Semi-intrusive Uncertainty Quantification for Multiscale models

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Multiscale models

Consider a PDE of the form

$$\frac{\partial u(x,t,\xi)}{\partial t} = \mathcal{L}(u(x,t,\xi),\xi),$$

where $\mathcal{L}$ is an operator acting in the space variable, and ξ denotes n-dimensional space of uncertain input. The analytical solution of this PDE satisfies

$$u(x,t+\Delta t,\xi) = e^{\Delta t \mathcal{L}} u(x,t,\xi).$$

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Let us assume for $\mathcal{L}$ a two-term splitting:

$$\mathcal{L} = \mathcal{L}^{\mu} + \mathcal{L}^{M},$$

where $\mathcal{L}^{\mu}$ and $\mathcal{L}^{M}$ are subscale models with micro and macro time scale, respectively. Thus, the equation can be rewritten

$$u(x, t + \Delta t, \xi) \approx e^{\Delta t \mathcal{L}^{\mathcal{M}}} e^{\Delta t \mathcal{L}^{\mu}} u(x, t, \xi)$$

Multiscale models

The original PDE can be approximated as a sequence of the following two sub-systems

$$\frac{\partial u^*(x,t,\xi)}{\partial t} = \mathcal{L}^{\mu} u^*(x,t,\xi), \text{ for } t_n < t < t_{n+1},$$

$$\text{with } u^*(x,t_n,\xi) \approx u(x,t_n,\xi)$$

$$\frac{\partial u^{**}(x,t,\xi)}{\partial t} = \mathcal{L}^{\mathcal{M}} u^{**}(x,t,\xi), \text{ for } t_n < t < t_{n+1},$$

$$\text{with } u^{**}(x,t_n,\xi) = u^*(x,t_{n+1},\xi)$$

Submodel Execution Loop

In general, a model with two or more different time scales can be illustrated by a *Submodel Execution Loop* [BFL$^+$13, CFHB11].

$\mathcal{L}^{\mu}$

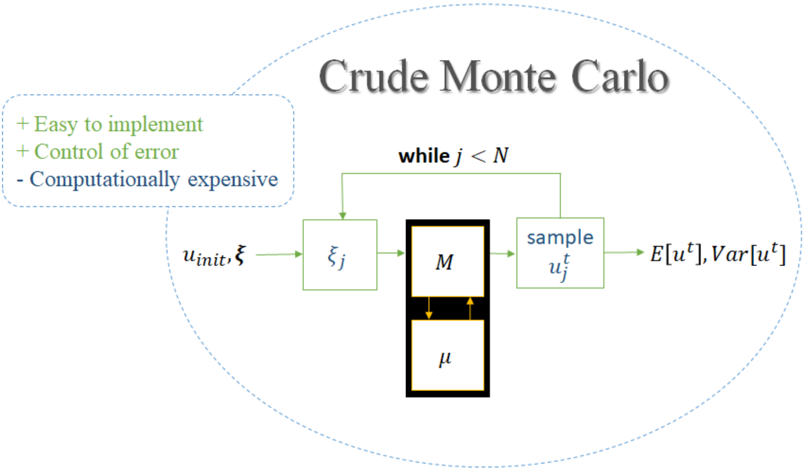
$u^t \leftarrow u_{init}$
$t \leftarrow t_1$
while $t_1 \leq t < t_1 + \Delta t_2$
  $O(u^t)$
  $u^{t+\Delta t_1} \leftarrow S(u^t)$
  $u^{t+\Delta t_1} \leftarrow B(u^{t+\Delta t_1})$
  $t \leftarrow t + \Delta t_1$
endwhile
$O(u^{t+n_1 \Delta t_1})$

$\mathcal{L}^{M}$

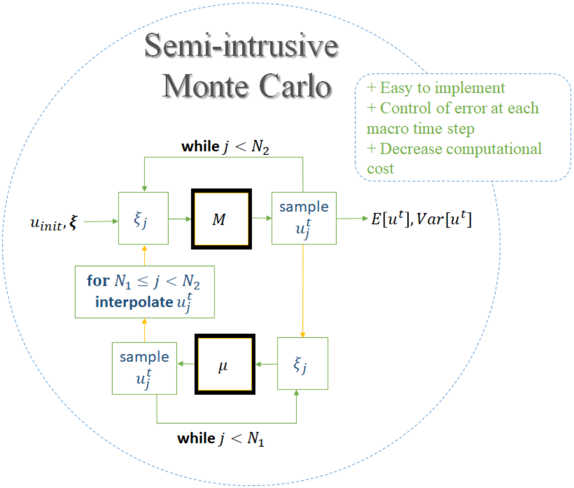
$u^0 \leftarrow u_{init}$
$t \leftarrow 0$
while $0 \leq t < t_{end}$
  $O(u^t)$
  $u^{t+\Delta t_2} \leftarrow S(u^{t+n_1 \Delta t_1})$
  $u^{t+\Delta t_2} \leftarrow B(u^{t+\Delta t_2})$
  $t \leftarrow t + \Delta t_2$
endwhile
$O(u^{t_{end}})$

u_{init} are some initial conditions for a sub-scale model, O is the observation of the current state, S is the solver, and B is the application of boundary conditions.

Black Box Monte Carlo



Semi-intrusive Monte Carlo



The Gray-Scott reaction diffusion model

$$\frac{\partial u(t,x,y,\xi)}{\partial t} = \underbrace{D_u(\xi_1)\nabla^2 u - uv^2}_{\text{Macro scale model}} + \underbrace{F(\xi_2)(1-u)}_{\text{Micro scale model}}$$

$$\frac{\partial v(t,x,y,\xi)}{\partial t} = \overbrace{D_v(\xi_3)\nabla^2 v + uv^2} - \overbrace{(F(\xi_2)+K(\xi_4))v}$$

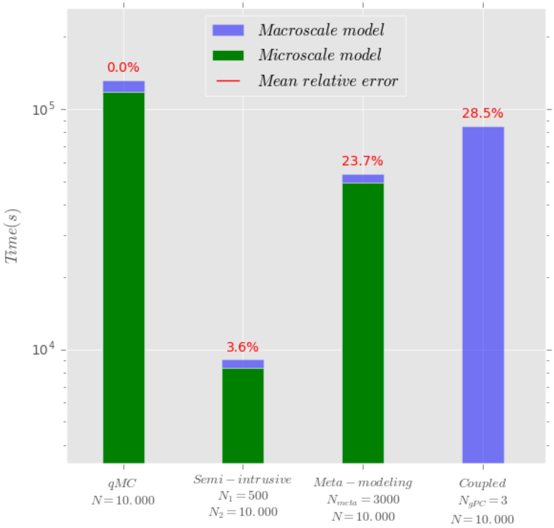
where the model reaction and diffusion coefficients contain 10% uncertainty with mean values

$$\mathbb{E}(D_u(\xi_1)) = 2\cdot 10^{-5},\ \mathbb{E}(F(\xi_2)) = 0.025,$$

$$\mathbb{E}(D_v(\xi_3)) = 1\cdot 10^{-5},\ \mathbb{E}(K(\xi_4)) = 0.053.$$

The model reproduces a compex pattern formation with a transition map studied in [HsQHS16].

Comparisomn of the performance of the UQ methods



References I

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