

Reduced-Order Models with Space-Adapted Snapshots

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Numerical
Analysis

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Quantification of Uncertainty: Improving Efficiency and Technology

Motivation



Heart for Adaptivity (coop. Deuflhard et al., ZIB)

Outline

- ▶ Implementation aspects
- ▶ Theoretical aspects
- ▶ Numerical example
- ▶ Conclusion

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- ▶ Theoretical aspects
- ▶ Numerical example
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Answer the following questions:

- ▶ How can the effort for creating reduced-order models with space-adapted snapshots be minimized?
- ▶ How can the union of all snapshot meshes be avoided?
- ▶ What is the main difference between static and adaptive snapshots in the error analysis of Galerkin reduced-order models?

POD with static snapshots

Given snapshots $u_1, u_2, \dots, u_N \in V_h \subset V$ equipped with $(\cdot, \cdot)_V, \|\cdot\|_V$

Find $V^R := \text{span}(\phi_1, \dots, \phi_R) \subset V_h$ with

- $R = \dim(V^R) \ll \dim(V_h) < \infty$
- $u_1^R, \dots, u_N^R \in V^R$ exist so that $u_n^R \approx u_n$ on average

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Solve minimization problem

$$\min_{\phi_1, \dots, \phi_R \in V_h} \sum_{n=1}^N \|u_n - \sum_{r=1}^R (u_n, \phi_r) \phi_r\|^2, \quad (\phi_i, \phi_j) = \delta_{ij}$$

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Eigenvalue decomposition of $G = ((u_i, u_j))_{i,j=1,\dots,N}$

Solve $Ga = \lambda a$

- eigenvalues: $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_D > 0 = \lambda_{D+1} = \dots = \lambda_N \in \mathbb{R}$
- eigenvectors: $a_1, \dots, a_N \in \mathbb{R}^N$

$$\min_{\phi_1, \dots, \phi_R \in V_h} \sum_{n=1}^N \left\| u_n - \sum_{r=1}^R (u_n, \phi_r) \phi_r \right\|^2, \quad (\phi_i, \phi_j) = \delta_{ij}$$

POD (proper orthogonal decomposition) basis functions

$$\phi_r = \sum_{i=1}^N \frac{a_r^n}{\sqrt{\lambda_r}} u_n, \quad r = 1, \dots, D$$

POD approximation $u^R \in V^R$ for $R \leq D$

$$u^R = \sum_{r=1}^R b_r \phi_r = \sum_{r=1}^R \sum_{n=1}^N \frac{b_r a_r^n}{\sqrt{\lambda_r}} u_n$$

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V -orthogonal projection of $v \in V$ onto V^R

$$P^R v := \sum_{r=1}^R (v, \phi_r) \phi_r, \quad \text{thus } b_r = (v, \phi_r)$$

$$\min_{\phi_1, \dots, \phi_R \in V_h} \sum_{n=1}^N \left\| u_n - \sum_{r=1}^R (u_n, \phi_r) \phi_r \right\|^2, \quad (\phi_i, \phi_j) = \delta_{ij}$$

Approximation property of V^R

$$\sum_{n=1}^N \|u_n - P^R u_n\|^2 = \sum_{n=R+1}^D \lambda_n$$

Observe: Error decreases monotonically with R and $P^D u_n = u_n$.

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Optimality of the P^R -projection

$$\|u - P^R u\| = \inf_{v \in V^R} \|u - v\|, \quad u \in V$$

Sirovich, Q. Appl. Math. 45 (1987), 561–590.

Holmes, Lumley, Berkooz, Cambridge University Press, 1996.

Kunisch, Volkwein, J. Optim. Theory Appl. 102 (1999), 345–371.

POD with adaptive snapshots

V_h is replaced by $\{V_n\}_{n=1,\dots,N}$ with $V_n \subset V$.

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Recall $\phi_r \in \text{span}(u_1, \dots, u_n)$. Work with common space $V_+ \subset V$:

$$V_1 + V_2 + \dots + V_N \subset V_+$$

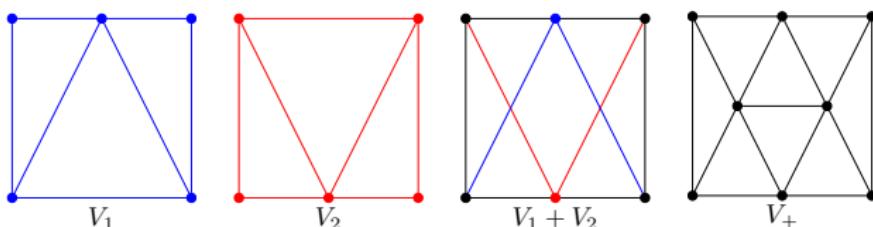
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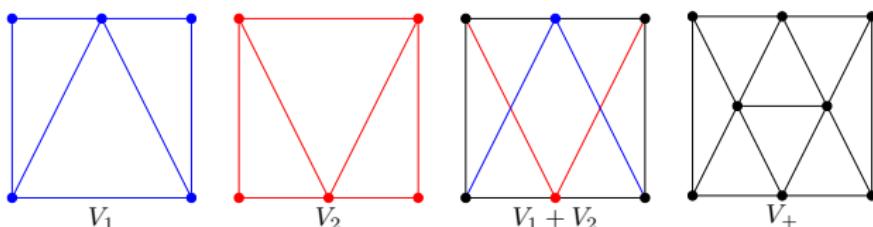
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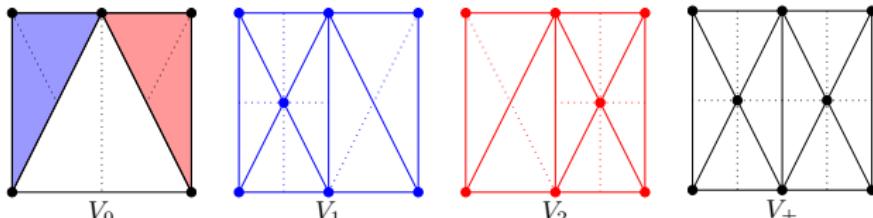
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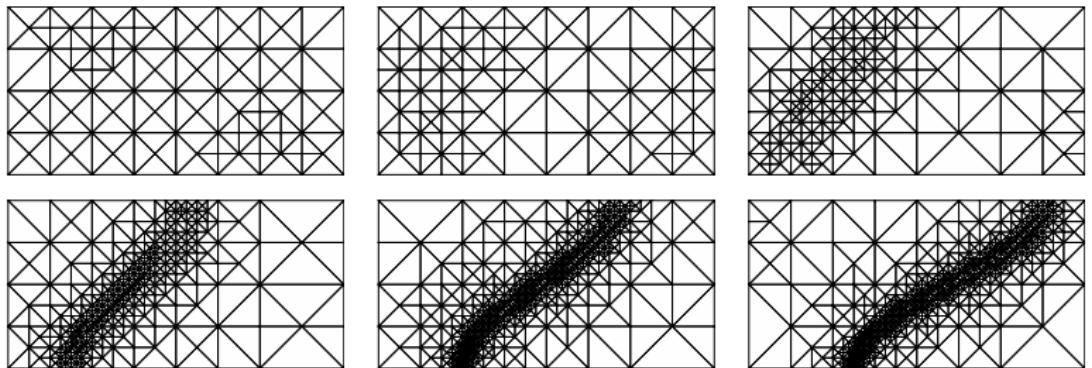
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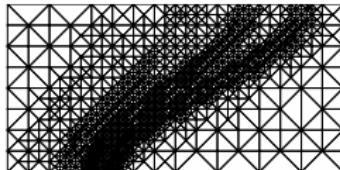
Newest vertex bisection (nested refinement)



$$V_1 + V_2 + \dots + V_N \subset V_+$$



Triangular mesh for common space V_+



$$u^R = \underbrace{\sum_{r=1}^R b_r \phi_r}_{A: \phi_r \in V_+} = \underbrace{\sum_{r=1}^R \sum_{n=1}^N \frac{b_r a_r^n}{\sqrt{\lambda_r}} u_n}_{B: u_n \in V_n}$$

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Approach A: Use common space V_+ .

- Set $V_h = V_+$ and interpret all $u_n \in V_h$. Proceed as with static snapshots to generate POD basis.
- Need N interpolations onto possibly large space V_+ .
- Appropriate if $\dim(V_+)$ is still moderate.

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Approach B: Use common space of pairs of snapshots.

- Define spaces V_{ij} with $V_i + V_j \subset V_{ij}$.
- Compute $((u_i, u_j))$ by interpreting $u_i, u_j \in V_{ij}$.
- Need $N(N - 1)/2$ interpolations onto not so large spaces.
- Method of choice if $\dim(V_+) \gg \dim(V_n)$.

POD-ROM: elliptic PDEs and FE

Original well-posed elliptic problem

For $\mu \in S \subset \mathbb{R}^K$, find $u(\mu) \in V$ such that

$$a(u, v; \mu) = f(v; \mu) \quad \forall v \in V$$

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Setting

Parameterized bilinear form $a(\cdot, \cdot; \mu) : V \times V \rightarrow \mathbb{R}$

$$a(v, v; \mu) \geq \alpha \|v\|^2 \quad \forall v \in V, \quad (\text{uniformly coercive})$$

$$|a(v, w; \mu)| \leq \gamma \|v\| \|w\| \quad \forall v, w \in V \quad (\text{uniformly continuous})$$

Parameterized linear form $f(\cdot; \mu) : V \rightarrow \mathbb{R}$

$$|f(v; \mu)| \leq \delta \|v\| \quad \forall v \in V \quad (\text{uniformly continuous})$$

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Snapshot computation

For $n = 1, \dots, N$, choose $\mu_n \in S$ and find $u_n \in V_n \subset V$ such that

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Reduced-order problem

For $\mu \in S$, find $u^R(\mu) \in V^R \subset \text{span}(u_1, \dots, u_N) \subset V$

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Set $\phi = \phi_1, \dots, \phi_R$ and substitute

$$u^R = \sum_{i=1}^R b_i \phi_i$$

so that for $r = 1, \dots, R$,

$$\sum_{i=1}^R \underbrace{a(\phi_i, \phi_r; \mu)}_{\phi_i, \phi_r \in V_+} b_i = \underbrace{f(\phi_r; \mu)}_{\phi_r \in V_+},$$

which gives the reduced-order model

$$A(\mu) b = F(\mu).$$

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Set $\phi = \phi_1, \dots, \phi_R$ and substitute

$$u^R = \sum_{i=1}^R \sum_{n=1}^N u_n \frac{a_i^n}{\sqrt{\lambda_i}} b_i, \quad \phi_r = \sum_{p=1}^N u_p \frac{a_r^p}{\sqrt{\lambda_r}}$$

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so that for $r = 1, \dots, R$,

$$\sum_{i=1}^R \sum_{n=1}^N \sum_{p=1}^N \underbrace{a(u_n, u_p; \mu)}_{u_n, u_p \in V_{np}} \frac{a_i^n a_r^p}{\sqrt{\lambda_i \lambda_r}} b_i = \sum_{p=1}^N \underbrace{f(u_p; \mu)}_{u_p \in V_p} \frac{a_r^p}{\sqrt{\lambda_r}},$$

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POD-ROM: Error Analysis

Error between original and reduced-order solution

Galerkin orthogonality: Since $V^R \subset V$, for any $\mu \in S$ it holds

$$a(u - u^R, \phi; \mu) = 0 \quad \forall \phi \in V^R.$$

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Cea's Lemma gives

$$\|u - u^R\| \leq \frac{\gamma}{\alpha} \inf_{v \in V^R} \|u - v\| = \frac{\gamma}{\alpha} \|u - P^R u\|$$

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and for the snapshot locations

$$\|\underbrace{u(\mu_n) - u^R(\mu_n)}_{POD ROM}\| \leq \frac{\gamma}{\alpha} \left(\|\underbrace{u(\mu_n) - u_n}_{FEM}\| + \|\underbrace{u_n - P^R u_n}_{POD}\| \right)$$

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Error between snapshots and reduced-order solution

Case: static snapshots

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and therefore

$$\left\| \underbrace{u_n - u^R(\mu_n)}_{POD\ ROM} \right\| \leq \frac{\gamma}{\alpha} \left\| \underbrace{u_n - P^R u_n}_{POD} \right\|$$

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Case: adaptive snapshots

no Galerkin orthogonality: $V^R \not\subset V_i$, $V^R \subset V_+$

$$\left\| \underbrace{u_n - u^R(\mu_n)}_{POD\ ROM} \right\| \leq \left(1 + \frac{\gamma}{\alpha}\right) \left\| \underbrace{u(\mu_n) - u_n}_{FEM} \right\| + \frac{\gamma}{\alpha} \left\| \underbrace{u_n - P^R u_n}_{POD} \right\|$$

POD-ROM: Numerical Example

Find $u(x, \mu) : \Omega \times S \rightarrow \mathbb{R}$, $\Omega = [0, 1] \times [0, 1]$, $S = [0, 1]$, such that

$$\begin{aligned} (\cos(\pi\mu/4), \sin(\pi\mu/4)) \cdot \nabla u - 0.01 \Delta u &= 1 && \text{in } \Omega \times S, \\ u &= 0 && \text{on } \partial\Omega \times S. \end{aligned}$$

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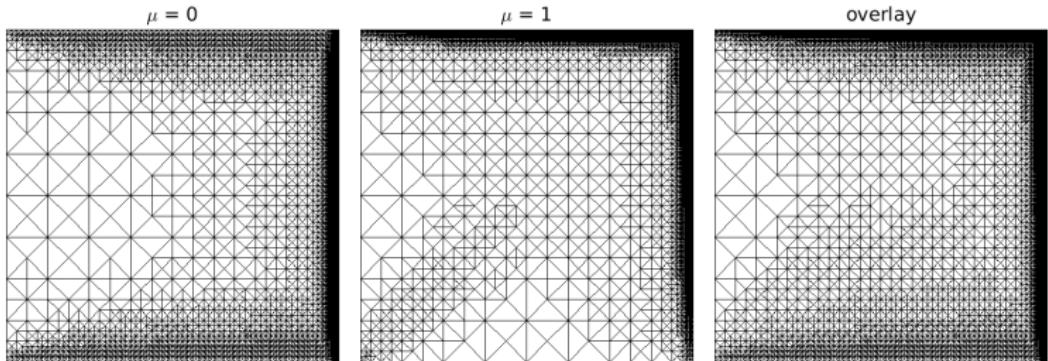
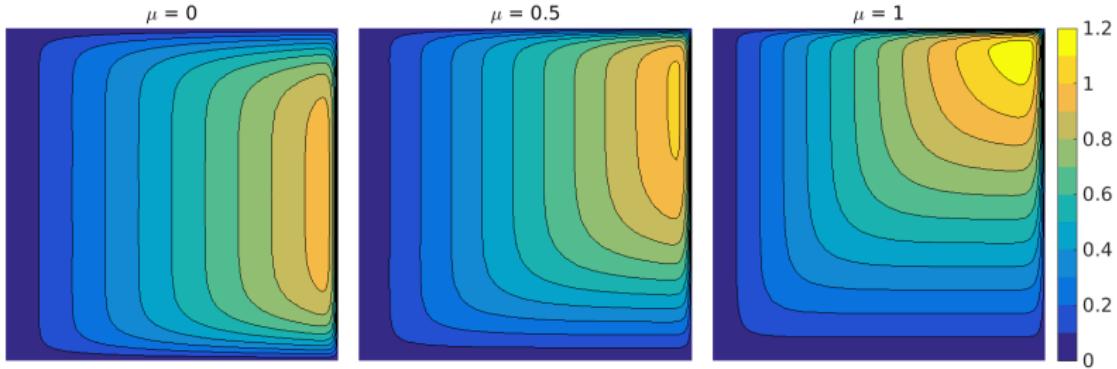
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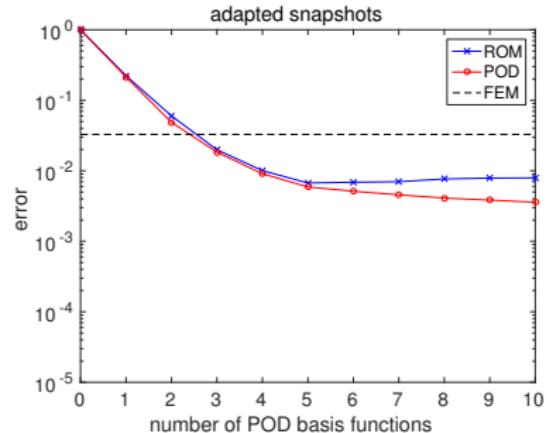
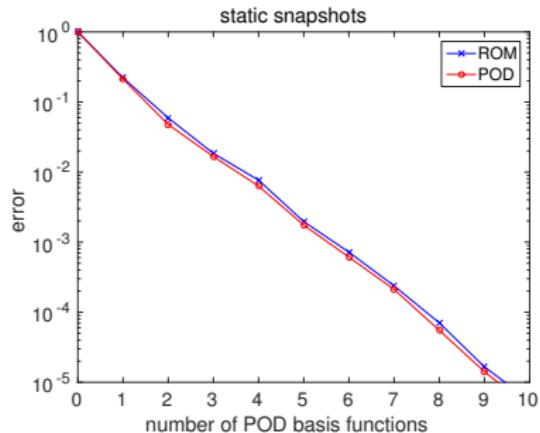
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- ▶ Linear finite elements on conforming triangulation.
- ▶ Error indicator for triangle K with Matlab `pdejmps`:

$$E(K) = \left(\frac{1}{2} \sum_{\tau \in \partial K} h_\tau^2 [\vec{n}_\tau \cdot (\nu \nabla u_h)]^2 \right)^{\frac{1}{2}}.$$

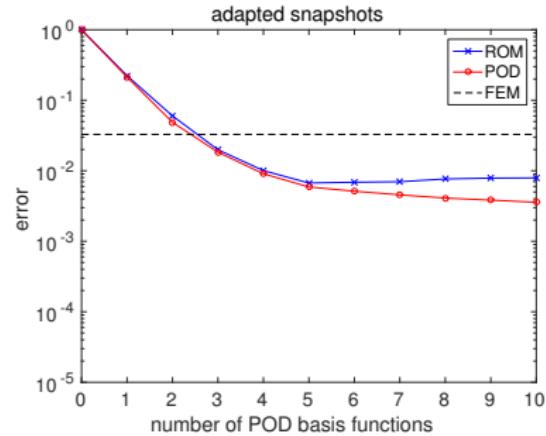
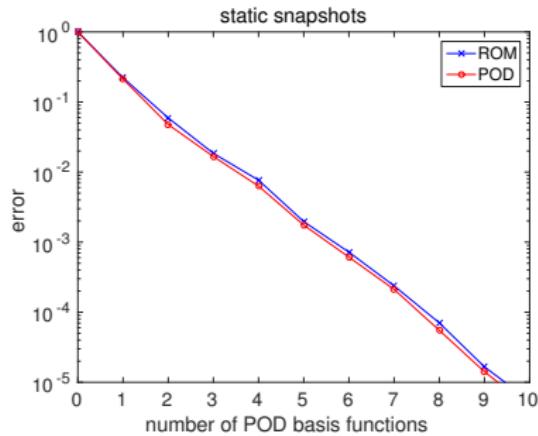
- ▶ Marking and termination with Matlab `pdeadgsc`.
- ▶ Refinement by newest vertex bisection.
- ▶ 33 equidistant parameter points.





Static snapshots:

$$\left\| \underbrace{u_n - u^R(\mu_n)}_{POD ROM} \right\| \leq \frac{\gamma}{\alpha} \left\| \underbrace{u_n - P^R u_n}_{POD} \right\|$$

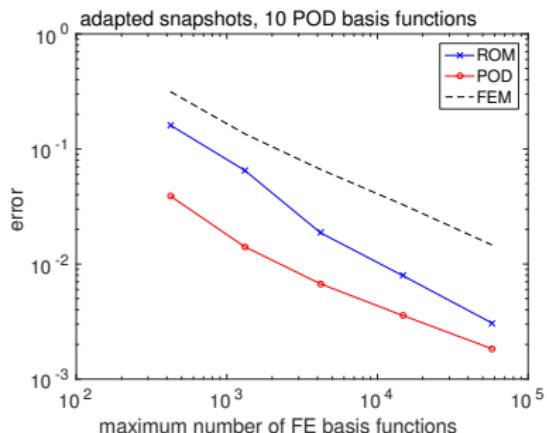
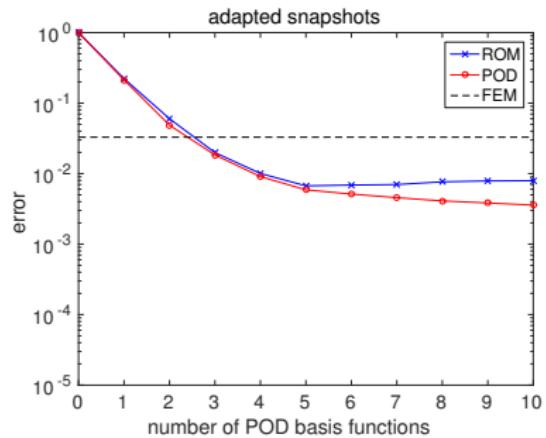
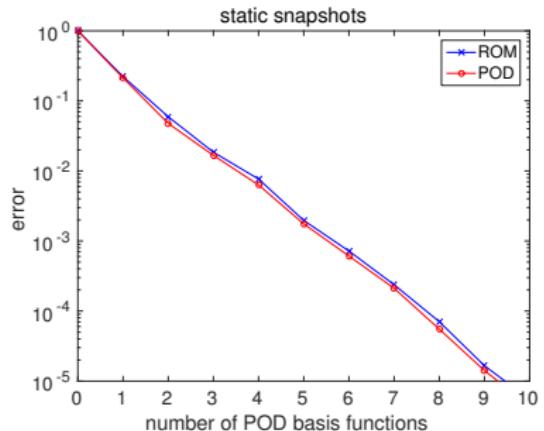


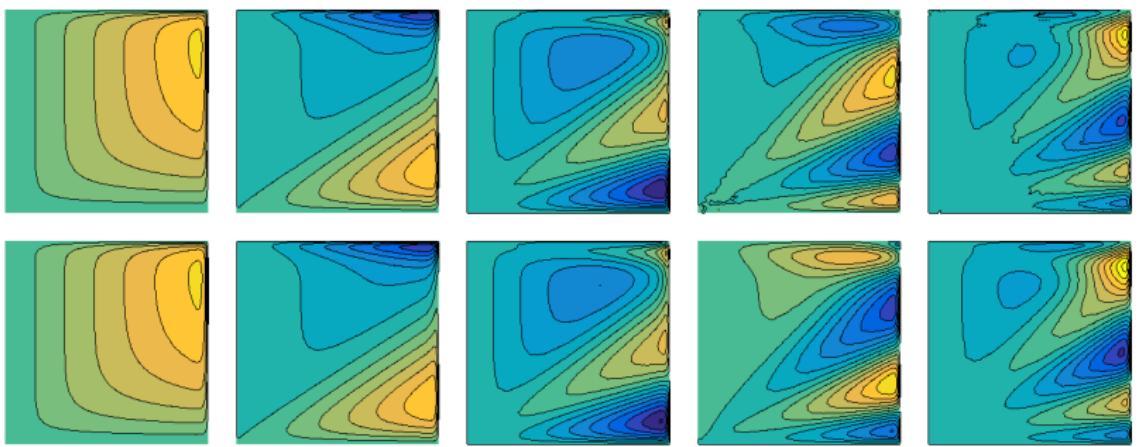
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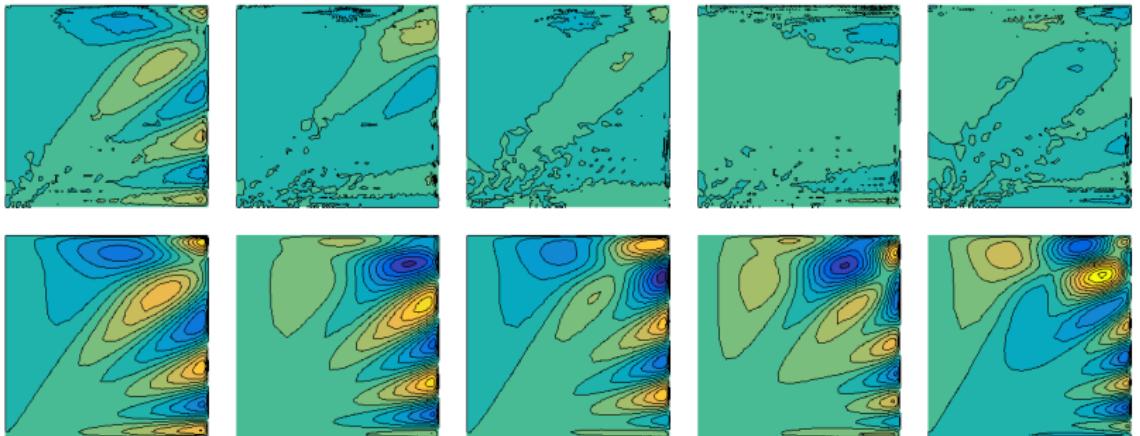
Adaptive snapshots:

$$\left\| \underbrace{u_n - u^R(\mu_n)}_{POD\ ROM} \right\| \leq \left(1 + \frac{\gamma}{\alpha} \right) \left\| \underbrace{u(\mu_n) - u_n}_{FEM} \right\| + \frac{\gamma}{\alpha} \left\| \underbrace{u_n - P^R u_n}_{POD} \right\|$$





POD ϕ_1, \dots, ϕ_5 (top), ϕ_6, \dots, ϕ_{10} (bottom): adaptive versus static



Summary and Outlook

- ▶ POD framework extended to adapted FE snapshots.
- ▶ Exact POD computation enabled by common FE spaces.
- ▶ Common FE space of all snapshots can be avoided.
- ▶ POD-ROM reproduces snapshots up to FE error.

Summary and Outlook

- ▶ POD framework extended to adapted FE snapshots.
- ▶ Exact POD computation enabled by common FE spaces.
- ▶ Common FE space of all snapshots can be avoided.
- ▶ POD-ROM reproduces snapshots up to FE error.

- ▶ Time adaptivity for space-time reduced basis methods?
- ▶ Ultimate goal: Balance POD and FE error contributions.
- ▶ Apply to problems with random data.

Ullmann, Rotkvic, L., JCP 325 (2016), 244–258.

Ali, Steih, Urban, Adv. Comput. Math. 43 (2017), 257–294.

Yano, M2AN 50 (2016), 163–185.