Optimal control for PDEs with uncertain coefficients

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Introduction

- optimal control of partial differential equations (PDEs) with uncertain parameters
- the random parameters are **not observable**
- look for a **deterministic robust** control
- introducing some **risk measure** to take care of all realizations of our uncertain parameters
- theoretical framework based on **adjoint calculus** to compute the gradient of the objective functional
- application in aerodynamics with particular emphasis on the optimal control of the shape of airfoils under variable operating condition (e.g. Mach, angle of attack, incoming pressure). Uncertainty: manufacture design. Robustness: passengers life.

Settings

First example: a simple elliptic PDE with

$$\begin{cases} -\operatorname{div}(a(x,\omega)\nabla y(x,\omega)) = g(x) + u(x), & x \in D,\ \omega \in \Gamma, \\ y(x,\omega) = 0, & x \in \partial D,\ \omega \in \Gamma. \end{cases} \tag{1}$$

- a is a random coefficient
- D is the spatial domain
- $(\Gamma, \mathcal{F}, P)$ is a complete probability space
- $y \in Y := L^2(\Gamma, H_0^1(D))$ is the state solution
- $u \in L^2(D)$ is the deterministic control term
- $g \in L^2(D)$ is a known source term

$$u \in \arg\min_{u \in U} J(u), \quad \text{s.t.} \quad y_\omega(u) \quad \text{solves} \quad (1) \quad \text{a.s.} \tag{2}$$

$J(u) := \mathbb{E}[f(u,\omega)]$ with $f(u,\omega) = \frac{1}{2}\|y_\omega(u) - z_d\|^2 + \frac{\alpha}{2}\|u\|^2$
z_d:target function
$\|\cdot\| = \|\cdot\|_{L^2(D)}$ the $L^2(D)$-norm induced by the inner product $\langle \cdot, \cdot \rangle$.

Numerical approximation

We will use three types of approximations:

- a Finite Element approximation, to discretize the physical domain D with a mesh of characteristic size h;
- a collocation method (e.g. Monte Carlo or sparse/full grid approximation) to approximate the expectation using realizations of the random variables;
- a steepest-descent type algorithm to compute an optimal function u. Here, we use a Fixed step-size gradient method.