Sparse grid approximation of elliptic PDEs with lognormal diffusion coefficient

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Outline

The lognormal problem

Elliptic PDE with lognormal diffusion coefficient

Approximate solution $u\colon \mathbb{R}^{\mathbb{N}} \mapsto H_0^1(D)$ of random elliptic PDE on $D \subset \mathbb{R}^d$

$$-\nabla \cdot (a(x,\boldsymbol{\xi})\, \nabla u(x,\boldsymbol{\xi})) = f(x), \qquad u(x,\boldsymbol{\xi}) = 0 \text{ on } \partial D$$

with lognormal diffusion coefficient

$$\log a(x,\boldsymbol{\xi}) = \phi_0(x) + \sum_{m=1}^{\infty} \phi_m(x)\, \xi_m, \qquad \boldsymbol{\xi} \sim \mu = \bigotimes_{m\geq 1} N(0,1),$$

where $\phi_0, \phi_m \in L^\infty(D)$ and series converges μ-a.e. in $L^\infty(D)$.

Under mild assumptions there holds

$$u \in L^2_\mu(\mathbb{R}^{\mathbb{N}}; H_0^1(D)) = \left\{ v\colon \mathbb{R}^{\mathbb{N}} \to H_0^1(D) \text{ s. t. } \int_{\mathbb{R}^{\mathbb{N}}} \|v(\boldsymbol{\xi})\|^2_{H_0^1(D)}\, \mu(d\boldsymbol{\xi}) < \infty \right\}$$

Part I

Sparse grids approximation of the lognormal problem

Existing results

Convergence analysis and algebraic convergence rates in infinite dimensions available so far for:

- **Best N-term approximations**
  - Bounded $\boldsymbol{\xi} \in [-1,1]^{\mathbb{N}}$: *Cohen et al., 2011*; *Bachmayr et al., 2016*
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- **Sparse grid quadrature methods**
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  - Bounded $\boldsymbol{\xi} \in [-1,1]^{\mathbb{N}}$: *Chkifa et al., 2014*; *Chkifa et al., 2015*
  - Gaussian $\boldsymbol{\xi} \in \mathbb{R}^{\mathbb{N}}$: **this talk!**

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Notation: in this talk:

- m, M refer to random variables;
- n, N to terms in expansion.

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  - associated sparse grid Ξ_Λ

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$$\|u - U_{\Lambda_N} u\|_{L^2_\mu} \leq C N^{-s}, \qquad s = \frac{1}{p} - \frac{1}{2}$$

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2. *Gauss-Hermite nodes satisfy the bound above with* $\theta = 1, K \geq 2.18e$

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- The proof is constructive, provides an estimate of the optimal set Λ_N

Sketch of proof - see also *Bachmayr et al., 2016 & Chen, 2016*

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$$\leq \left(\sum_{\boldsymbol{\nu}\in\mathcal{F}} b_{\boldsymbol{\nu}}\|f_{\boldsymbol{\nu}}\|^2_{H^1_0(D)}\right)^{1/2} \cdot \left(\sum_{\boldsymbol{\nu}\in\mathcal{F}\setminus\Lambda_N} \hat{c}^2_{\boldsymbol{\nu}}\right)^{1/2}, \ \hat{c}_{\boldsymbol{\nu}} \text{ is monotone} \Rightarrow \text{use Stechkin}$$

$$\leq C(N+1)^{-(1/p-1/2)}, \text{ then use bound } |\Xi_{\Lambda_N}| = \mathcal{O}(N^2)$$

Numerical results

lognormal problem on $D = [0, 1]$ (*Bachmayr et al., 2016*)

$$-\frac{\mathrm{d}}{\mathrm{d}x}\left(a(x,\boldsymbol{\xi})\,\frac{\mathrm{d}}{\mathrm{d}x}u(x,\boldsymbol{\xi})\right) = 0.03\sin(2\pi x), \qquad u(0,\boldsymbol{\xi}) = u(1,\boldsymbol{\xi}) = 0$$

where $\log a$ behaves like a smoothed Brownian bridge:

$$\log a(x,\boldsymbol{\xi}) = 0.1\sum_{m=1}^{\infty} \underbrace{\frac{\sqrt{2}}{(\pi m)^q}\sin(m\pi x)}_{=:\phi_m(x)}\,\xi_m, \qquad q \geq 1.$$

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Predicted convergence rate

$$\|u - U_{\Lambda_N}u\|_{L^2_\mu} \;\leq\; CN^{-(q-1.5)} \;\leq\; C|\Xi_{\Lambda_N}|^{-\left(\frac{q-1.5}{2}\right)}.$$

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Sparse grid software

Sparse grids Matlab kit, available at `https://csqi.epfl.ch/`.
Latest version: 17-5

Algorithm to generate the sets Λ_N

Algorithm based on *Gerstner & Griebel, 2003*

Build up Λ_N by subsequently adding new multiindex from neighborhood

$$\Lambda_{N+1} := \Lambda_N \cup \{\boldsymbol{\nu}_N^*\}, \qquad \boldsymbol{\nu}_N^* = \underset{\boldsymbol{\nu} \in \mathcal{N}(\Lambda_N)}{\arg\max}\, h(\boldsymbol{\nu})$$

where $h\colon \mathcal{F} \to \mathbb{R}$ is a heuristic (for improvement by adding $\boldsymbol{\nu}$) and

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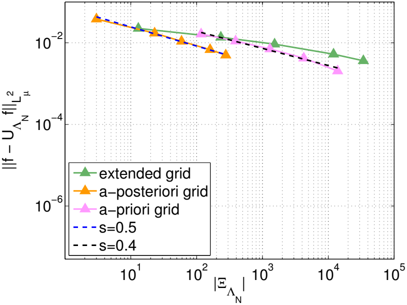
Two choices for $h(\boldsymbol{\nu})$

- **A-priori heuristic:** $h(\boldsymbol{\nu}) = \hat{c}_{\boldsymbol{\nu}}$ from the constructive proof
- **Adaptive heuristic:** $h(\boldsymbol{\nu}) \approx \dfrac{\|\Delta_{\boldsymbol{\nu}} u\|_{L^2_\mu}}{|\Xi^{(\boldsymbol{\nu})}|}$

  evaluated by quadrature, given evaluations of u at tensor grid $\Xi^{(\boldsymbol{\nu})}$.

Results: convergence wrt $|\Xi_{\Lambda_N}|$



$q = 1$
expect no convergence;
a-priori $s = 0.4$;
a-posteriori $s = 0.5$

- Extended grid = a-posteriori with evaluations in the neighbourhood
- Expected rate smaller than observed:
  - summability argument could be improved
  - bound between number of elements and points not sharp

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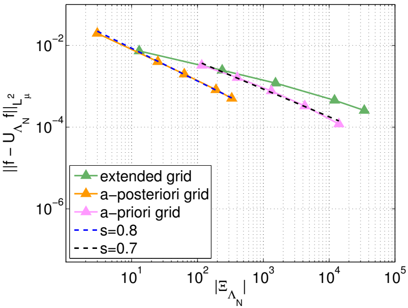
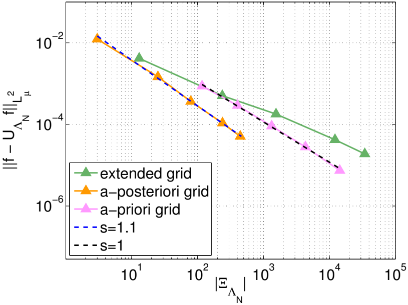


$q = 1.5$
expect $s = 0$;
a-priori $s = 0.7$;
a-posteriori $s = 0.8$

- Extended grid = a-posteriori with evaluations in the neighbourhood
- Expected rate smaller than observed:
  - summability argument could be improved
  - bound between number of elements and points not sharp

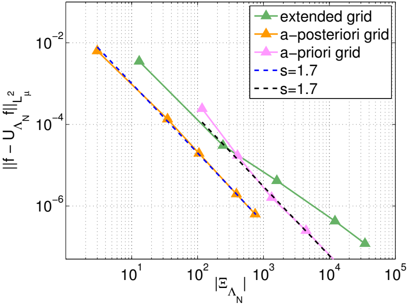
Results: convergence wrt $|\Xi_{\Lambda_N}|$



$q = 2$
expect $s = 0.25$;
a-priori $s = 1.0$;
a-posteriori $s = 1.1$

- Extended grid = a-posteriori with evaluations in the neighbourhood
- Expected rate smaller than observed:
  - summability argument could be improved
  - bound between number of elements and points not sharp

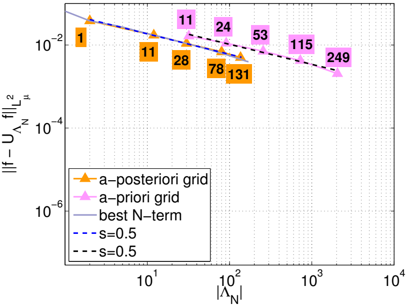
Results: convergence wrt $|\Xi_{\Lambda_N}|$



$q = 3$
expect $s = 0.75$;
a-priori $s = 1.7$;
a-posteriori $s = 1.7$

- Extended grid = a-posteriori with evaluations in the neighbourhood
- Expected rate smaller than observed:
  - summability argument could be improved
  - bound between number of elements and points not sharp

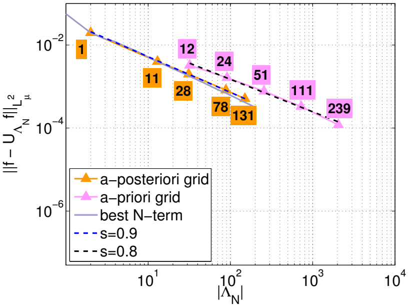
Results: convergence wrt $|\Lambda_N|$



$q = 1$
expect no convergence;
a-priori $s = 0.5$;
a-posteriori $s = 0.5$

- Labels show the number of activated random variables
- Similar rate to before $\Rightarrow$ growth of points linear in $|\Lambda_N|$
- best-N-terms obtained by converting sparse grid into Hermite polynomials and sorting the coefficients

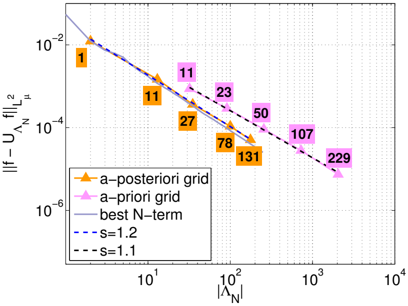
Results: convergence wrt $|\Lambda_N|$



$q = 1.5$
expect $s = 0$;
a-priori $s = 0.8$;
a-posteriori $s = 0.9$

- Labels show the number of activated random variables
- Similar rate to before $\Rightarrow$ growth of points linear in $|\Lambda_N|$
- best-N-terms obtained by converting sparse grid into Hermite polynomials and sorting the coefficients

Results: convergence wrt $|\Lambda_N|$



$q = 2$
expect $s = 0.5$;
a-priori $s = 1.1$;
a-posteriori $s = 1.2$

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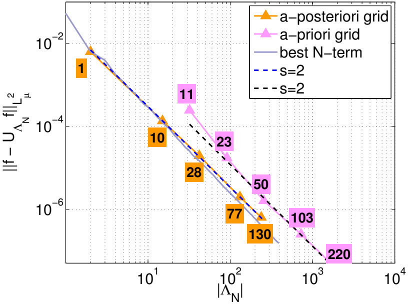
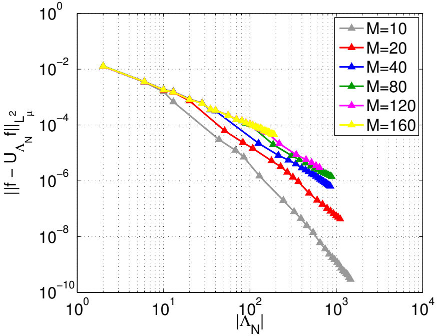


$q = 3$
expect $s = 1.5$;
a-priori $s = 2$;
a-posteriori $s = 2$

- Labels show the number of activated random variables
- Similar rate to before $\Rightarrow$ growth of points linear in $|\Lambda_N|$
- best-N-terms obtained by converting sparse grid into Hermite polynomials and sorting the coefficients

Results: convergence wrt $|\Lambda_N|$ for several M



Convergence of the sparse grid approximation with increasingly larger number of dimensions: the asymptotic rate wrt to $|\Lambda_N|$ is not constant with respect to M (but the rate for $M \to \infty$ is finite).

Part II

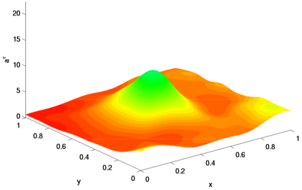
Monte Carlo Control variate approximation of the lognormal problem

Monte Carlo Control Variate

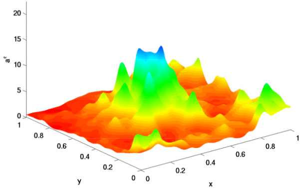
For rough random fields sparse grids may be non-effective.

Remedy: use sparse grids as **control var.** (preconditioner) for MC

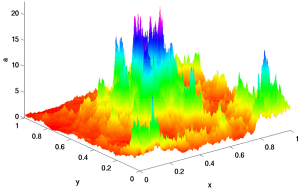
1. Consider a **smoothed field** a^ϵ, such that $\mathcal{Q}_{\mathcal{I}}[u^\epsilon] \to \mathbb{E}[u^\epsilon]$ quickly.



smoothed field, $\epsilon = 1/2^4$



smoothed field $\epsilon = 1/2^6$



non-smoothed field, $\epsilon = 0$

Monte Carlo Control Variate

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Remedy: use sparse grids as **control var.** (preconditioner) for MC

1. Consider a **smoothed field** a^ϵ, such that $\mathcal{Q}_\mathcal{I}[u^\epsilon] \to \mathbb{E}[u^\epsilon]$ quickly.
2. Define $u_{CV} = u - u^\epsilon +$ "$\mathbb{E}[u^\epsilon]$". There holds

$$\mathbb{E}[u_{CV}] = \mathbb{E}[u], \quad \mathbb{V}\mathrm{ar}(u_{CV}) = \mathbb{V}\mathrm{ar}(u) + \mathbb{V}\mathrm{ar}(u^\epsilon) - 2\mathrm{cov}(u, u^\epsilon)$$

Thus, the smaller ϵ, the smaller the MC error, but slower the convergence $\mathcal{Q}_\mathcal{I}[u^\epsilon] \to \mathbb{E}[u^\epsilon]$.

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   Thus, the smaller ϵ, the smaller the MC error, but slower the convergence $\mathcal{Q}_\mathcal{I}[u^\epsilon] \to \mathbb{E}[u^\epsilon]$.

3. Set $\mathbb{E}[u_{CV}] \approx \frac{1}{M}\sum_{i=1}^{M} u^{CV}(\omega_i) = \frac{1}{M}\sum_{i=1}^{M}(u(\omega_i) - u^\epsilon(\omega_i)) + \mathcal{Q}_\mathcal{I}^m[u^\epsilon]$.

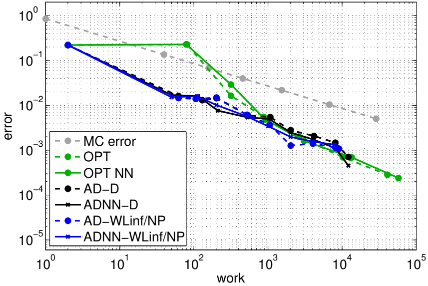
   M can be chosen balancing either the works or the errors of MC and sparse grids.

Numerical results - Part II

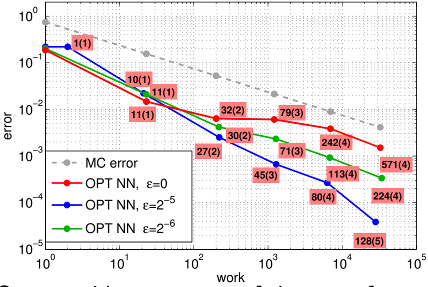
Field data: exponential covariance, $\sigma = 1$, corr. length $L_c = 0.5$

Sparse grids used here:

- **OPT**:** a-priori ("quasi-optimal") construction as in *Beck et al, 2012*
- **AD**:** a-posteriori construction as in *Nobile et al, 2014*



MCCV error for adaptive and quasi-optimal sparse grids. ~ 30 r.v. activated.



Sparse grid component of the error for different values of ϵ. The performance deteriorates as $\epsilon \to 0$

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