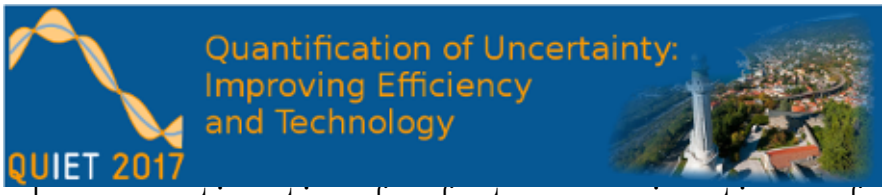

Goal-oriented error estimation for fast approximations of nonlinear problems

A. Janon[1], M. Nodet[2], Cr. Prieur[3], and Cl. Prieur[2]

[1]Université Paris-Sud, France
[2]Université Grenoble Alpes and INRIA, France
[3]CNRS, Grenoble, France

Numerical simulation is a key component of numerous domains: industry, environment, engineering, physics for instance. In some cases time is the limiting factor, and the numerical simulation should be very fast and accurate. The computing time must be very short, either because the computation is repeated many times in a relatively short interval (*many-query context*) or because the result cannot wait (*real-time context*).

To tackle this issue, several procedures of accelerating existing numerical models have been proposed. The general idea of such procedures consists in replacing the existing model, called the *full* model, by a fast and accurate approximation, called *metamodel*, or *surrogate model*. It is possible in some cases to design metamodels which include a certified error bound. In this latter case, the user does not know exactly the approximation error, but the error is guaranteed to be lower than the provided bound. Moreover, the error bound computation is included in the metamodel, so that its computational burden stays small compared to the full model. For example, we can cite [3] where the authors provide such bounds in the framework of the reduced basis method (dimension reduction).

Providing such error bound for nonlinear problems is the aim of this work. More precisely, let $\mathcal{P} \subset \mathbb{R}^d$ denote a parameter space, and let P be a probability distribution on $\mathcal{P}$. Let X (*resp.* Y) be a finite dimensional vector space endowed with a scalar product $\langle , \rangle_X$ (*resp.* $\langle , \rangle_Y$). We consider a nonlinear function $\mathcal{M} : \mathcal{P} \times X \to Y$. Given a parameter $\mu \in \mathcal{P}$, we denote by $u(\mu) \in X$ a solution to the equation:

$$\mathcal{M}(\mu, u(\mu)) = 0, \tag{1}$$

and we define the *output* by $s(\mu) = \langle \ell, u(\mu) \rangle_X$, for a given $\ell \in X$.

We assume that for every $\mu \in \mathcal{P}$, Equation (1) admits a unique solution in X, so that the application $s : \mathcal{P} \to \mathbb{R}$ is well-defined. Denote $\mathcal{N}$ the dimension of X.

As already mentioned, it is common in a *many-query context*, or in a *real-time context*, to call for model reduction (*metamodelling*). More precisely, let $\widetilde{X}$ be a subspace of X, of dimension N such that $N << \mathcal{N}$. We consider $\widetilde{u} : \mathcal{P} \to \widetilde{X}$ a surrogate model (in a very wide sense of the term) of $u : \mathcal{P} \to X$. Let us define the approximate output $\widetilde{s}(\mu)$ by $\widetilde{s}(\mu) = \langle \ell, \widetilde{u}(\mu) \rangle_X$.

The objective is then to provide some probabilistic error bound between $s(\mu)$ and $\widetilde{s}(\mu)$. In other words, one accepts the risk of this bound $\epsilon(\mu; \alpha)$ being violated for a set of parameters having "small" probability measure $\alpha \in (0, 1)$: $\big(|s(\mu) - \widetilde{s}(\mu)| \geq \epsilon(\mu; \alpha)\big) \leq \alpha$. This quantity $\epsilon(\mu; \alpha)$ is a so-called *goal-oriented probabilistic error bound*.

The methodology we propose here in the nonlinear framework [2] is an extension of the one in [1].

References

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