Stochastic Collocation Methods for Stability Analysis of Dynamical Systems

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Problem Definition

Dynamical system: $u_t = f(u), \quad f = f_\nu$, parameter ν

Discrete (in space) form: $M\mathbf{u}_t = \mathbf{f}(\mathbf{u}), \quad \mathbf{f} = \mathbf{f}_\nu$

Steady discrete solution $\mathbf{u}^{(s)} = \mathbf{u}^{(s)}(\cdot,\nu), \quad \frac{\partial \mathbf{u}^{(s)}}{\partial t} = 0$

Question: Is $\mathbf{u}^{(s)}$ stable with respect to perturbation?

Given a (small) perturbation $\boldsymbol{\delta}$

If $\mathbf{u}^{(s)} + \boldsymbol{\delta}$ is specificied as an initial condition, does the resulting $\mathbf{u}(\cdot, t; \nu)$

- revert to $\mathbf{u}^{(s)}(\cdot,\nu)$ (decay in time) $\Longrightarrow$ stable
- not revert to $\mathbf{u}^{(s)}(\cdot,\nu)$ $\Longrightarrow$ unstable

Stability Analysis

Jacobian $J \equiv J_\nu(\mathbf{u}^{(s)}) = \frac{\partial \mathbf{f}}{\partial \mathbf{u}}(\mathbf{u}^{(s)})$

Eigenvalue problem $J\mathbf{v} = \lambda M\mathbf{v}$

Solution $\mathbf{u}^{(s)}(\cdot, \nu)$ is **linearly stable**: $Re(\lambda) < 0$ for all λ
linearly unstable: $Re(\lambda) > 0$ for some λ

Linear instability $\Longrightarrow$ nonlinear instability
Implications of linear stability not as clear

Examples (Trefethen, Trefethen, Reddy & Driscoll):

- Linearly stable models that exhibit large *transient* growth
  plane Couette flow, plane Poiseuille flow
- Explored using *pseudospectra*
  (Trefethen, et al., Trefethen & Embree)

Pseudospectra

Eigenvalue problem $J\mathbf{v} = \lambda M\mathbf{v}$

For $M = \boldsymbol{I}$, perturbed problem

$$(J + E)\mathbf{v} = \lambda\mathbf{v}$$

E a perturbation, $\|E\| \leq \epsilon$

Explore pseudospectra $\cup_{\|E\| \leq \epsilon}\sigma(E)$

More in sync with transient growth

Expensive

Cf. Embree & Keeler for $M \neq \boldsymbol{I}$

Use of Surrogate Models

Eigenvalue problem $J\mathbf{v} = \lambda M\mathbf{v}$, recalling $J = J(\mathbf{u}_\nu^{(s)})$

Let $\mathbf{u}_\nu^{(s)} + \boldsymbol{\delta}$ be a perturbation of the steady solution $\mathbf{u}_\nu^{(s)}$

Idea: consider perturbed eigenvalue problem $\hat{J}(\mathbf{u}_\nu^{(s)}, \boldsymbol{\delta})\,\mathbf{v} = \hat{\lambda}\, M\mathbf{v}$
Generate perturbation $\boldsymbol{\delta} \equiv \boldsymbol{\delta}(\boldsymbol{\xi})$ in a systematic way,
depending on some (other) parameters $\boldsymbol{\xi} \equiv (\xi_1, \ldots, \xi_m)^T$

$$\hat{J}(\mathbf{u}_\nu^{(s)}, \boldsymbol{\delta}(\boldsymbol{\xi}))\,\mathbf{v} = \hat{\lambda}\, M\mathbf{v} \tag{1}$$

Let $g(\boldsymbol{\xi}) \equiv$ rightmost eigenvalue of (1)
$g^{(I)}(\boldsymbol{\xi}) \equiv$ surrogate approximation of $g(\boldsymbol{\xi})$, cheaper to compute

Pseudo-spectral experiment: study values of $g^{(I)}(\boldsymbol{\xi})$

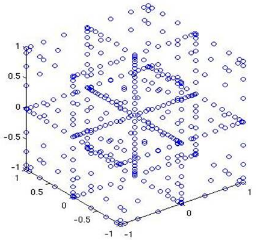
Done by *sampling* $\boldsymbol{\xi}$

Background: Collocation for Surrogate Model

To approximate a function $g(\boldsymbol{\xi})$ of m parameters $\xi_1, \ldots, \xi_m$:

- Choose particular realizations of $\boldsymbol{\xi}$: $\{\boldsymbol{\xi}^{(k)}, k = 1, 2, \ldots, n_{\boldsymbol{\xi}}\}$
  Evaluate $\{g(\boldsymbol{\xi}^{(k)})\} =$
  $\{$rightmost eigenvalues of $\hat{J}(\mathbf{u}_\nu^{(s)}, \delta(\boldsymbol{\xi}^{(k)}))\, \mathbf{v} = \hat{\lambda}\, M\mathbf{v}\}$
  Entails solving eigenvalue problems for realizations $\boldsymbol{\xi} = \boldsymbol{\xi}^{(k)}$
  N.B. Not cheap, but can use unperturbed eigenvalue as a shift

- Surrogate $g^{(I)}(\boldsymbol{\xi})$ taken to be the interpolate



$$g^{(I)}(\boldsymbol{\xi}) = \sum_{k=1}^{n_{\boldsymbol{\xi}}} g(\boldsymbol{\xi}^{(k)}) L_{\boldsymbol{\xi}^{(k)}}(\boldsymbol{\xi})$$ — very cheap

For samples: use sparse grid points

Used to approximate functions on high-dimensional spaces (Barthelmann, Novak, & Ritter)

Benchmark Problem: Incompressible Navier-Stokes

Navier-Stokes equations:

$$\begin{aligned} \vec{u}_t - \nu\nabla^2\vec{u} + \vec{u}\cdot\nabla\vec{u} + \nabla p = \vec{f} \quad &\text{in} \quad \mathcal{D} \\ \nabla\cdot\vec{u} = 0 \quad &\text{in} \quad \mathcal{D} \end{aligned}$$

Posed on $\mathcal{D} \subset \mathbb{R}^d$ with suitable boundary conditions
Steady velocity solution $\vec{u}^{(s)}$, perturbation $\vec{u}^{(s)} + \vec{\delta}$

Equations of linear stability analysis: eigenvalue problem

$$\begin{aligned} -\nu\nabla^2\vec{\delta} + \vec{u}^{(s)}\cdot\nabla\vec{\delta} + \vec{\delta}\cdot\nabla\vec{u}^{(s)} + \nabla\eta \quad &= -\lambda\vec{\delta} \quad \text{in } \mathcal{D} \\ \nabla\cdot\vec{\delta} \quad &= \;0 \qquad \text{in } \mathcal{D} \end{aligned}$$

Discrete version

$$\underbrace{\begin{pmatrix} F(\mathbf{u}^{(s)}) & B^T \\ B & 0 \end{pmatrix}}_{J(\mathbf{u}^{(s)})} \underbrace{\begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix}}_{\mathbf{v}} = \lambda \underbrace{\begin{pmatrix} -Q & \alpha B^T \\ \alpha B & 0 \end{pmatrix}}_{M} \underbrace{\begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix}}_{\mathbf{v}}$$

Nonzero α (using $\alpha = -.1$) makes M nonsingular (Cliff, Garrett & Spence)

Perturbed Eigenvalue Problem

Steady flow $\vec{u}^{(s)}$ gives

$$\begin{pmatrix} F(\mathbf{u}^{(s)}) & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \lambda \begin{pmatrix} -Q & \alpha B^T \\ \alpha B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix}$$

Discrete perturbed flow $\mathbf{u}^{(s)} + \boldsymbol{\delta} \longrightarrow$ perturbed eigenvalue problem

$$\begin{pmatrix} F(\mathbf{u}^{(s)}) + N & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \lambda \begin{pmatrix} -Q & \alpha B^T \\ \alpha B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix}$$

For the perturbation (E. Phillips): $\boldsymbol{\delta}(\boldsymbol{\xi}) \equiv curl_h\, \phi(\cdot, \boldsymbol{\xi})$,

$$\phi(x, \boldsymbol{\xi}) \equiv \sigma \sum_{\ell=1}^{m} \sqrt{\mu_\ell}\, \phi_\ell(x)\, \xi_\ell, \quad \text{Karhunen-Loève expansion}$$

Perturbed eigenvalue problem

$$\begin{pmatrix} F(\mathbf{u}^{(s)}) + N(\boldsymbol{\xi}) & B^T \\ B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \lambda \begin{pmatrix} -Q & \alpha B^T \\ \alpha B & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix}$$

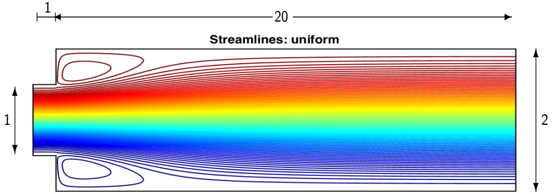
Key points:

- The perturbation is (discrete) divergence free:
  $div_h\, \boldsymbol{\delta}(\boldsymbol{\xi}) = div_h\, curl_h\, \phi(\boldsymbol{\xi}) = 0$ for any $\boldsymbol{\xi}$
- The perturbation is non-dissipative: $N(\boldsymbol{\xi}) = -N(\boldsymbol{\xi})^T$ for any $\boldsymbol{\xi}$
- Compute pseudospectra using surrogate function (interpolant):
  $$g^{(I)}(\boldsymbol{\xi}) = \sum_{k=1}^{n_{\boldsymbol{\xi}}} g(\boldsymbol{\xi}^{(k)}) L_{\boldsymbol{\xi}^{(k)}}(\boldsymbol{\xi})$$
- Collocation points $\{\boldsymbol{\xi}^{(k)}\}$ chosen using sparse-grid collocation.
  In experiments, using package `spinterp` (Klimke & Wohlmuth)

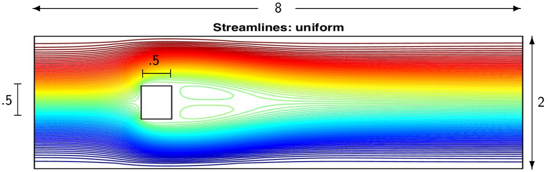
Comparison: Eigenvalue Analysis and Simulation in Time

Two benchmark problems:

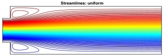
(1) Flow in expanding step
Critical viscosity $\nu \approx 1/220.5$
Real rightmost eigenvalue
Pitchfork bifurcation

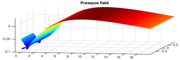


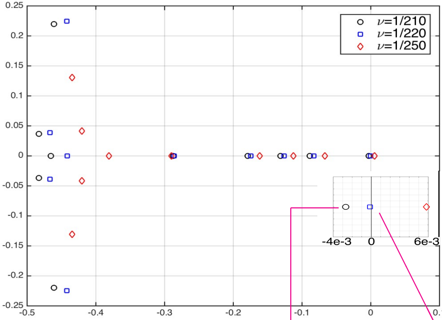
(2) Flow around square obstacle
Critical viscosity $\nu \approx 1/186$
Complex conjugate rightmost eigenvalues, Hopf bifurcation



Eigenvalues for step problem



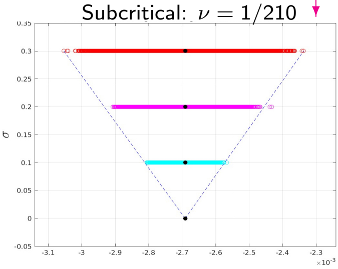


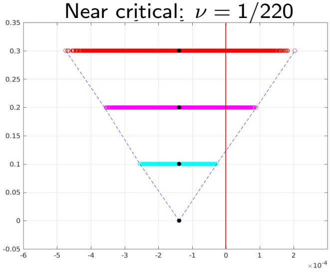


ν	λ
$1/210$	-2.7×10^{-3}
$1/220$	-1.4×10^{-4}
$1/250$	5.8×10^{-3}

Perturbed eigenvalues

For this:
- Solve 760 perturbed eigenvalue problems
- Sample surrogate
- 1M samples, ∼5 min





Simulation in Time

Experiment: Simulate laboratory scenario

1. Start from quiescent state, integrate to steady state
   Done using adaptive stabilized trapezoidal rule
   (Gresho, Griffiths, Silvester)

2. Perturb the velocity and continue the integration until either
   – flow returns to steady state, or
   – something else happens

Assessed using

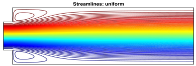
Acceleration $a(t) = \sqrt{\int_{\mathcal{D}} \left(\frac{\partial \vec{u}_h}{\partial t}\right)^2}$, small if velocity $\vec{u}_h$ is steady

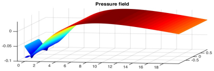
Mean vorticity $\omega(t) = \int_{\mathcal{D}} \nabla \times \vec{u}_h(\cdot, t) = \int_{\partial \mathcal{D}_{\mathcal{N}}} u_y(\cdot, t)\, \mathrm{ds}$,

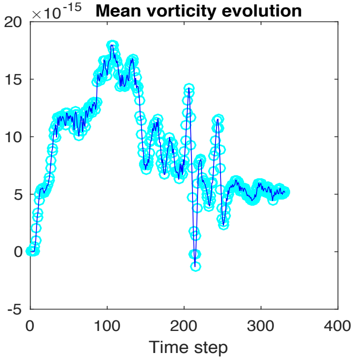
avg vertical velocity at outflow, 0 for reflectionally symmetric flow

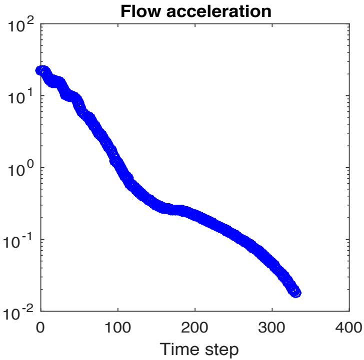
Preliminary: What happens for *supercritical* viscosity, $\nu = 1/250$?

Answer: Steady-state solution is *nearly* found

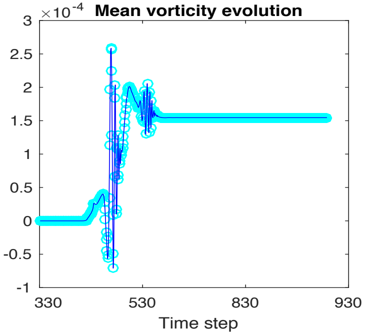


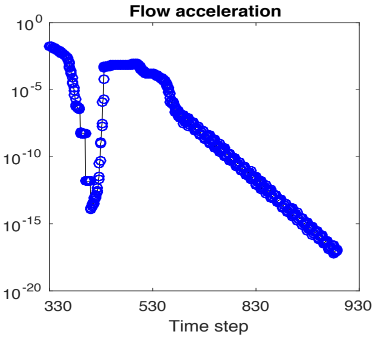




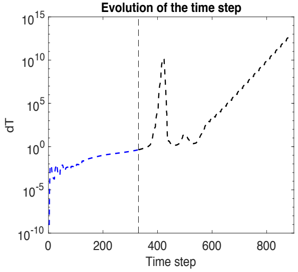


What happens next, after interrupt, w/o perturbation?

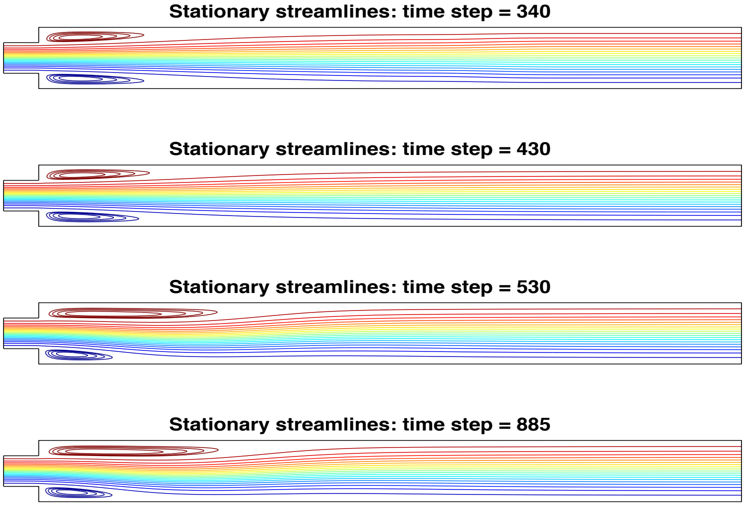




Additional insight from automatic time stepping:

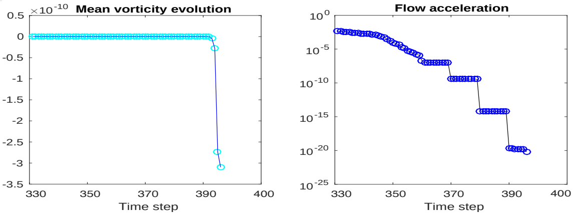


Solution obtained: symmetry breaking

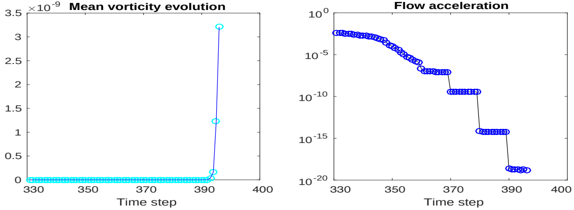


Repeat experiment for **subcritical** $\nu = 1/210$

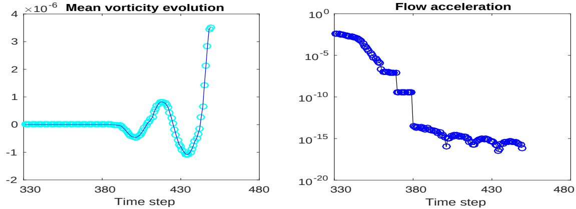
Long-term behavior, no perturbation



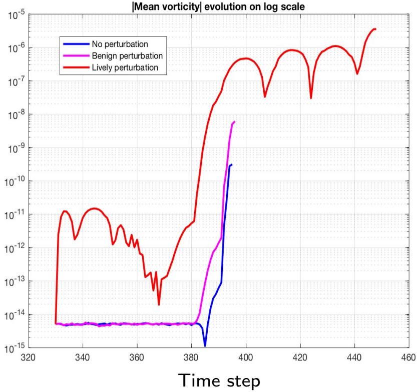
Long-term behavior, perturbation #1 (benign)

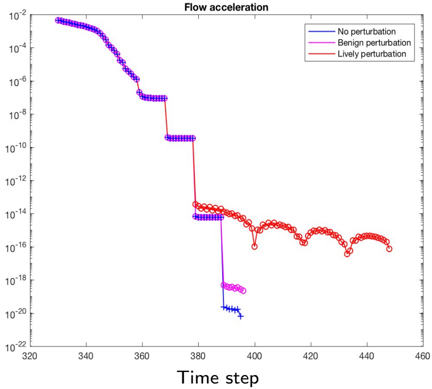


Long-term behavior, perturbation #2 (lively)



Display these results differently:



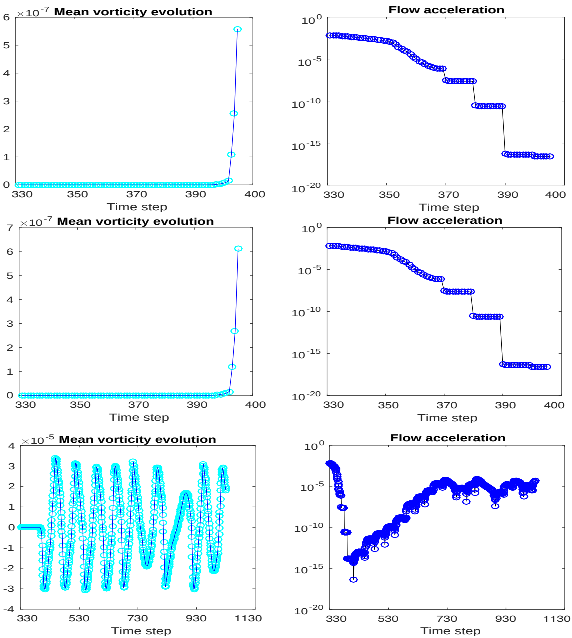


Repeat experiment for **near critical** ν=1/220

Long-term behavior, no perturbation

Long-term behavior, perturbation #1 (benign)

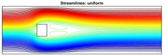
Long-term behavior, perturbation #2 (lively)

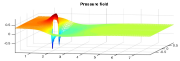


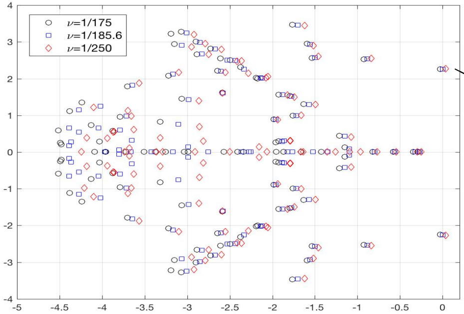
Summarizing these results, for flow in expanding step:

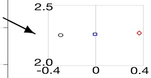
- Transient iteration is consistent with perturbation analysis
  – Instability for near-critical parameter is displayed
  – Flow for sub-critical (but barely so) parameter is stable but slight leanings to instability can be observed
- Symmetry-breaking for super-critical parameter
- Effects can also seen in time step choices made by a good integrator

Eigenvalues for obstable problem







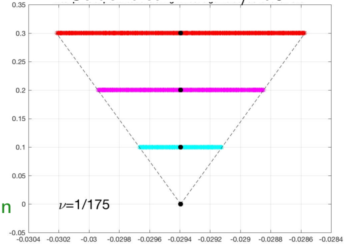


ν	$\mathrm{Re}(\lambda)$
$1/175$	-2.9×10^{-2}
$1/185.6$	-3.0×10^{-4}
$1/200$	3.7×10^{-2}

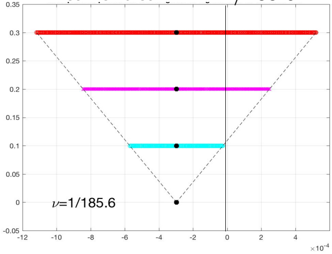
Perturbed eigenvalues

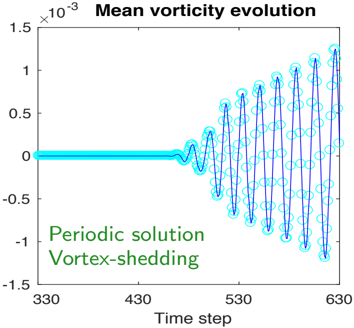
For this:
Solve 760 perturbed eigenvalue problems
Sample surrogate
100K samples, ∼1 min

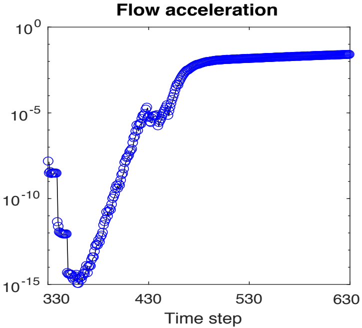
Subcritical: $\nu = 1/175$



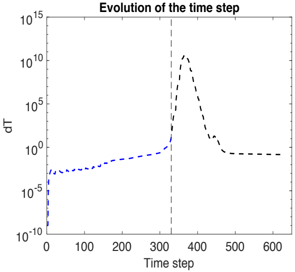
Near critical: $\nu = 1/185.6$







Simulation for obstacle,
super-critical $\nu = 1/200$
after interrupt

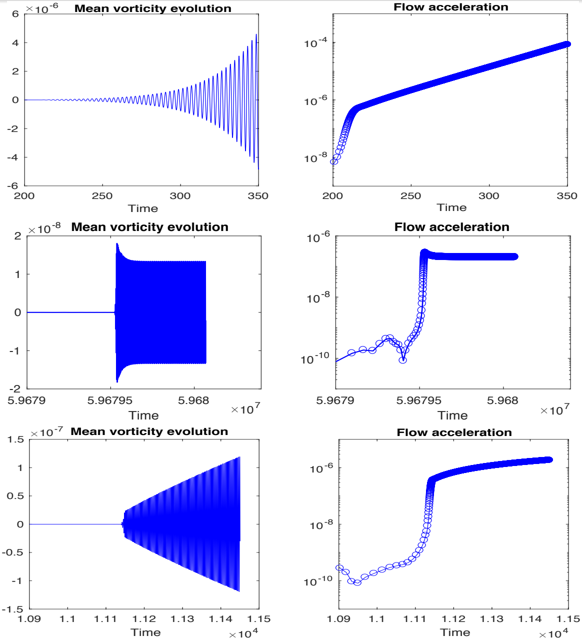


Highlights of evolution for three parameters, with **no perturbation**

Super-critical, $\nu = 1/200$

Sub-critical, $\nu = 1/175$

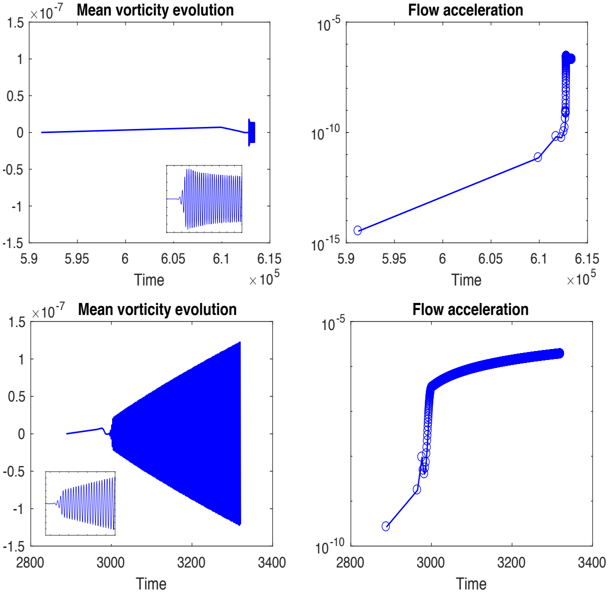
Near-critical, $\nu = 185.6$



Impact of perturbation (lively)

Subcritical parameter
$\nu = 1/175$

Near-critical parameter
$\nu = 1/185.6$



Summarizing these results, for flow around obstacle:

- Transient iteration is again consistent with perturbation analysis
  - For sub-critical parameter, performance with perturbation is like that for no perturbation
  - For near-critical parameter, performance with perturbation is like that for super-critical regime
- Results affected by delicacy of stability analysis
  - Some instability is seen even for subcritical parameters
    Caused by truncation error in transient iteration

For both benchmark problems:

- New relatively cheap method for finding pseudospectra is predictive of behavior of simulation in time
- Refined understanding of simulation in time near stability limit