# Analytic continuations, black holes perturbations and CFT

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# Introduction

We will discuss the problem of the analytic continuation of solutions of ODEs.

This problem has a lot of applications, for both mathematics
(spectral problems, monodromy problems, ..)
and physics
(black hole perturbations, holography, CFT, gauge theories, ..).

We will formulate the problem, see how ideas from physics (CFT) can help to find a solution, and take a look at applications.

# Plan of the talk

- Formulation of the problem and applications
- Hypergeometric equation
- Heun equation

# Formulation of the problem and applications

ODEs are ubiquitous in mathematics and physics.

Very rarely one can find explicit solutions

$$\left(\partial_z^2 - 1\right)\psi(z) = 0 \Rightarrow \psi(z) = c_1 e^{z} + c_2 e^{-z}\,.$$

More often, no explicit solution is known.

For concreteness we consider the case

$$\left(\partial_z^2 - V(z)\right)\psi(z) = 0\,,$$

where

$$V(z) \sim \frac{\#}{(z - z_i)^2}\,, \quad i = 1, \ldots, n$$

This class goes under the name of *Fuchsian ODEs*. Of course in general explicit solutions are not known.

However, it's always easy to write down local solutions (Frobenius expansions). The prescription to write down such a solution, say for $z \sim z_i$, is the following: we write the ansatz

$$\psi(z) = (z - z_i)^\alpha \sum_{n \geq 0} c_n (z - z_i)^n \,,$$

plug it into the ODE and solve for $\alpha, c_n$. Since we are working with 2nd order ODEs we will find two solutions

$$(\alpha^+, c_n^+), \;\; (\alpha^-, c_n^-)$$

Finally one can write

$$\psi(z) = \mathcal{A}^+ (z - z_i)^{\alpha^+} \sum_{n \geq 0} c_n^+ (z - z_i)^n + \mathcal{A}^- (z - z_i)^{\alpha^-} \sum_{n \geq 0} c_n^+ (z - z_i)^n$$

This is enough if we want to extract information locally, close to some singularity.

What about the **global** structure of the solutions?

Let's choose for simplicity $\mathcal{A}^- = 0$, $\mathcal{A}^+ = 1$, so that

$$\psi(z) = (z - z_i)^{\alpha^+} \sum_{n \geq 0} c_n^+ (z - z_i)^n \, .$$

Our solution around another singularity, say $z \sim z_j$, will look like this:

$$\begin{aligned} \psi(z) &= (z - z_i)^{\alpha^+} \sum_{n \geq 0} c_n^+ (z - z_i)^n \\ &= \mathcal{B}^+ (z - z_j)^{\beta^+} \sum_{n \geq 0} k_n^+ (z - z_j)^n + \mathcal{B}^- (z - z_j)^{\beta^-} \sum_{n \geq 0} k_n^+ (z - z_j)^n \end{aligned}.$$

$\mathcal{B}^{\pm}$ are the so called **connection coefficients** of the ODE. They encode information about the global structure.

Why are they interesting?

They allow you to analytically continue the solution form one singularity to the other, eventually covering all the Riemann sphere.

You can use them to compute monodromies around several singularities just by carrying the solution around the complex plane.

You can use them to solve spectral problems. Imagine

$$\alpha^{\pm} = \pm\mathbb{R}_{>0}\,, \quad \beta^{\pm} = \pm\mathbb{R}_{>0}\,.$$

Then you can search for a bounded ($\sim L_2$) solution as follows:

$$\psi(z) = (z - z_i)^{\alpha^+} \sum_{n \geq 0} c_n^+ (z - z_i)^n = \mathcal{B}^+ (z - z_j)^{\beta^+} \sum_{n \geq 0} k_n^+ (z - z_j)^n\,,$$

This requires

$$\mathcal{B}^{+} = 0\,,$$

that is a condition on the parameters of the ODE. If you think at this as a Schrodinger problem, $\mathcal{B}^{+} = 0$ is the equation for the discrete eigenvalues (energies).

This is one way this problem has many application in physics.

The setting we are mostly interested into is the one of **black hole perturbations**.

Black hole perturbations satisfy some PDE, that typically separates and reduces to a Fuchsian equation.

In black hole problems we typically ask for perturbations to fall off into the horizon, since classically nothing comes out of the horizon, and we want to know how the perturbations looks like at infinity ($\sim$ observer).

The problem again reduces to computing some connection coefficients:

$$\psi(r) = (r - r_{hor})^{\#_{in}} (1 + \dots) = \mathcal{A} r^{\#_{in}} (1 + \dots) + \mathcal{B} r^{\#_{out}} (1 + \dots) \ .$$

All physical informations are hidden in the connection coefficients.

One immediate example is the one of quasinormal modes.

When you perturb a black hole, it will just radiate away the perturbation away after some time.

This corresponds to impose the following boundary condition:

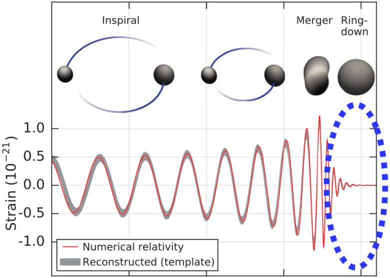
$$\psi(r) = (r - r_{hor})^{\#_{in}} (1 + \dots) = \mathcal{B} r^{\#_{out}} (1 + \dots) \ ,$$

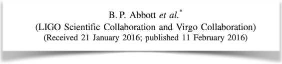
that is

$$\mathcal{A} = 0 \, .$$

This is again some kind of spectral problem solved by these quasinormal modes.

This frequencies are interesting for observations

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damped oscillations

$$\sim \sum_n A_n \exp[-i\omega_n t], \quad \mathrm{Im}(\omega_n) < 0$$

but also for more abstract problems related to holography and thermal CFT.

I hope I convinced you that the connection coefficients of Fuchsian ODEs are very interesting objects.

The natural question at this point is: can we compute them?

In one easy case classical methods in mathematics allow us to compute these connection coefficients, but in general the situation is much more complicated and requires more sophisticated techniques.

Let's start by discussing the easy case.

# The hypergeometric equation

This is the case where

$$V(z) \simeq \frac{\#}{(z - z_i)^2}\,, \quad i = 1, 2, 3\,.$$

This is an ODE on a three-punctured sphere. Since we can always fix 3 points on a sphere, we set

$$z_1 = 0,\; z_2 = 1,\; z_3 = \infty\,.$$

We can try to write again a Frobenius ansatz

$$\psi(z) = z^{\alpha} \sum_n c_n z^n\,.$$

In this case this admits the following solution:

$$\psi(z) = z^{\#_\pm}(1-z)^{\#}\,{}_2F_1\left(a_\pm, b_\pm; c_\pm, z\right)$$

where

$${}_2F_1\left(a, b; c, z\right) = \sum_{n\geq 0} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}\,, \quad (x)_n = \frac{\Gamma\left(x+n\right)}{\Gamma\left(x\right)}\,.$$

Note that to know the solution for the n-th coefficient is a very rare situation.

In some sense we can expect that knowing the asymptotic form of the coefficients is equivalent to recover some global properties of the solution of the ODE.

In this case, we have a powerful tool at our disposal: ${}_2F_1$ admits an integral representation

$$
{}_2F_1\left(a, b; c, z\right) \propto \int_0^1 x^{b-1}\left(1-x\right)^{c-b-1}\left(1-zx\right)^{-a} \, .
$$

Expanding the integral for small $z$ we recover the previous series expansion, but we can also the the expansion close to other singularities, for example for $z \sim 1$. In particular,

$$
\int_0^1 x^{b-1}\left(1-x\right)^{c-b-1}\left(1-x\right)^{-a} = \frac{\Gamma\left(b\right)\Gamma\left(c-b-a\right)}{\Gamma\left(c-a\right)} \, .
$$

Note that the integral for $z \sim 0$ and $z \sim 1$ are basically the same up to reshuffling the exponents.

This allows us to connect the expansion close to the 0 with the one close to 1. One finds

$$
{}_2F_1(a,b,c,z) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}{}_2F_1(a,b,a+b+1-c,1-z) +
$$

$$
+\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}(1-z)^{c-a-b}\,{}_2F_1(c-a,c-b,-a-b+1+c,1-z)\ .
$$

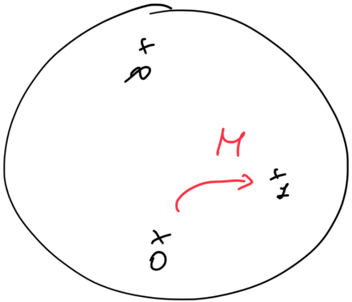
These bunch of Gamma functions

$$
(M_{++}, M_{+-}) = \left(\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\right)
$$

are the *hypergeometric connection coefficients*.

$M_{\pm\pm}$ have two indices because they come from a 2nd order ODE:

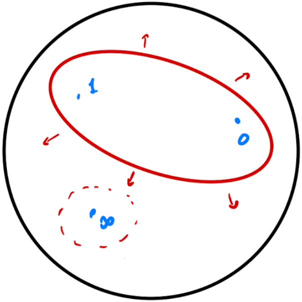
$$\begin{pmatrix} {}_2F_1^+(z) \\ {}_2F_1^-(z) \end{pmatrix} = \begin{pmatrix} M_{++}\,{}_2F_1^+(1-z) + M_{+-}\,{}_2F_1^-(1-z) \\ M_{-+}\,{}_2F_1^+(1-z) + M_{--}\,{}_2F_1^-(1-z) \end{pmatrix} .$$



Note that the hypergeometric equation only depends of 3 parameters $(a, b, c)$.

They correspond to the 3 local monodromies around the singularities at $0, 1, \infty$.

Monodromy around two singularities can be reduces to the monodromy around only one singularity.



An example of a physical application of these connection formulas is the following: free particle in $AdS_5$ of mass $\Delta(\Delta - 4)$.

The radial wave function is

$$\psi(r) = (1-z)^{\frac{\Delta-1}{2}} z^{1+\frac{\ell}{2}} {}_2F_1\left(\frac{1}{2}(\ell+\Delta-\omega), \frac{1}{2}(\ell+\Delta+\omega), 2+\ell, z\right),$$

$$z = \frac{r^2}{1+r^2}.$$

This is given as a series for $r \sim 0$ (center of $AdS_5$).

Note that

$$\psi(0) = 0.$$

In order for the wave function to be $L_2$ we need to study the behavior at $r \to \infty$ (that is $z \to 1$).

$$\psi(r) \sim M_{++}(1-z)^{\frac{\Delta-1}{2}}(1+\dots) + M_{+-}(1-z)^{\frac{-\Delta-1}{2}}(1+\dots)\,.$$

Since $\Delta > 1$ in order for the solution to be bounded we need

$$M_{+-} = 0\,.$$

Since $\Gamma$ functions have only poles, the only way this can happen is by making some $\Gamma$ function in the denominator of $M_{+-}$ blow up.

We need

$$\Gamma\left(\frac{\ell+\Delta-\omega_n}{2}\right)^{-1} = 0 \Rightarrow \frac{\ell+\Delta-\omega_n}{2} = -n\,, n \in \mathbb{Z}_{\geq 0}\,.$$

This gives

$$\omega_n = 2n + \Delta + \ell\,,$$

and since $\omega \sim$energy, this is the energy spectrum of a free particle in $AdS_5$.

What happens if we put a black hole in the center of $AdS_5$?

From a physics perspective, this is a very interesting questions for various reasons. Mathematically, the presence of a black hole adds a singularity to the ODE.

# The Heun equation

Now we consider the potential

$$V(z) \simeq \frac{\#}{(z-z_i)^2}\,, \quad z_i = 0, 1, \infty, t\,.$$

Since we can only fix 3 point on a sphere, the position of the 4th singularity is new parameter of the problem. This complicates the treatment greatly.

To have an idea why, let's write down the full equation:

$$\left(\partial_z^2 + \frac{\frac{1}{4}-a_1^2}{(z-1)^2} + \frac{\frac{1}{4}-a_t^2}{(z-t)^2} + \frac{\frac{1}{4}-a_0^2}{z^2} + \right.$$
$$\left. - \frac{\frac{1}{2}-a_1^2-a_t^2-a_0^2+a_\infty^2+u}{z(z-1)} + \frac{u}{z(z-t)}\right) f(z) = 0$$

Let's count the parameters:

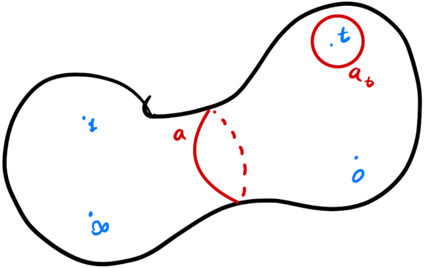
$$\text{position of the new singularity}: \; t\,,$$
$$\text{local monodromies}: \; a_0,\, a_1,\, a_t,\, a_\infty\,,$$

but there is an extra independent parameter: the *accessory parameter* $u$.

This new parameter is signaling that the monodromy around two points can't be written in terms of the local monodromies around one singularities as it happened for the hypergeometric.

We need new information.

Unfortunately, the relation between $u$ and and this combined monodromy is not easy. Let's call $a$ this monodromy:




The relation $u(a)$ is very complicated. When $t \ll 1$,

$$u = \sum_{n \geq 0} c_n(a) t^n$$

When it comes to the Heun equation, everything we've said for the hypergeometric breaks down:

we don't have an explicit solution for the Frobenius series,

we don't have an integral representation,

monodromies are difficult.

How can we solve the connection problem in this case?

Since classical mathematical ideas don't help, let's see if we can steal something from physics.

In the context of 2d Liouville CFT a very similar problem has been solved in the 90's (by Dorn, Otto, Zamolodchikov and Zamolodchikov).

The problem was very roughly the following. In 2d CFT you typically want to compute correlation functions of local operators

$$\langle V_1(z_1) V_2(z_2) \dots V_n(z_n) \rangle \,,$$

where $z_i$ are holomorhic coordinates on the Riemann sphere. Since the theory enjoys a lot of symmetries (Virasoro), series expansions of correlators are easy: in the expansion

$$\langle V_1(z_1) V_2(z_2) \dots V_n(z_n) \rangle \sim \sum_{n,i,j} c_n (z_i - z_j)^{\# + n}$$

the coefficients $c_n$ are computable up to an overall normalization.

Turns out you can do various kinds of series expansions: you can expand for $z_1 \sim z_2$, for $z_1 \sim z_3$ and so on.

Since this correlator is a physical object, consistency of the theory requires that all the different expansions have to agree (*crossing symmetry*). This fact may seem trivial, but is actually a very strong constraint.

In particular, this fixes the normalization of the previous series expansion.

This problem looks very similar to the one we need to solve: we have local series centered around different points in the sphere, and we want something that relates them.

This problem has been solved already, so if we can rephrase our problem in the language of Liouville CFT, maybe we can use this solution to compute our connection coefficients.

The question is: can we find a correlation function that solves the Heun equation?

The answer is yes: in fact, in Liouville CFT there are zero norm states. We want correlation functions involving such states to vanish in order to preserve unitarity of the theory.

This turns into a differential equation acting on correlators involving special operators. We call this special operator $\phi_{2,1}$.

Correlation functions made up of $\phi_{2,1}$ and $n$ other insertions

$$\langle \phi_{2,1}(z) V_1(z_1) V_2(z_2) \dots V_n(z_n) \rangle$$

will satisfy a Fuchsian ODE with $n \sim (z - z_i)^{-2}$ singularities. To recover the Heun equation we need to consider

$$\langle V_\infty(\infty) V_1(1) V_t(t) \phi_{2,1}(z) V_0(0) \rangle \,.$$

Turns out, crossing symmetry of this correlator imposes constraints that are strong enough to allow us to solve explicitly for the connection coefficients.

Before sketching how the solution looks like, there is an important remark to make. The CFT is sensible to the monodromies of the equation.

In particular, our solution for the connection coefficients will depend on the monodromy around two points: $a$. We've seen that $a$ doesn't enter the equation, but $u$ does, and

$$u = \sum_n c_n(a) t^n \,.$$

In order to write down a solution that depends on $u$ we need to invert this complicated relation:

$$u(a) \Rightarrow a = \sum_n c_n(u) t^n \,.$$

This looks complicated, and one can ask how concrete our solution can be made.

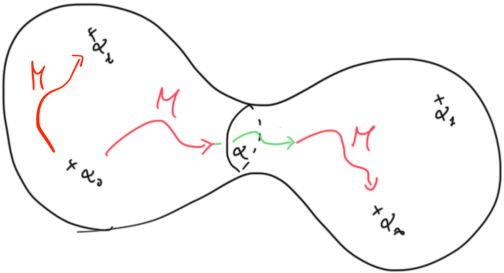
Luckily, there is a very convenient way to compute this $u$ as a function of $a$, and this provides an algorithmic procedure to find $a(u)$.

In fact this $u(a)$ appears as an integral in a completely different theory that is *AGT-dual* to our Liouville CFT: this is a 4d supersymmetric gauge theory.

This integral can be computed by localization, and the answer is given as a combinatorial series in $t$. This is very convenient, since it provides a completely algebraic way of inverting this relation.

Of course one could compute everything without having to resort to this gauge theory, but since this provides a completely algebraic algorithm, it looks the most convenient way to practically do computations.

That said, the solution (very) schematically looks like this:



If we go back to the particle in $AdS_5$ problem, now can solve the problem also when there is a black hole sitting in the center of $AdS$.

We can for example compute the energies:

$$\omega_n = \Delta + \ell + 2 + \sum_{n \geq 1}^{\ell} \mu^n c_n + i\mu^\ell c^* + \dots .$$

The solution is given as a $\mu$-series, where $\mu$ is related to the mass of the black hole. When $\mu = 0$ we recover the previous result.

These are the quasinormal modes of the black hole. In fact, they have a nonzero imaginary part that says that the particle will eventually decay in the horizon.

# Conclusions

- We outlined a procedure that allow us to compute the connection coefficients of the Heun equations from CFT correlators.
- We schematically discussed some physical and mathematical applications.
- Nothing prevents us to adding singularities to the ODE.
- We can also consider confluent limits. This is particularly interesting since it is related to asymptotically flat black holes.

Thanks for the attention!