$ algebra:

$$\mathrm{F}(\vec{\sigma}_{n-3}, \dots; \{z_k\}) = \langle \vec{\theta}_\infty | V_{a_{n-2}\vec{\omega}_1}(z_{n-2}) \mathrm{P}_{\vec{\sigma}_{n-3}} \dots \mathrm{P}_{\vec{\sigma}_1} V_{a_1\vec{\omega}_1}(z_1) | \vec{\theta}_0 \rangle$$

# Torus (simplest example)

- Non-autonomous Calogero-Moser system:

$$\boxed{(2\pi i)^2 \frac{d^2 Q(\tau)}{d\tau^2} = m^2 \wp'(2Q(\tau)|\tau)}$$

- Tau function:

$$\partial_\tau \log \mathcal{T}_{CM}(\tau) = (2\pi i \partial_\tau Q(\tau))^2 - m^2 \wp(2Q(\tau)|\tau) + 4\pi i m^2 \partial_\tau \log \eta(\tau)$$

- Isomonodromy–CFT relation (Bonelli, Del Monte, PG, Tanzini):

$$\eta(\tau)^{-2}\theta_1(\rho + Q(\tau))\theta_1(\rho - Q(\tau))\mathcal{T}_{CM}(\tau) := \mathcal{T}_{(1,1)}(\tau) =$$

$$= \sum_{n_1,n_2\in\mathbb{Z}} e^{4\pi i(\rho+\frac{1}{2})\frac{n_1+n_2+1}{2}} e^{\frac{i(n_1-n_2)\eta}{2}} \mathrm{F}\big((a,-a)+(n_1,n_2), m; \tau\big),$$

where $\mathrm{F} = \mathrm{tr}\, q^{L_0} V_m(0)$ is a toric Virasoro $\oplus$ Heisenberg conformal block.

# Torus (general example)

- We should consider $N \times N$ connection matrix on a torus with some number of simple poles and non-trivial twist $\boldsymbol{Q}$.
- Residues at all points should be "identity + rank 1".
- Good object is $\mathcal{T}_{(1,n)}$:

$$\mathcal{T}_{(1,n)} = \eta(\tau)^{-N} \prod_{i=1}^{N} \theta_1(Q_i - \rho)\, \mathcal{T} \tag{1}$$

- $\mathcal{T}_{(1,n)}$ is a Fourier series of $W_N \oplus$ Heisenberg toric conformal blocks.

# Irregular cases

Hajime Nagoya; Bonelli, Lisovyy, Maruyoshi, Sciarappa, Tanzini; ...

- Painlevé equations other from PVI have regions with irregular expansions (like for Bessel functions at infinity).
- For some cases CFT counterparts are know, for some, no.
- The simplest example (to write) is $\mathrm{PIII}_3$ tau function at infinity (Its, Lisovyy, Tykhyy):

$$\tau^{\infty}(\rho, \nu, r) = e^{\frac{r^2}{16}} r^{\frac{1}{4}} \sum_{n\in\mathbb{Z}} e^{4\pi i n\rho} e^{(\nu+in)r} r^{\frac{1}{2}(\nu+in)^2} \mathcal{B}^{\infty}(\nu+in, r),$$

$$\tau_1^{\infty}(\rho, \nu, r) = e^{\frac{r^2}{16}} r^{\frac{1}{4}} \sum_{n\in\mathbb{Z}} (-1)^n (\ldots),$$

where $r = t^{1/4}/8$, and $\mathcal{B}^{\infty}$ has no CFT (or any other) formula yet.

# q-deformation: q-$\mathrm{PIII}_3$

q-Painlevé $\mathrm{III}_3$:

$$\boxed{G(Zq^{-1})^{\frac{1}{2}}G(Zq)^{\frac{1}{2}} = \frac{G(Z)+Z}{G(Z)+1}}$$

Expression in terms of tau functions $G(Z) = -Z^{1/2}\frac{\tau(Z)^2}{\tau_1(Z)^2}$ (Bershtein, Shchechkin):

$$\tau(Z) = \sum_{n\in\mathbb{Z}} e^{4\pi i n\eta} Z^{(\sigma+n)^2}\mathcal{B}_q(\sigma+n, Z),$$

$$\tau_1(Z) = \sum_{n\in\frac{1}{2}+\mathbb{Z}} e^{4\pi i n\eta} Z^{(\sigma+n)^2}\mathcal{B}_q(\sigma+n, Z),$$

where $\mathcal{B}_q$ are q-deformed conformal blocks

# q-deformation: general case

- Large class of q-difference systems can be obtained from deautonomized discrete flows in Goncharov–Kenyon integrable systems (Bershtein, PG, Marshakov).
- Initial combinatorial datum is a Newton polygon.
- On the isomonodromic side it defines dimer lattice, quiver and related cluster algebra, and then discrete flows (=quiver automorphisms).
- On the "CFT" side it defines partition function of topological strings on toric $CY_3$, the analog of conformal block.
- Fourier transformation of TS partition function solves the discrete flow.

# Quantum deformation

- Should not be confused with q-deformation.
- Corresponds to arbitrary central charge.
- Will come after some motivation.

# Plan

# Proofs of the Kyiv formula

1. From quantum monodromies of conformal blocks:
   1. Initial $2 \times 2$ problem: Iorgov, Lisovyy, Teschner
   2. $W_N$ case: PG, Iorgov, Lisovyy
   3. q-difference case: Jimbo, Nagoya, Sakai
   4. Toric case: Bonelli, Del Monte, PG, Tanzini
2. From $\mathbb{C}^2/\mathbb{Z}_2$ blow-up relations (Bershtein, Shchechkin)
3. Fully rigorous: from the Fredholm determinant:
   1. Spherical case, $2 \times 2$: PG, Lisovyy
   2. Spherical case, $N \times N$: PG, Iorgov, Lisovyy
   3. Toric case: Del Monte, Desiraju, PG
4. From $\mathbb{C}^2$ blow-up relations (Nekrasov). Blow-up relations still have to be proved.

# Iorgov, Lisovyy, Teschner proof

1. Consider conformal block with insertion of two $\phi_{(2,1)}$ degenerate fields $\phi_i$ at $y$ and $y_0$:
$\Psi_{ij}(y, y_0) = \langle\theta_\infty|\phi_i(y)\phi_{-j}(y_0)V(z_{n-2})\mathrm{P}_{\sigma_{n-3}}V(z_{n-3})\ldots\mathrm{P}_{\sigma_1}V(z_1)|\theta_0\rangle$
2. Compute monodromies of $\phi_{(2,1)}$ around all points. They are operator-valued functions $\hat{M}_\nu = \hat{M}_\nu(\{e^{2\pi i b\sigma_k}\}, \{e^{b\partial_{\sigma_k}}\})$.
3. For $b^2 = -1$ diagonalize $e^{2\pi i b\sigma}$ and $e^{b\sigma}$ by Fourier transformation.
4. Express Fourier transformation of $\Psi(y, y_0)$ in terms of solution of the linear system $Y(y)$.
5. Identify tau function.

What about other central charges?

# Tau function as a conformal block

- Tau function is a concrete element in the space of $c = 1$ conformal blocks.
- It diagonalizes the action of Verlinde loop operators (see previous slide).
- It also simplifies the action of Moore-Seiberg gruppoid:

$$\tau(\ldots, 1-t) = \chi_{01}\, \tau(\widetilde{\ldots}, t).$$

  $\chi_{01}$ is called connection constant.
- Knowledge of $\chi_{01}$ gives fusion matrix for conformal blocks. It is conjectured by Iorgov, Lisovyy, Tykhyy, and proved by Its, Lisovyy, Prokhorov.

# Bershtein–Shchechkin proof

1. Substitute Fourier-type ansatz into the equation.
2. Get bilinear equations for conformal blocks ($\mathbb{C}^2/\mathbb{Z}_2$ blow-up relations).
3. Prove these relations from conformal field theory, or from Nakajima-Yoshioka ($\mathbb{C}^2$) blow-up relations.

These relation exist for arbitrary central charge ($\epsilon_1 \neq -\epsilon_2$).
Typical form of $\mathbb{C}^2/\mathbb{Z}_2$ blow-up relation:

$$\sum_{2n \in \mathbb{Z}} \hat{\mathrm{D}}\Big(\mathsf{F}(\sigma + 2n\epsilon_1, \mathbf{2}\epsilon_1, \epsilon_2 - \epsilon_1 | t), \mathsf{F}(\sigma + 2n\epsilon_2, \epsilon_1 - \epsilon_2, \mathbf{2}\epsilon_2 | t)\Big) = \mathsf{F}_{NSR}(= 0)$$

It is used only for $\epsilon_1 = -\epsilon_2$ $(c = 1)$. What about other central charges?

# Nakajima–Yoshioka blow-up relations

Typical Nakajima–Yoshioka, or $\mathbb{C}^2$, blow-up relation:

$$\mathrm{F}(\sigma, \epsilon_1, \epsilon_2) = \sum_{n\in\mathbb{Z}} \mathrm{F}(\sigma + n\epsilon_1, \epsilon_1, \epsilon_2 - \epsilon_1 | t)\mathrm{F}(\sigma + n\epsilon_2, \epsilon_1 - \epsilon_2, \epsilon_2 | t).$$

- For $\epsilon_2 = 0$ we have relation between $c = \infty$ and $c = 1$ conformal blocks.
- Such kind of relations were used by Nekrasov to relate Litvinov, Lukyanov, Nekrasov, Zamolodchikov paper about $c = \infty$ conformal blocks to GIL.
- $c = \infty$ conformal block describe vanishing of $c = 1$ tau function, and so spectral problems for something like cos-potential.
- For $\epsilon_1 = -\epsilon_2$ there is a formula $\tau^{PIII_3} = \tau_+\tau_-$, where $\tau_\pm$ are $\underline{c = -2}$, or $(\varepsilon_1, \varepsilon_2) = (-1, 2)$ tau functions (Bershtein, Shchechkin).

# Plan

# Motivations for quantum deformation

1. Iorgov, Lisovyy, Teschner quantization of monodromies.
2. Bershtein–Shchechkin $\mathbb{C}^2/\mathbb{Z}_2$ bilinear relations for arbitrary $c$.
3. Very natural quantization of cluster discrete flows.

Consequences:

1. $\sigma$ and $\eta$ should be replaced with $\hat{\sigma}$ and $\hat{\eta}$.
2. It's better to start from q-difference equations.

# Quantum q-PIII$_3$ equation

The equation:

$$\begin{cases} \hat{G}(Zq^{-1})^{\frac{1}{2}}\hat{G}(Zq)^{\frac{1}{2}} = \dfrac{\hat{G}(Z)+pZ}{\hat{G}(Z)+p}, \\ \hat{G}(Z)\hat{G}(q^{-1}Z) = p^4\hat{G}(q^{-1}Z)\hat{G}(Z). \end{cases}$$

where $p^2 = e^{l_5(\epsilon_1+\epsilon_2)}$, $q = e^{2l_5\epsilon_2}$. Its solution:

$$\hat{G}(Z)^{\frac{1}{2}} = \pm i p^{\frac{1}{2}} Z^{\frac{1}{4}} \left( \sum_{n\in\frac{1}{2}+\mathbb{Z}} e^{4\pi i\hat{\eta}n} \mathsf{F}_{5d}\left(\hat{\sigma}+2n\epsilon_2, \epsilon_1-\epsilon_2, 2\epsilon_2|Z\right)\right)^{-1}$$

$$\sum_{n\in\mathbb{Z}} e^{4\pi i\hat{\eta}n} \mathsf{F}_{5d}\left(\hat{\sigma}+2n\epsilon_2, \epsilon_1-\epsilon_2, 2\epsilon_2|Z\right),$$

Commutation relation: $[\hat{\sigma}, \hat{\eta}] = \frac{\epsilon_1+\epsilon_2}{2\pi i}$.

# Quantum PIII$_3$ at infinity: new conformal blocks

- $q \to 1$ limit of quantum equation:

$$\begin{cases} 4\epsilon_2^2 t \dfrac{d}{dt}\left(t\dfrac{d\hat{w}}{dt}\cdot \hat{w}^{-1}\right) = \dfrac{2t}{\hat{w}} - 2\hat{w}, \\ \left[\hat{w}^{-1}, t\dfrac{d\hat{w}}{dt}\right] = 2(\epsilon_1+\epsilon_2). \end{cases}$$

- Its solution (PG, Marshakov, Stoyan):

$$\hat{w}(r)^{\frac{1}{2}} = \pm\frac{r}{8}\left(\sum_{n\in\mathbb{Z}}(-1)^n e^{4\pi i n\hat{\rho}}\mathrm{F}^{\infty}\left(\hat{\nu}+2in\epsilon_2, \epsilon_1-\epsilon_2, 2\epsilon_2|r\right)\right)^{-1}$$
$$\sum_{n\in\mathbb{Z}} e^{4\pi i n\hat{\rho}}\mathrm{F}^{\infty}\left(\hat{\nu}+2in\epsilon_2, \epsilon_1-\epsilon_2, 2\epsilon_2|r\right),$$

- $\mathrm{F}^{\infty}$ are the arbitrary-$c$ analogs of Its, Lisovyy, Tykhyy "conformal blocks". Cannot be found in any other way yet.

# Perspectives

- There is a lot of conjectures still to be proved, like conjecture about solution for the general non-autonomus Goncharov–Kenyon integrable system, also the formulas for q-deformed Fredholm determinants.
- Formulas for more irregular expansions still have to be found. Also, it is unclear if there is any phenomenon of this kind for q-difference equations.
- There should be also generalizations to higher genus, to Lie algebras of other series (Del Monte, et al.)
- The story about other central charges should be understood better: what is the correct quantization of other Painlevé equations, what is the role of $b^2 = -n$, what are the linear systems in all these cases, etc.

Thank you for your attention!